GALOIS GROUPS, GROUP HOMOLOGY, AND RECIPROCITY: EXERCISES

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Let Γ be a group, and let R be a ring (commutative, with identity). Let A be a $R\Gamma$-module.

(1) Use the standard resolution to show that $H_1(\Gamma, \mathbb{Z})$ is naturally isomorphic to the abelianization $\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma]$.

(2) Suppose $\Gamma \subseteq G$. Suppose $g \in G$ normalizes Γ. Let $[z] \in H_r(\Gamma, A)$. Prove that zg is again a cycle and its class $[zg]$ depends only on $[z]$.

(3) Let S be a semigroup such that $\Gamma \subseteq S \subseteq G$, and for all $s \in S$, we have $\Gamma \cap s\Gamma s^{-1}$ and $\Gamma \cap s^{-1}\Gamma s$ are finite index in Γ. If $[z] \in H_r(\Gamma, A)$ is represented by $z = \sum f_i \otimes_{R\Gamma} a_i$, for each $s \in S$, define a Hecke operator T_s by

$$T_s([z]) = [\sum_i \sum_\alpha f_i s_\alpha \otimes_{R\Gamma} a_i s_\alpha],$$

where

$$\Gamma s \Gamma = \coprod_\alpha s_\alpha \Gamma.$$

Show that this description of Hecke operator is well-defined. Specifically, prove T_s sends cycles to cycles and boundaries to boundaries.

(4) Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. You may assume Γ is the free product of $\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (of order 2) and $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ (of order 3) So

$$\Gamma^{\mathrm{ab}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

Using the standard resolution of $\mathrm{PGL}_2(\mathbb{Q})$, compute the Hecke operator T_ℓ on $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$ for odd ℓ and $\ell = 2$. Find the attached Galois representation.

Hints: Let x_Γ denote the image of x in the coinvariants M_Γ for x in a Γ-module M.

(a) $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ generated by the cycle $[1, \sigma]_\Gamma$ in the standard resolution of $\mathrm{PGL}_2(\mathbb{Q})$.

(b) If $s \in \mathrm{PGL}_2(\mathbb{Q})$ and $\gamma \in \Gamma$, then $[s, s\gamma]_\Gamma$ is homologous to $[1, \gamma]_\Gamma$.

Warning: This is a hard and lengthy exercise.

(5) Work out the well-rounded retract and Voronoi tessellation for $n = 2$. See Example A.15 in the Appendix of Modular Forms, a Computational Approach if you get stuck.