GALOIS REPRESENTATIONS, GROUP HOMOLOGY, AND RECIPROCITY

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Abstract. Notes by Dan Yasaki from three lectures by Avner Ash at the 2014 UNCG Summer School in Computational Number Theory: Modular Forms and Geometry.

1. What is a reciprocity law?

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. The automorphism group of $\overline{\mathbb{Q}}$ is the *absolute Galois group* of $\mathbb{Q}$,

$$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Any automorphism in $G_{\mathbb{Q}}$ fixes $\mathbb{Q}$ pointwise and acts on roots of irreducible polynomials with $\mathbb{Z}$-coefficients. This gives restriction maps. Specifically, let F be the splitting field of f. Then, we have a map

$$r_F \colon G_{\mathbb{Q}} \to \mathrm{Gal}(F/\mathbb{Q})$$

given by restriction. We will use a subscript of the field name to denote restriction when convenient. The restrictions are compatible in the sense that if $F \subset F'$, we have $r_{F'/F} \circ r_{F'} = r_F$ so that if $\sigma \in G_{\mathbb{Q}}$,

$$r_F(\sigma_{F'}) = \sigma_F.$$

Conversely, every compatible system of automorphisms $\{\sigma_F\}$ over all Galois number fields F defines an element of $G_{\mathbb{Q}}$. In other words, $G_{\mathbb{Q}}$ is an inverse limit of Galois groups,

$$G_{\mathbb{Q}} = \varprojlim_{F} \{\mathrm{Gal}(F/\mathbb{Q})\}.$$

As the inverse limit of finite groups, $G_{\mathbb{Q}}$ is compact.

A *reciprocity law* is a correspondence between

$$\{\text{some properties of } G_{\mathbb{Q}}\} \longleftrightarrow \{\text{some properties of something else}\}.$$

The “something else” is hopefully something computable. It is important to remember that this is a 2-way street.

This notion is vague, so we consider three examples in which we can be a bit more precise.

Example 1.1. Let F be a field. Then we have the reciprocity law

$$\{\text{certain continuous homomorphisms from } G_{\mathbb{Q}} \to F^*\} \longleftrightarrow \{\text{homomorphisms from } (\mathbb{Z}/N\mathbb{Z})^* \to F^*\}.$$

The left side is the set of characters of the Galois group. This corresponds to the right side by class field theory for $\mathbb{Q}$.

Example 1.2. Let $\mathbb{Q}_p$ denote the p-adics, and let $\overline{\mathbb{Q}}_p$ denote the algebraic closure of $\mathbb{Q}_p$. Then we have the reciprocity law

$$\{\text{certain homomorphisms } G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)\} \longleftrightarrow \{\text{certain modular forms for } \Gamma_0(N)\}.$$

This is nice because the right side is computable.

Example 1.3. Let $\mathbb{F}_p$ be the finite field with p elements, and let $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$. Let Γ be a $\Gamma_0(N)$ type of group in $\mathrm{SL}_n(\mathbb{Z})$, and let V be a mod p coefficient module. Then we have the reciprocity law

$$\{\text{certain homomorphisms } G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)\} \longleftrightarrow \{\text{Hecke eigenclasses in } H_*(\Gamma, V)\}.$$

Remark 1.4. Mod p versions of examples 1.1 and 1.2 can be rewritten in the form of 1.3 using group homology.

2. Review of group homology

Let Γ be a group, and let R be a ring (commutative with identity). Suppose we have a short exact sequence of right $R\Gamma$-modules

$$0 \to A \to B \to C \to 0.$$

If we tensor this over $R\Gamma$ with R (recall that tensor is right exact) we get

$$A \otimes_{R\Gamma} R \to B \otimes_{R\Gamma} R \to C \otimes_{R\Gamma} R \to 0.$$

Prepend with kernels to complete this to long exact sequence. Complete this functorially to get

$$\cdots \to H_i(\Gamma, A) \to H_i(\Gamma, B) \to H_i(\Gamma, C) \to \cdots \to H_0(\Gamma, A) = A \otimes_{R\Gamma} R \to \cdots .$$

To compute cohomology $H^*(\Gamma, V)$, we use Hom instead of tensor. Then $H^0(\Gamma, V)$ is the space of Γ-invariants of V

$$H^0(\Gamma, V) = V^\Gamma.$$

How do we compute these homology groups? Take an exact sequence of right $R\Gamma$-modules,

$$\cdots \to F_2 \to F_1 \to F_0 \to R \to 0.$$

This is called a *resolution of* R. Find a resolution of R such that the F_i are all projective (take them free for simplicity). This is a *projective resolution*. Tensor (over $R\Gamma$) the resolution F_* with A to get a new chain complex. The resulting homology is isomorphic to group homology

$$H_*(F_* \otimes_{R\Gamma} A) \simeq H_*(\Gamma, A).$$

Remark 2.1. If we can find such a resolution F_*, then we can just write out the sequence and compute homology.

How do you find F? There are many techniques. Here is one way. Find a connected cell complex X on which Γ acts cellularly. i.e., Γ sends k-cells to k-cells.

Example 2.2. Let $\Gamma = \mathbb{Z}$, and let $X = \mathbb{R}$ viewed as a cell complex made of 1-cell segments with vertices at the integers, where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation.

Take the chain complex associated to X, $C_k = C_k(X, R)$. It is a $R\Gamma$-module. We get a chain complex

$$C_n \to C_{n-1} \to \cdots \to C_0 \to R \to 0, \tag{1}$$

where the map $C_0 \to R$ is summing the coefficients, $\sum a_p p \mapsto \sum a_p$.

Assume X is contractible. Then (1) is exact, since it computes homology of X. Now assume Γ acts freely on X. i.e., the stabilizers of cells are trivial. Then C_i are free. Thus we get a free resolution of R.

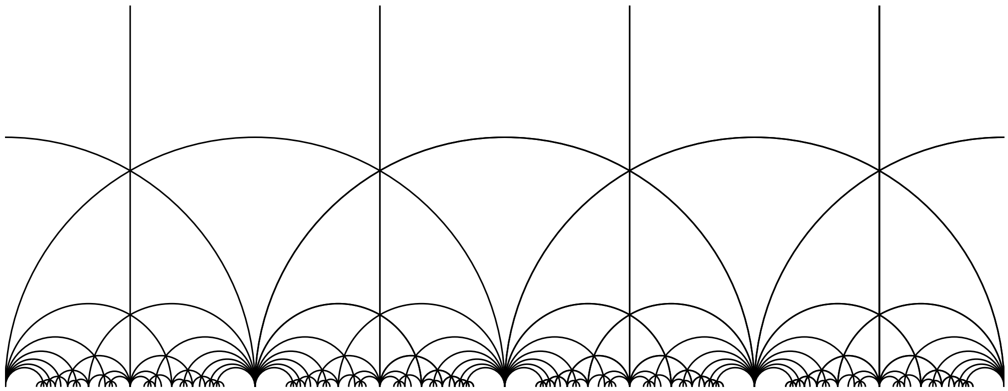

FIGURE 1. Tessellation of the upper halfplane by a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.

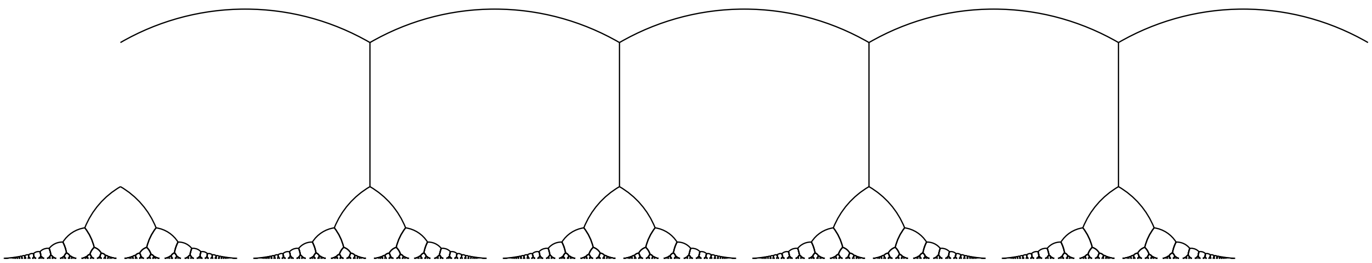

FIGURE 2. The well-rounded retract for $\mathrm{SL}_2(\mathbb{Z})$ is a trivalent tree in the upper halfplane.

Example 2.3. Fix $N \geq 3$, and let $\Gamma = \Gamma(N)$ be the principal congruence

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\}.$$

Let $X = \mathfrak{h}$ be the upper halfplane

$$\mathfrak{h} = \{x + iy \in \mathbb{C} \colon y > 0\}.$$

Then $\Gamma(N)$ acts on $\mathfrak{h}$ by fractional linear transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d},$$

preserving the tessellation, see Figure 1, of $\mathfrak{h}$ by translates of fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$.

Example 2.4. Let $X = \mathfrak{h}$ and $\Gamma = \Gamma(N)$ as above. There is a $\mathrm{SL}_2(\mathbb{Z})$-equivariant deformation retract of $\mathfrak{h}$ onto a trivalent tree $W \subset \mathfrak{h}$ shown in Figure 2. The tree W is the *well-rounded retract* for $\mathrm{SL}_2(\mathbb{Z})$. This is a special case of a more general construction described in Section 4.

Example 2.5. The following construction works for any group Γ and is known as the *standard resolution*. Let C_0 be the free R-module generated the elements of Γ, C_1 the free R-module generated by ordered pairs in $\Gamma \times \Gamma$. In general, C_k is the free R-module generated by ordered $k+1$-tuples $(\gamma_0, \gamma_1, \ldots, \gamma_k)$. The boundary operator $\partial \colon C_k \to C_{k-1}$ is given on a basis element by

$$\partial((\gamma_0, \dots, \gamma_k)) = \sum_{i=0}^{k} (-1)^i (\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_k),$$

where $\hat{\cdot}$ means omit $\cdot$.

Exercise 1. Use standard resolution to show that $H_1(\Gamma, \mathbb{Z})$ is naturally isomorphic to the abelianzation $\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma]$.

3. Hecke operators

Suppose Γ is a subgroup of a bigger group G. Suppose $g \in G$ normalizes Γ. Then we get an action of g on the group homology $H_r(\Gamma, A)$, where A is a RG-module.

How does this work? Let $[z] \in H_r(\Gamma, A)$ be a homology class. If in the standard resolution $z = \sum(\gamma_{0,i}, \dots, \gamma_{r,i}) \otimes a_i$ is a cycle representing $[z]$, then $[z]g = [zg]$, where $zg = \sum(\dots, g^{-1}\gamma_{k,i}g, \dots) \otimes a_i g$. This construction is a "proto-Hecke operator".

Exercise 2. Check that zg is again a cycle and its class $[zg]$ depends only on $[z]$.

From now on, let R be a field. Let S be a semigroup such that $\Gamma \subset S \subset G$, and for all $s \in S$, the subgroups $\Gamma \cap s\Gamma s^{-1}$ and $\Gamma \cap s^{-1}\Gamma s$ are finite index in Γ.

Example 3.1. $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $S = \mathrm{GL}(n, \mathbb{Q}) \cap M(n, \mathbb{Z})$, and $G = \mathrm{GL}_n(\mathbb{Q})$.

Exercise 3. Check this. Namely, in the previous example, verify that for all $s \in S$, the subgroups $\Gamma \cap s\Gamma s^{-1}$ and $\Gamma \cap s^{-1}\Gamma s$ are finite index in Γ.

Remark 3.2. A Hecke operator is an operator that averages over an object in a case where you don't have normality.

E.g. you have a space X and a finite to one map, $X \to X$. You pull back and average.

Define Hecke operators. A is an RS-module. Let $s \in S$.

$$T_s \colon H_r(\Gamma, A) \to H_r(\Gamma, A).$$

You can define it in terms of the remark with restriction and transfer maps.

$$H_r(\Gamma) \xrightarrow{\mathrm{tr}} H_r(\Gamma \cap s\Gamma s^{-1}) \xrightarrow{\mathrm{Ad}(s)} H_r(s^{-1}\Gamma s \cap \Gamma) \xrightarrow{\mathrm{cores}} H_r(\Gamma), \qquad T_s \colon H_r(\Gamma) \to H_r(\Gamma)$$

where cores is corestriction. $\mathrm{Ad}(s)$ is conjugation by s. The first map tr is the averaging process called *transfer*. The composition of these maps is T_s.

Here is an easy way to think of T_s: Let $F_* \to R \to 0$ be a projective resolution of $R\Gamma$-modules such that each F_k is a RS-module in a compatible way.

Example 3.3. Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, and let $S = \mathrm{GL}(n, \mathbb{Q}) \cap M(n, \mathbb{Z})$. A first thought would be to take F to be the standard resolution of Γ. This won't work since S does not act on it. Instead, take F to be the standard resolution of $\mathrm{GL}(n, \mathbb{Q})$.

If $[z]$ is represented by $\sum f_i \otimes_{R\Gamma} a_i$, then

$$T_s([z]) = \sum_i \sum_\alpha f_i s_\alpha \otimes_{R\Gamma} a_i s_\alpha,$$

where

$$\Gamma s\Gamma = \coprod_{\text{finite}} s_\alpha \Gamma.$$

Exercise 4. Show this makes sense. Specifically, show cycles go to cycles and boundaries go to boundaries.

Suppose

$$\Gamma_0(N) = \left\{ \gamma \in \mathrm{SL}_n(\mathbb{Z}) \colon \gamma \equiv \begin{bmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & \cdots & * \end{bmatrix} \bmod N \right\}.$$

Fix a prime ℓ not dividing N. Let $s_{\ell,k}$ be the diagonal matrix with k ℓs and $n-k$ 1s

$$s_{\ell,k} = \begin{bmatrix} \ell & & & & & \\ & \ddots & & & & \\ & & \ell & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}.$$

Let $T_{\ell,k}$ denote the Hecke operator $T_{s_{\ell,k}}$ associated to $s_{\ell,k}$.

Example 3.4. The finite list of s_α for $\mathrm{SL}(3,\mathbb{Z})$ and $k=1$ is

$$\{s_\alpha\} = \left\{ \begin{bmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ a & \ell & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & \ell \end{bmatrix} : a,b,c = 0,1,\ldots,\ell-1 \right\}.$$

From now on, let $R = \overline{\mathbb{F}}_p$. $\ell \nmid N$. Let E be a finite dimensional vector space over $\overline{\mathbb{F}}_p$ on which S acts via mod p. We usually assume that E is an irreducible module for $\mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$, in which case we call E a *weight*. Generalizes integer k from Paul's talks. Choose a character $\chi \colon S \to \mathbb{F}_p^*$ which is the application of a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}_p^*$ to the $(1,1)$ entry of any matrix in S. It is called a nebentype character.

Form the module $E \otimes \chi = E_\chi$.

Assume $p \nmid N$. Form $H_r(\Gamma, E_\chi)$ We get action of $T_{\ell,k}$.

These all commute with each other. We get eigenvalues and eigenvectors z. $z|T_{\ell,k} = a_{\ell,k} z$. $a_{\ell,k}$ are mod p.

Next we bundle these numbers into a polynomial which we hope is a characteristic polynomial of Frobenius.

Consider $H_r(\Gamma, E_\chi)$. If we assume Γ is torsion free, or if the torsion primes in Γ are invertible on E, then there exists an integer A for all E, such that $H_r(\Gamma, E_\chi) = 0$ if $r > A$. Smallest such A is called the cohomological dimension of Γ.

$v \in H_r(\Gamma_0(N), E_\chi)$ is a Hecke eigenclass.

$T_{\ell,k} v = a_{\ell,k} v$ for all $\ell \nmid pN$, $k = 0, \ldots, n$. Form a Hecke polynomial

$$P_{v,\ell}(T) = \sum_{k=0}^{n} (-1)^k a_{\ell,k} \ell^{k(k-1)/2} T^k.$$

For $n = 2$, this is the Euler factor $1 - a_{\ell,1}T + \ell a_{\ell,2}T^2$ from Paul's lecture. In elliptic curve case, $a_{\ell,2} = 1$.

All of our computations are designed to compute these polynomials. What is the significance of these polynomials? Go back to Galois side. Suppose we have a continuous representation of the absolute Galois group

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p).$$

The topology on $\overline{\mathbb{F}}_p$ is discrete so that ρ continuous means there exists a finite Galois extension $K/\mathbb{Q}$, unramified outside pN, such that ρ factors through $\mathrm{Gal}(K/\mathbb{Q})$

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\rho} & \mathrm{GL}_n(\overline{\mathbb{F}}_p) \\ \downarrow{\scriptstyle r_K} & \nearrow & \\ \mathrm{Gal}(K/\mathbb{Q}) & & \end{array}.$$

If $\ell \nmid pN$, then $\det(I - \rho(\mathrm{Frob}_\ell)T)$ is well-defined.

Remark 3.5. If we know all of these polynomials, we know ρ up to semisimplification.

Definition 3.6. We say ρ is attached to v if for every ℓ not dividing pN,

$$P_{v,\ell}(T) = \det(I - \rho(\mathrm{Frob}_\ell)T)$$

If ρ is attached to v, that gives a reciprocity law.

Class field theory over $\mathbb{Q}$ (Kronecker-Weber) can be interpreted in this way.

Conjecture (Ash 1990) For any v, there exists a ρ attached to v. Now proven by Peter Scholze (2013), (conditional on stabilization of a twisted trace formula)

Now we know they are attached to Galois representations, so we want to compute them.

The proof of Scholze's theorem implies any such ρ must be odd (since it comes from geometry.)

Definition of "odd": Let c = complex conjugation in $G_{\mathbb{Q}}$. $p = 2$: every ρ is odd. If $p > 2$: $\rho(c)$ is a linear map of order 2 and we say ρ is odd if $\rho(c)$ is conjugate to $\pm diag(1, -1, 1, -1, ...)$

Conjecture (Serre $n = 2$, Ash-Sinnot, refined Ash-D. Pollack- Doud, Herzig) Weak form: for all ρ, odd, there exists v with ρ attached. Strong form: you can also specify level N, weight E, and nebentype χ. For $n = 1$, this is class field theory again. For $n = 2$, this is Serre's conjecture. Proved by Khare-Wintenberger. With this, there is a very short proof of FLT (but depends on Wiles's techniques). For $n > 2$, this is wide open. Even Scholze doesn't know how to do it.

$n = 3$ example. Let ω = cyclotomic character:

$$\omega\colon G_{\mathbb{Q}} \to \mathbb{F}_p^*$$
$$\omega(\mathrm{Frob}_\ell) = \ell, \quad \text{if } \ell \neq p.$$

$$\rho = 1 + \omega + \omega^2.$$

Let's find v with ρ attached.

$$\rho(\mathrm{Frob}_\ell) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & \ell^2 \end{bmatrix}$$

$$\det(I - \rho(\mathrm{Frob}_\ell)T) = 1 - (1 + \ell + \ell^2)T + \ell(1 + \ell + \ell^2)T^2 - \ell^3T^3.$$

Let v generate $H_0(\mathrm{SL}_3(\mathbb{Z}), \mathbb{F}_p) \simeq \mathbb{F}_p$. Recall Hecke matrices from before. Call these s_α.

Since the action of S is trivial, $v|T_{\ell,1} = \sum vs_\alpha = (1+\ell+\ell^2)v$ since there are $1+\ell+\ell^2$ matrices.

In general, given v, we can find a ρ which is unique up to semisimplification. Given an odd ρ, we expect there to be many different v.

4. How to compute Hecke operators for $\mathrm{GL}(n)/\mathbb{Q}$

We would like to compute the Hecke action on $H_r(\Gamma_0(N), E_\chi)$. We can break this up into two main problems:

(1) Compute homology.
(2) Compute Hecke action.

Compute $H_r(\Gamma)$, $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ One way to do this is to find a nice resolution of $\mathbb{Z}$ by Γ-modules. Assume Γ is torsion free, for simplicity. Assume finite index. Note: if $n > 2$, this implies Γ is a congruence subgroup.

Start with a contractible space on which Γ acts nicely. X = positive definite symmetric $n \times n$ matrices/homotheties. $\mathrm{GL}(n, \mathbb{Q})$ acts on X on the right. $A \mapsto g^t A g$. X contains an optimal subset W of dimension equal to the cohomological dimension. This is the well-rounded retract. This has dimension $(1/2)n(n-1) = \mathrm{cd}(\Gamma)$ and is a Γ-equivariant retract of X. W is a Γ-cell complex. We can use the chains on W, $C_*(W)$, to compute the homology $H_*(\Gamma)$

Let y be a positive definite symmetric $n \times n$ matrix. Set $m(y) = min\{\lambda^t y \lambda,\, \lambda \in \mathbb{Z}^n \setminus \{0\}\}$. This is called the arithmetic minimum $m(y)$. $M(y)$ = the set of minimal vectors, on which the minimum is attained.

def. y mod homotheties is in W if and only if $M(y)$ spans $\mathbb{R}^n$ over $\mathbb{R}$. E.g. $n = 2$, W is trivalent tree in $\mathfrak{h}$. Dual to W is the Voronoi cellulation of X. A Voronoi cell is gotten by taking a y, list the λ in $M(y)$, take convex conical hull of the $\lambda\lambda^t$.

Upshot: Use well-rounded or Voronoi to compute.

Now we have to compute Hecke operators. $H = H(\Gamma, S)$=algebra of double cosets $\Gamma/S\backslash\Gamma$, S a semigroup in $\mathrm{GL}(n, \mathbb{Q})$. Trouble: S does not preserve W. Two approaches to fix this.

(1) W is a retraction. Act by s and flow back to W. (McConnell, Macpherson) This does not work well.
(2) Use a larger resolution of R on which S acts. Sharbly resolution.

    Take a Voronoi cell written $[\lambda_1^t, \ldots, \lambda_m^t]$ where $\{\pm\lambda_1, \ldots, \pm\lambda_m\} = M(y)$. If this is a simplex, we can view it as a sharbly. Sharblies are defined as follows:

    A k-sharbly for $\mathrm{GL}(n)/\mathbb{Q}$ is an R-linear combination of symbols $[v_1, \ldots, v_{n+k}]$, $(k \geq 0)$ v_i any nonzero row vectors in $\mathbb{Q}^n$, modulo relations:

    (a) $[v_1, \ldots, v_{n+k}] = [av_1, \ldots, v_{n+k}]$ for a in $\mathbb{Q}^*$
    (b) permutation by σ in indices picks up $\mathrm{sgn}(\sigma)$
    (c) degeneracy $[v_1, \ldots, v_{n+k}]$=0 if the vectors do not span $\mathbb{Q}^n$ over $\mathbb{Q}$

    boundary map:

$$\partial\colon \mathrm{Sh}_k \to \mathrm{Sh}_{k-1}$$

    in usual way.

    Elements of Sh_k are called k-sharblies.

E.g. $n = 2$ X = positive definite symmetric matrices modulo homotheties

$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in X$ if $ac - b^2 > 0$ X is circular cone. mod homotheties, we get the disc.

$$\text{well-rounded Voronoi cell} \longleftrightarrow W\text{-cell}$$

Subexample:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \longleftrightarrow 2x^2 + 2y^2 - 2xy = x^2 + y^2 + (x-y)^2$$

$e_1, e_2, e_1 + e_2$ are minimal vectors. (Any rank 1 matrix vv^t is on the boundary of the cone.) The corresponding Voronoi cell σ is a triangle and

$$\partial\sigma = \tau_1 + \tau_2 + \tau_3$$

where τ_i has minimal vectors equal to a two-element subset of $\{e_1, e_2, e_1 + e_2\}$.

Top dimensional cells in Voronoi correspond to vertices in W. In general, the dimensions are complementary.

Sharblies:

Sh_1 contains $A = [e_1, e_2, e_1 + e_2]$

$$\partial(A) = B - C + D.$$

We can compute the homology using Voronoi, well-rounded retract, or sharblies. The shambles give a $\mathrm{GL}(n, \mathbb{Q})$-equivariant exact sequence where $[v_1, \dots, v_{n+k}]g = [v_1 g, v_2 g, \dots, v_{n+k} g]$:

$$\cdots \to \mathrm{Sh}_k \to \mathrm{Sh}_{k-1} \to \dots \xrightarrow{\partial_0} \mathrm{Sh}_0$$

Define $\operatorname{coker} \partial_0 = \mathrm{Sh}_0$ mod boundaries $= \mathrm{St}_n$ (Steinberg module) for $\mathrm{GL}(n)/\mathbb{Q}$ and $\mathrm{GL}(n, \mathbb{Q})$ acts on it.

Given a 0-sharbly, $[v_1, \dots, v_n] \in \mathrm{Sh}_0$, the image in coker is denoted $[[v_1, ..., v_n]] \in \mathrm{St}_n$, so we have an exact sequence:

$$\cdots \to \mathrm{Sh}_k \to \mathrm{Sh}_{k-1} \to \dots \xrightarrow{\partial_0} \mathrm{Sh}_0 \to \mathrm{St}_n \,.$$

The Sh_* complex gives a resolution of St_n

History: St_n invented first, Lee-Sczarba found the sharblies later. (They did not call them that.)

Assume Γ is torsion-free subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$. Exercise: Sh_k is a free $R\Gamma$-module.

Let's compute $H^*(\Gamma, E)$.

One approach is to use the chain complex on the Voronoi decomposition.

Now compute Hecke operators. These do not preserve the Voronoi cells. Let's use the sharbly complex to see what we get.

If Sh_* was a resolution of R, then we could use it to compute the cohomology. But it is the resolution of St_n.

Compute $H_r(\mathrm{Sh}_k \otimes_{R\Gamma} E) \simeq H_r(\Gamma, \mathrm{St}_n \otimes_R E)$.

By Borel-Serre duality, $H_r(\Gamma, \mathrm{St}_n \otimes_R E) \simeq H^{\nu - r}(\Gamma, E)$, where $\nu = \mathrm{cd}(\Gamma) = (1/2)n(n-1)$. This is a form of Lefschetz duality.

Now put everything together for Γ in $\mathrm{GL}(n, \mathbb{Z})$ for $n = 2, 3, 4$, all the Voronoi cells are simplicial, except for one $\mathrm{SL}_n(\mathbb{Z})$-orbit of Voronoi cells if $n = 4$.

Say $n = 2$ or 3. We have a map from the Voronoi to the sharbly complex: see AGM. The Voronoi sharblies after reindexing give us a resolution:

$$\cdots \to V_1 \to V_0 \to \mathrm{St}_n$$

which can thus be compared to the sharbly resolution:

$$\cdots \to \mathrm{Sh}_1 \to \mathrm{Sh}_0 \to \mathrm{St}_n .$$

To compute $H^*(\Gamma, E)$ as a Hecke module, we compute $H_*(\Gamma, \mathrm{St}_n \otimes E)$ By homological algebra, we can compute the latter by Voronoi or Sharbly resolutions. Throw away St, tensor over Γ with complex (take coinvariants) and compute kernel modulo image.

V_m/Γ is a finite dimensional R-vector space (assume R is a field). This is good, but it is NOT stable under $\mathrm{GL}(n, \mathbb{Q})$. For 0-sharblies we use "modular symbol algorithm", developed by Ash-Rudolph in 1979. (It is a generalization of continued fractions. Works for any n.)

Allows us to compute $H_0(\Gamma, \mathrm{St} \otimes E)$, so we can compute in $H^\nu(\Gamma, E)$.

Using the algorithm for modular symbols (i.e. 0-sharblies, we can compute as Hecke module:

- $n = 2$: $H^1(\Gamma, E)$ which includes cusp forms;
- $n = 3$: $H^3(\Gamma, E)$ which includes cusp forms;
- $n = 4$: $H^6(\Gamma, E)$ in which the modular forms are like Eisenstein series. They are not the cusp forms we want. We need H^5 to get cusp forms.

We need a reduction algorithm for 1-sharblies. We use Gunnells' algorithm for one below the cohomological dimension.

Open problems:

(1) Prove that the Gunnells algorithm terminates.
(2) Find reduction algorithms in the other dimensions.

(Note added in 2015: I have heard a rumor that Gunnells, working with some other mathematicians, may have made progress on the second problem.)

Now let's think about the problem of finding a Hecke eigenclass corresponding to a given Galois representation.

We can try to make some progress for reducible representations.

Let $\rho = \sigma_1 \oplus \cdots \sigma_k$ be a direct sum of reducible representations.

$$\rho\colon G_{\mathbb{Q}} \to (\mathrm{GL}(n_1) \times \mathrm{GL}(n_2) \times \cdots \times \mathrm{GL}(n_k))(\overline{\mathbb{F}_p})$$

$n = n_1 + \cdots + n_k$.

Simplest question. Suppose $\sigma_1, \ldots, \sigma_k$ are attached to Hecke eigenclasses in the expected homology. Show that ρ is also attached. This has the same feel as Eisenstein series.

In any case we must assume that ρ is odd.

Theorem 4.1 (Ash-Doud)**.** *Let $\overline{\mathbb{F}}_p$ be an algebraic closure of a finite field of characteristic p. Let ρ be a continuous homomorphism from the absolute Galois group of $\mathbb{Q}$ to $\mathrm{GL}(3, \overline{\mathbb{F}}_p)$ which is isomorphic to a direct sum of a character and a two-dimensional odd irreducible representation. Under the condition that the Serre conductor of ρ is squarefree, ρ is attached to a Hecke eigenclass in the homology of an arithmetic subgroup Γ of $\mathrm{GL}(3, \mathbb{Z})$. In addition, the coefficient module needed is, in fact, predicted by a conjecture of Ash, Doud, Pollack, and Sinnott.*

(Note added in 2015: Doud and I can do various other cases of reducible representations also.)

Idea of proof: Let σ_1 be a 2-dimensional, odd, irreducible Galois representation. Let σ_2 be a character. Then $\sigma_1 \oplus \sigma_2$ should be attached to a GL(3) homology class.

Khare-Wintenberger: σ_1 is known to be attached to class in $H_1(\Gamma_1, E)$, $\Gamma_1 \subset \mathrm{GL}_2(\mathbb{Z})$

Class field theory says σ_2 is attached to GL(1)

Let P be the parabolic subgroup $\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}$

One could try to put classes on boundary and extend to interior of cone. This is difficult because the homology of the boundary is not known for mod p coefficients and it is hard to control the level, nebentype and weight. Instead, to find a homology Hecke eigenclass attached to ρ, we take a certain resolution

$$\cdots \to X_1 \to X_0 \to \overline{\mathbb{F}}_p$$

This is an exact sequence of $\mathrm{GL}_3(\mathbb{Q})$-modules. The key new ideas here is that the X_i are not free.

You don't need to use free resolutions. When you compute $H_*(X_j \otimes_\Gamma E)$ and X_j are not free, you get a spectral sequence that computes $H_*(\Gamma, E)$. X_0 has stabilizers equal to conjugates of arithmetic subgroups of P. Use Shapiro's lemma to jack up the homology to Γ.

Point: We can use nonfree resolutions to capture certain homology classes cleanly. (Although there is quite a bit of work in chasing down the details.)

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