ALGEBRAIC MODULAR FORMS – EXERCISES AND ACTIVITIES

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$\mathbf{H}$ denotes Hamilton's quaternion $\mathbb{R}$-algebra: $\mathbf{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.

1) Show that the Laplace operator

$$\Delta = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

commutes with the action of SO(3). What about O(3)? GO(3)?

2) Let

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\tfrac{1+i+j+k}{2}$$

and set

$$\theta_{\mathcal{O}}(q) = \sum_{x \in \mathcal{O}} q^{N(x)} = \sum_{n=0}^{\infty} r_{\mathcal{O}}(n) q^n,$$

where $r_{\mathcal{O}}(n) = |\{x \in \mathcal{O} : N(x) = n\}|$. What is are the level and weight of $\theta_{\mathcal{O}}$? Show that the space of modular forms of this level and weight is 1-dimensional and find an Eisenstein series spanning it. Now find a closed-form expression for $r_{\mathcal{O}}(n)$. Are there other theta functions for which you can find nice formulas like these? Are there infinitely many?

3) Write a program to compute spaces of homogeneous, harmonic polynomials in three variables. Extend your program to facilitate computing the Hecke operators on these spaces (really, the subspaces invariant under $R^\times$ where R is the Hurwitz order). As a check on your work, check that the Hecke operators commute. Compute some systems of Hecke eigenvalues occuring in spaces of harmonic polynomials and match them up with systems of Hecke eigenvalues occuring in spaces of modular forms.

4) Identify S^2 with $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by stereographic projection. The group $\mathbf{H}_1^\times$ of norm 1 elements of Hamilton's quaternions acts on $\mathbb{P}^1(\mathbb{C})$ via the identification $\mathbf{H}_1^\times \cong \mathrm{SU}_2 \subset \mathbf{GL}_2(\mathbb{C})$. By the stereographic projection isomorphism $S^2 \cong \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, we get an action of $\mathbf{H}_1^\times$ on S^2. How does this action relate to the action of $\mathbf{H}_1^\times$ on S^2 obtained via the embedding $\mathbf{H}_1^\times \hookrightarrow \mathrm{SO}(3)$ as described in the lecture?

Let $B = \mathbf{M}_2(E)$. The determinant map $\det : B \to E$ is a quadratic form. Hence, $(B, \det)$ is a 4-dimensional quadratic space over E. When $E = \mathbb{R}$, what is the signature of B? Let

$$\langle A, B \rangle = \tfrac{1}{2}(\det(A + B) - \det(A) - \det(B))$$

be the associated symmetric, bilinear form. Show that

$$\langle A, B \rangle = \tfrac{1}{2} \operatorname{Tr} A\overline{B}$$

where

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Show that we have an orthogonal decomposition

$$B = E \perp B^0 \tag{1}$$

where E is identified with the scalar matrices in B and B^0 is the trace-zero subspace of B. Further, show that the restriction of $\langle \cdot, \cdot \rangle$ to B^0 is nondegenerate. Consider the following basis of B^0:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

What is the dual basis of $\{X, Y, Z\}$ with respect to $\langle \cdot, \cdot \rangle$?

Let $B^\times$ act on B from the left by the rule

$$x \cdot b = xb\overline{x}.$$

Show that (1) is $B^\times$-stable. How does $B^\times$ act on E? Show that in the basis $\{X, Y, Z\}$ the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B$ on $B^\times$ is given by the matrix

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} := \begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad + bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix}.$$