ALGEBRAIC MODULAR FORMS – EXERCISES AND ACTIVITIES II

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(1) Let $E = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ with $ij = k = -ji$.
  (a) Suppose $i^2 = j^2 = 1$. Show that $E \cong \mathrm{M}_2(\mathbb{Q})$.
  (b) Suppose $i^2 = -1$ and $j^2 = -p$ with $p \equiv 3 \pmod 4$. Show that the E you get for different p are nonisomorphic. (Hint: Consider discriminants.)
  (c) Suppose $i^2 = -1$ and $j^2 = -p$ with $p \equiv 1 \pmod 4$. Show that $E \cong \mathbf{H}$.

(2) (a) Let E be the quadratic field extension of F obtained by adjoining a root α of the irreducible quadratic polynomial $f(x)$. Let K be a field extension of F in which $f(x)$ factors. Show that $E \otimes_F K \cong K \oplus K$.
  (b) Show that $\mathbf{H} \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-1}) \cong \mathrm{M}_2(\mathbb{Q}(i))$.

(3) Suppose $(V, h) = (E^n, \sum_{i=1}^n x_i \overline{y}_i)$. Show that

$$\mathrm{GU}(V, h) = \{A \in \mathrm{GL}_n(E) : A\overline{A}^t = cI \text{ for some } c \in F^\times\},$$
$$\mathrm{U}(V, h) = \{A \in \mathrm{GL}_n(E) : A\overline{A}^t = I\}.$$

(4) (a) Suppose $E = F \times F$ and $(V, h) = (E^n, \sum_{i=1}^n x_i \overline{y}_i)$. Show that $\mathrm{U}(V, h)$ is canonically isomorphic to $\mathrm{GL}_n(F)$.
  (b) (Lengthy) Suppose $E = \mathrm{M}_2(F)$ and let (V, h) be a Hermitian E-space of E-rank n. Let $e_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and let $V' = e_{11}V$. Show that $\dim_F V' = 2n$. Show that there is a unique function $h' : V' \times V' \longrightarrow F$ such that $h(x, y) = h'(x, y)e_{12}$ for all $x, y \in V'$. Show that h' is symplectic and conclude that (V', h') is a symplectic space of dimension $2n$. Show that $(V, h) \mapsto (V', h')$ extends to an equivalence of the category of Hermitian E-spaces with the category of symplectic F-spaces (i.e., Explain how to map morphisms). Conclude that there is a canonical isomorphism $\mathrm{GU}(V, h) \to \mathrm{GSp}(V', h')$ that restricts to an isomorphism $\mathrm{U}(V, h) \to \mathrm{Sp}(V', h')$.