UNCG SUMMER SCHOOL PROBLEM LIST

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(1) (a) Prove that the action of $SL_2(\mathbb{R})$ on $\mathfrak{H}$ by fractional linear transformations is a left action.
(b) Prove that the action is transitive, and that the stabilizer of i is isomorphic to $SO(2)$.
(c) Prove that the left action of $SL_2(\mathbb{R})$ on $\mathfrak{H}$ preserves the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$ and the area $dx\,dy/y^2$.
(d) Show that the region $D = \{z = x + iy \mid x^2 + y^2 \geq 1, -1/2 \leq x \leq 1/2\}$ is a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$, and compute its area (with the hyperbolic measure). (Hint: look at the action of the matrices $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T^{\pm 1} = \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$ on $\mathfrak{H}$ and argue that they generate $SL_2(\mathbb{Z})$.)

(2) Let $\Gamma(N) \subset SL_2(\mathbb{Z})$ be the principal congruence subgroup of level N.
(a) Show that $\Gamma(N)$ is torsion-free if $N > 2$. (Hint: use the fact that $\Gamma(N)$ is normal and that all torsion elements of $SL_2(\mathbb{Z})$ are the center and the conjugates of the subgroups generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$).
(b) Show that the map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective (look at Shimura if you get stuck).

(3) The *Farey tessellation* is the tiling of $\mathfrak{H}$ by the $SL_2(\mathbb{Z})$-translates of the geodesic triangle with vertices at 0, 1, and ∞.
(a) Show that if $N > 2$, a fundamental domain of $\Gamma(N)$ can be built from tiles in the Farey tessellation.
(b) Draw pictures of $X(N) = \Gamma(N)\backslash\mathfrak{H}^*$ for $N = 3, 4, 5, 6, 7$ with the triangulation induced from the Farey tessellation. (Hint: The vertices of this triangulation lie at the cusps. $3, 4, 5$ are going to look very familiar. For $6, 7$ you probably just want to draw a picture of a union of triangles with identifications on the boundary. It also helps to know that the cusps of $\Gamma(N)$ are in bijection with nonzero pairs $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2$ with $gcd(a, b, N) = 1$ modulo the relation $(a, b) \simeq (-a, -b)$.)

(4) To get presentation for a group using a fundamental domain, one can use the following theorem:

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Theorem 1. *Let $\Gamma \subset PSL_2(\mathbb{R})$ be a discrete group acting properly discontinuously on $\mathfrak{H}$. Let $V \subset \mathfrak{H}$ be an open connected subset such that*

$$\mathfrak{H} = \bigcup_{\gamma \in \Gamma} \gamma V,$$
$$\Sigma = \{\gamma \mid V \cap \gamma V \neq \emptyset\} \text{ is finite.}$$

Then a presentation for Γ can by constructed by taking generators to be symbols $[\gamma]$ for $\gamma \in \Sigma$ subject to the relations $[\gamma][\gamma'] = [\gamma\gamma']$ if $V \cap \gamma V \cap \gamma' V \neq \emptyset$.

Use the theorem to get a presentation of $PSL_2(\mathbb{Z})$. (Hint: take V to be a slight "thickening" of the fundamental doman D from class.)

(5) Let $\mathfrak{H}_3$ be hyperbolic three-space. An "upper halfspace" model for $\mathfrak{H}_3$ can by gotten by taking the points $(z, r) \in \mathbb{C} \times \mathbb{R}_{>0}$ and using the metric $ds^2 = (dx^2 + dy^2 + dr^2)/r^2$ (here we are writing $z = x + iy$). We can also think of $\mathfrak{H}^3$ as being the subset of quaternions $\mathbf{H} = \{x + iy + rj + tk \mid x, y, r, t \in \mathbb{R}\}$ with $r > 0$ and $t = 0$. Write $P = P(z, r)$ for the quaternion corresponding to $(z, r) \in \mathfrak{H}_3$.

Let $G = SL_2(\mathbb{C})$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, define a transformation of $\mathfrak{H}_3$ by

$$M \cdot P = (aP + b)(cP + d)^{-1}.$$

In this defintion the operations on the right are to be computed in $\mathbf{H}$.

(a) Show that this is a left action of G on $\mathfrak{H}_3$.
(b) Show that the action is transitive.
(c) Show that the stabilizer of $(0, j)$ is isomorphic to

$$SU(2) = \{M \in G \mid M\bar{M}^t = I\}.$$

(6) (a) Let $\Gamma = PSL_2(\mathbb{Z}[i]) \subset PSL_2(\mathbb{C})$. Then Γ acts on $\mathfrak{H}_3$. Show that the set

$$D_{\sqrt{-1}} = \{(x + iy, r) \in \mathfrak{H}_3 \mid 0 \leq |x|, y \leq 1/2, x^2 + y^2 + r^2 \geq 1\}$$

is a fundamental domain for the action of Γ on $\mathfrak{H}_3$. (Hint: generalize the algorithm from above that used $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T^{\pm 1} = \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$ to move points into the fundamental domain for $PSL_2(\mathbb{Z})$).

(b) If you try to construct a fundamental domain of $\Gamma' = SL_2(\mathbb{Z}[\sqrt{-5}])$ using something like the above, it doesn't work. What goes wrong?

(c) Use the fundamental domain $D_{\sqrt{-1}}$ to find a presentation of Γ (you probably want to use a computer, but by hand is possible too).

(7) (a) Suppose a Dirichlet series $\sum_{n \geq 1} a(n)/n^s$ with $a(1) = 1$ can be written as an infinite product of the form

$$\sum_{n \geq 1} \frac{a(n)}{n^s} = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1},$$

where the product is taken over primes p and k is a fixed positive integer (This is an example of an *Euler product*). Prove

(i) $a(mn) = a(m)a(n)$ if m and n are relatively prime.

(ii) $a(p^n)$ can be computed in terms of $a(p^{n-1})$ and $a(p^{n-2})$. Compute the explicit formula for $a(p^n)$.

(b) Verify that the Dirichlet series attached to the Eisenstein series $E_k(z)$, $k \geq 4$, has an Euler product. (Hint: relate the Dirichlet series to the Riemann zeta function somehow.)

(c) Let $\Delta = q\prod_{n\geq 1}(1-q^n)^{24}$ be the cuspform of weight 12 for $SL_2(\mathbb{Z})$. Putting $\Delta(q) = \sum_{n\geq 1} a(n)q^n$, check the recursion for the p-power coefficients $a(p^n)$ for Δ for all p powers less than 100 (you probably want to use a computer for this).

(8) The coefficients in the q-expansion of Δ are called the *Ramanujan τ-function.* In particular one writes

$$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n.$$

Here are some amusing facts about τ that can be verified (up to some point) by computer experiments:

(a) $\tau(n) \equiv \sigma_{11}(n) \mod 691$.

(b) $\tau(n) \equiv n\sigma_9(n) \mod 7$ for $n \equiv 0, 1, 2, 4 \mod 7$.

(c) For $p \neq 23$ a prime, we have (i) $\tau(p) \equiv 0 \mod 23$ if $\left(\frac{p}{23}\right) = -1$; (ii) $\tau(p) \equiv \sigma_{11}(p) \mod 23^2$ if p is of the form $a^2 + 23b^2$; and (iii) $\tau(p) \equiv -1 \mod 23$ otherwise.

(d) (Lehmer's conjecture) $\tau(n) \neq 0$.

(9) The E_8 root lattice Λ_8 can be described as the set of vectors in $\mathbb{R}^8$ with all components x_i either integral or half-integral (meaning odd integer/2) and such that $\sum x_i$ is an even integer. (Note that the x_i can't be a mixture of integers and half-integers ... only one or the other). For instance $(1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2) \in \Lambda_8$, as is $(1, 1, 0, 0, 0, 0, 0, 0)$.

(a) Verify that the q-expansion for the modular form built from the theta series for Λ_8 agrees with the Eisenstein series $E_4(z)$ up to as high a q-power as you dare. (q^n for $n \leq 3$ is probably possible without too much trouble by hand, but beyond this is probably going to require a computer.)

(b) Challenge: is it possible to prove equality of this modular form with $E_4(z)$, without using $M_4(SL_2(\mathbb{Z})) = \mathbb{C}E_4(z)$?

(10) (a) Use a computer and the fact that $M_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$ to find a basis of $M_k(SL_2(\mathbb{Z}))$ for $k \leq 36$. (Take each basis vector to be a q-series up to q^{30}.)

(b) Find an expression for the theta series of the Leech lattice in terms of your basis (this is an even unimodular lattice, in fact the unique one in $\mathbb{R}^{24}$ up to rotation and scaling such that all minimal nonzero vectors have squared length 2). You can get the first coefficients of the theta series at `http://oeis.org/A008408`, but don't look at the bottom of this encyclopedia entry, or you'll see spoilers.

(c) Find the unique polynomials in E_4, E_6 giving the q-expansions you found in parts (b) and (c) (if that wasn't what you did for part (a)).

(11) Let $\theta(z)$ be the classical theta function

$$\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

(a) Show that $\theta(z)^m = 1+\sum_{k\geq 1}\rho_m(k)q^k$, where $\rho_m(k)$ is the number of ways of representing k as a sum of m squares.

(b) One can show that $\theta(z)^4$ is a modular form of weight 2 for the group $\Gamma_0(4)$. Furthermore, one knows that the space $M_2(\Gamma_0(4))$ is spanned by the two weight two Eisenstein series $E_2^*(z)-2E_2^*(2z)$ and $E_2^*(z)-4E_2^*(4z)$, where $E_2^*(z) = 1-24\sum_{n\geq 1}\sigma_1(n)q^n$. (In particular these combinations of E_2^*, a scalar multiple of E_2, are actually modular.) Write $\theta(z)^4$ in terms of these Eisenstein series.

(c) Use part (b) to prove a famous formula of Jacobi:

$$\rho_4(n) = 8\sum_{\substack{d\mid n\\ d\not\equiv 0 \bmod 4}} d.$$

(d) Deduce Lagrange's theorem: every positive integer can be written as a sum of four squares.

(12) The notation for this problem is taken from Problem (11). This time we consider $\theta(z)^8 \in M_4(\Gamma_0(4))$. This space of modular forms is spanned by $E_4(az)$ for $a = 1, 2, 4$. Prove

$$\rho_8(n) = 16\sum_{\substack{d\mid n\\ d\not\equiv 2 \bmod 4}} d^3 + 12\sum_{\substack{d\mid n\\ d\equiv 2 \bmod 4}} d^3.$$

(13) Let $\eta(z)$ be the Dedekind eta-function $\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$. Let $f(z)$ be the eta-product $(\eta(z)\eta(11z))^2$.

(a) Compute the q-expansion of f up to q^{100}.

(b) Verify that the coefficients of this q-expansion agree with the coefficients $a(n)$ of the L-function of the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$

up to q^{100}. In particular, these coefficients are determined by

$$\sum a(n)n^{-s} = (1-11^{-s})^{-1} \cdot \prod_{p \neq 11} (1-a(p)p^{-s}+p^{1-2s})^{-1}$$

where the product is taken over primes $p \neq 11$, and for $p \neq 11$ we put $a(p) = p + 1 - \#E(\mathbb{F}_p)$. (Hint: $\#E(\mathbb{F}_p)$ is the number of solutions of $y^2 + y = x^3 - x^2 - 10x - 20 \bmod p$ plus 1; the extra solution comes from the “point at infinity” on the projecive closure of the this affine curve.)

(14) The ring of *quasi-modular forms* on $SL_2(\mathbb{Z})$ is the polynomial ring $QM_* = \mathbb{C}[E_2, E_4, E_6]$ (in particular, there are no polynomial relations among these Eisenstein series). Define *Ramanujan's theta operator* Θ by

$$\Theta(f) = q\frac{df}{dq}.$$

f Thus if $f(z) = \sum_n a(n)q^n$, then $\Theta(f) = \sum na(n)q^n$. Show that Θ takes QM_* into itself. (Hint: to show two modular forms are equal you can check equality of q-expansions up to some degree. Going up to q^{50} is more than enough.)

(15) For $k > 2$ an even integer and for any nonnegative integer Δ, define

$$f_k(\Delta, z) = \sum_{\substack{a,b,c \in \mathbb{Z} \\ b^2-4ac=\Delta}} \frac{1}{(az^2+bz+c)^k}.$$

(We omit $a, b, c = 0$ if $\Delta = 0$). This sum converges absolutely.

(a) Show that f_k vanishes unless $\Delta \equiv 0, 1 \bmod 4$.
(b) Show that $f_k(\Delta, z)$ satisfies the transformation law of a modular form of weight $2k$ on $SL_2(\mathbb{Z})$. (In fact f_k is a modular form.)
(c) Show that $f_k(0, z)$ is a constant multiple of the Eisenstein series $E_{2k}(z)$.

(16) In this problem, we learn more about Dirichlet series built from elementary arithmetic functions. (If you don't know the definitions, try Wikipedia.) For each multiplicative function $f(n)$ below, express the associated Dirichlet series $\sum f(n)n^{-s}$ in terms of the Riemann zeta function. (Hint: these functions are all multiplicative, so you just have to match the factors for the primes p. It might help first to work out the coefficients of a product of two Dirichlet series.)

(a) $\mu(n)$, the Möbius function. (Hint: to get you started on this problem, the answer is $1/\zeta(s)$.)
(b) $d(n)$, the number of positive divisors of n.
(c) $\varphi(n)$, Euler's phi-function.
(d) $\lambda(n) = (-1)^{\nu(n)}$, where $\nu(p_1^{r_1} \cdots p_k^{r_k}) = r_1 + \cdots + r_k$.
(e) $\mu(n)^2$.
(f) $d(n)^2$.

(g) $d(n^2)$.

(17) Let $G(n)$ be the number of finite abelian groups of order n, up to isomorphism. Build the associated Dirichlet series $L(s,G) = \sum_{n\geq 1} G(n)n^{-s}$.
  (a) Prove that $G(n)$ is multiplicative.
  (b) Find a formula for $G(p^r)$ in terms of partitions.
  (c) Show (at least formally) that $L(s,G) = \zeta(s)\zeta(2s)\zeta(3s)\zeta(4s)\cdots$.
  (d) Prove that $\sum_{n<X} G(n) \sim Cn$, where C is a constant. (Hint: Tauberian theorem.)
  (e) Compute the constant C accurate to 5 places past the decimal.

(18) Show that the Hecke operators T_n satisfy

$$T_n T_m = \sum_{d|n,m} T_{mn/d^2}$$

when applied to any modular form for $SL_2(\mathbb{Z})$.

(19) This problem gives another perspective on Hecke operators. Fix an integer q. Let $\mathcal{T}$ be the infinite tree of degree $q+1$. Thus $\mathcal{T}$ is a graph with infinitely many vertices and no cycles; each vertex of $\mathcal{T}$ is joined to $q+1$ others. Define the distance $d(v,v')$ between two vertices v, v' to be the length of the shortest path connecting them, where each edge is defined to have length 1. Finally define two sequences of correspondences $\theta_k, T_k, k \geq 0$ on the set of vertices of $\mathcal{T}$ by

$$\theta_k(v) = \sum_{d(v,v')=k} v'$$

and

$$T_k = \theta_k + T_{k-2} \quad (k \geq 2),$$

with the initial conditions $T_0 = \theta_0$, $T_1 = \theta_1$.
  (a) Show that the θ_k satisfy

$$\theta_1\theta_1 = \theta_2 + (q+1)\theta_0,$$
$$\theta_1\theta_k = \theta_{k+1} + q\theta_{k-1} \quad (k \geq 2).$$

  (b) Show that the T_k satisfy

$$T_k T_1 = T_{k+1} + qT_{k-1} \quad (k \geq 1).$$

(20) Let $f \in M_k(N)$ have Fourier expansion $\sum a_n q^n$. Show that $T_n f = \sum b_m q^m$, where

$$b_m = \begin{cases} a_0 \sum\limits_{\substack{d|n \\ (d,N)=1}} d^{k-1} & \text{if } m = 0, \\ a_n & \text{if } m = 1, \\ \sum\limits_{\substack{d|m,n \\ (d,N)=1}} d^{k-1} a_{mn/d^2} & \text{otherwise.} \end{cases}$$

(21) Fix a level N and consider the Hecke operators acting on weight k modular forms of level N. Write U_p for the operator T_p when $p|N$.
  (a) Show that the operators satisfy
    (i) $T_p T_{p^r} = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}$ if $(p, N) = 1$,
    (ii) $T_{p^r} = (U_p)^r$ if $p|N$.
    (iii) $T_m T_n = T_{mn}$ if $(m, n) = 1$.
  (b) Conclude that if $f = \sum a_n q^n$ is a simultaneous eigenform for all T_p, U_p, and $a_1 = 1$, then $L(s, f)$ has the Euler product

$$L(s, f) = \prod_{(p,N)=1} (1 - a_p p^{-s} + p^{k-1-2s})^{-1} \prod_{p|N} (1 - a_p p^{-s})^{-1}.$$

(22) (a) Suppose $N'|N$ and $M|(N/N')$. Suppose $f(z) \in S_k(N')$. Prove that $f(Mz) \in S_k(N)$.
  (b) Suppose that f is a Hecke eigenform for all Hecke operators T_p with $(p, N) = 1$. Then prove each $f(Mz)$ is a Hecke eigenform with the same eigenvalues for all T_p with $(p, N) = 1$.

(23) Define $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ to be the space of primitive pairs $(\alpha, \beta) \in (\mathbb{Z}/N\mathbb{Z})^2$ modulo the action of $(\mathbb{Z}/N\mathbb{Z})^\times$. (Primitive means that after lifing back to $\mathbb{Z}^2$, we have $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathbb{Z}$). Write $(\alpha : \beta)$ for the point corresponding to the pair (α, β). Show that $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ is in bijection with the cosets $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ via the "bottom row map:" $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c : d)$.

(24) Let p be a prime and count the orbits of $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ in $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$. (Hint: the answer depends on $p \bmod 12$.)

(25) Implement the modular symbol algorithm in the computer software of your choice: given a modular symbol $\{\alpha, \beta\}$ that is not unimodular, write $\{\alpha, \beta\} = \sum_i \{\alpha_i, \beta_i\}$ where each modular symbol on the right hand side is unimodular.

(26) Show that there are no weight 2 cusp forms of level 13. (Hint: $X_0(p)$ has 2 cusps for p prime.)

(27) Compute the space of modular symbols $\mathcal{M}_2(23)$ and the eigenvalues of the operators T_2, T_3, T_5. Show that there is a weight 2 cuspidal eigenform of level 23 whose q-expansion begins $q + \alpha q^2 - (2\alpha + 1)q^3 - (\alpha + 1)q^4 + 2\alpha q^5 + (\alpha - 2)q^6 + \cdots$ where $\alpha^2 + \alpha - 1 = 0$.

(28) Use modular symbols to show that there are no weight 2 cusp forms of level $2 \leq N \leq 10$. (You will need to look up how many cusps $X_0(N)$ has.)

(29) Use modular sumbols to find the weight 4 cusp form of level 5.