UNCG Summer School in
Computational Number Theory

Zeta Functions – New Theory and Computations

May 18 to May 22, 2015

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

Speakers

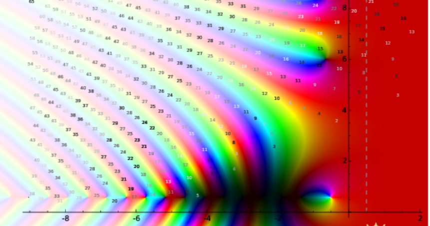
Fredrik Johansson (INRIA Bordeaux-Sud-Ouest)

Yuri Matiyasevich (Steklov Institute of Mathematics)

Filip Saidak (UNC Greensboro)

Cem Yildırım (Bogaziçi University, Istanbul)

Peter Zvengrowski (University of Calgary)



Organizers: Sebastian Pauli, Filip Saidak, Brett Tangedal, Dan Yasaki

www.uncg.edu/mat/numbertheory/summerschool

Investigation on the *abc*-Conjecture and Ruderman's problem

Arnab Bose
(Advisor: Prof. Amir Akbary)

University of Lethbridge

UNCG Summer School in Computational Number Theory, 2015

The *abc*-Conjecture (J.Osterlé, D.Masser, 1985)

- Given any $\epsilon > 0$, there is a constant $K(\epsilon) > 0$ such that for every triple of coprime integers a, b, c, satisfying $a + b = c$, we have,
$$\max(|a|, |b|, |c|) \leq K(\epsilon) N^{1+\epsilon},$$
where
$$N = rad(abc) = \prod_{\substack{\text{p prime} \\ p|abc}} p.$$
- Easy to state, difficult to verify.
- Implications: $abc \Rightarrow$ the Fermat equation $x^n + y^n = z^n$ has at most finitely many integer solutions.
- $abc \Rightarrow$ Faltings' Theorem on Mordell's Conjecture. (N. Elkies, 1991)

Refinements to the *abc*-Conjecture

- Notation: Let $\omega(n)$ denote the number of distinct prime factors of an integer n and define $\omega = \omega(abc)$.
- Refinement (A. Baker, 1996):

$$\max(|a|, |b|, |c|) \ll N(\log N)^{\omega}/\omega!,$$

  where the implied constant is absolute.
- Explicit *abc*-Conjecture (A. Baker, 2004): For any coprime integers a, b, c satisfying $a + b = c$, we have,

$$\max(|a|, |b|, |c|) < \frac{6}{5} N(\log N)^{\omega}/\omega!.$$

A Problem of Ruderman

- Selfridge noticed that $2^2 - 2 | x^2 - x$, $2^{2^2} - 2^2 | x^{2^2} - x^2$ and $2^{2^{2^2}} - 2^{2^2} | x^{2^{2^2}} - x^{2^2}$ for all $x \in \mathbb{N}$, and asked the following question: Find all pairs (m, n) such that $2^m - 2^n | x^m - x^n$ for all $x \in \mathbb{N}$.
- This is true for $(m, n) \in S$, with $|S| = 14$. (Sun Qi, Zhang Ming Zhi, 1985)
- There is a finite set S' such that for $m > n \geq 0$,

$$2^m - 2^n | 3^m - 3^n \iff (m, n) \in S'.$$

  (Murty and Murty, 2011)
- **H. Ruderman (1974)**: If $m > n \geq 0$ are integers such that $2^m - 2^n | 3^m - 3^n$, then $2^m - 2^n | x^m - x^n$ for all $x \in \mathbb{N}$.
- Note that Ruderman's problem is true iff $S' = S$.

My goal is to employ Baker's explicit *abc*-Conjecture to find S', and hence resolve Ruderman's problem.

UNCG Summer School in Computational Number Theory, 2015

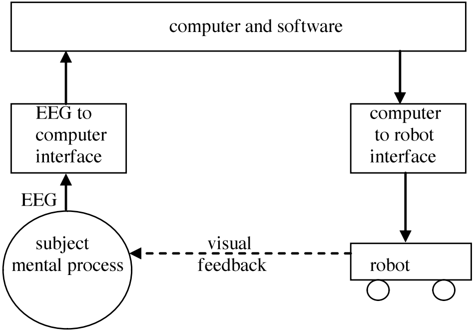
Brain-Robot Interface: Controlling robots using energy emanating from a human brain

Stevo Bozinovski

South Carolina State University

The pioneering work, 1988

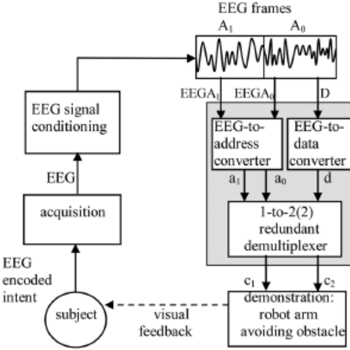
- S. Bozinovski, M. Sestakov, L. Bozinovska,. "Using EEG alpha rhythm to control a mobile robot," In G. Harris, C. Walker (eds.) Proc 10th Annual Conference of IEEE Engineering in Medicine and Biology Society, New Orleans, USA, vol 3, pp. 1515-1516, 1988 (non-invasive recording, subjects were humans)
- The second work on Brain-Robot interface was 11 years later (Chapin et al 1999), (invasive inside brain recording, subjects were rats).
- Those were the only works in brain-robot interface in 20th century. In 21st century there is explosion of works in this area.

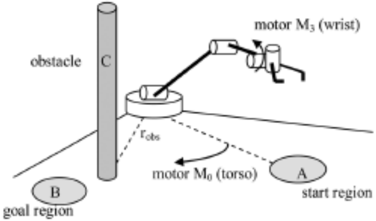


Solving the problem of psychokinesis, using EEG-based psychokinesis

Recent work, 2015

- S. Bozinovski, A. Bozinovski "Mental States, EEG Manifestations, and Mentally Emulated Digital Circuits for Brain-Robot Interaction" IEEE Transactions on Autonomous Mental Development 7(1): 39-51, 2015





Controlling several robot motors using single EEG channel

Things I Like to Work on

Sneha Chaubey

Department of Mathematics, UIUC

UNCG Computational Number Theory Summer School
May 18-22, 2015

Circle Packings and the Riemann zeta function

- Study the distribution of Ford circles in a Ford Circle Packing and obtain asymptotics of various geometric statistics associated to these circles.
- Obtain similar results for other kinds of packings like the Apollonian Circle Packing (ACP) and generalized ACPs.
- Monotonicity properties of derivatives of Riemann zeta function and L-functions associated to modular forms.
- Zeros on the critical line of bounded vertical shifts of the Riemann zeta function.

Pair Correlation of Sequences

- Study the distribution of gaps between members of a given arithmetic sequence.
- Provide examples of sequences whose pair correlation behaves as that of random sequences.
- For instance, Rudnick, Sarnak and Zaharescu showed that the pair correlation function for $\{\alpha x_n\}$, when $\{x_n\}$ is a lacunary sequence which coincides with that of a random sequence, for almost all real numbers α.
- We consider the fractional parts coming from a class of sequences that take rational values and show that their pair correlation behaves in the same way as that of a uniformly distributed random sequence.

Introduction and Interests

Kenneth Chilcoat
East Carolina University

UNCG Summer School in Computational Number Theory 2015

May 18, 2015

Research Background

- Undergraduate Work - Image Processing

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- Recent Work - Digital Signal Processing
  - Attempt to develop a criterion for choosing a packet size of partitioning signals of length $N = 2^p$ and compressing using the Discrete Haar Wavelet Transform.
  - Obtained better (though not phenomenally) results than standard Fourier Transform based compression however criterion is again elusive and signals needed to be optimized manually.
  - Both projects under Dr. Gail Ratcliff.

Research Interests

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    - Computability
    - Theoretical Computer Science – logic, algorithms, language theory etc.

My Research Interest

Lance Everhart

Department of Mathematics and Statistics
University of North Carolina at Greensboro

May 13, 2015

Let d be a squarefree positive integer and $\mathcal{O}_d$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$.

In my research for my thesis I am working towards computing and tabulating congruence subgroups of

$$PSL(2, \mathcal{O}_d) = SL(2, \mathcal{O}_d)/\{\pm 1\}$$

(Hilbert modular group) using Magma.

Some interesting past work of mine:

- Multi-user Dynamic Proofs of Data Possession using Trusted Hardware
  - Crytography and programming
  - Published by CODASPY
- 3D engine for possible future virtual tours of UNCG
  - Calculus application
  - Linear algebra based engine
  - Curve fitting with B-spline curves

Fractional Derivatives of Hurwitz Zeta Functions

Ricky Farr Joint Work With Sebastian Pauli

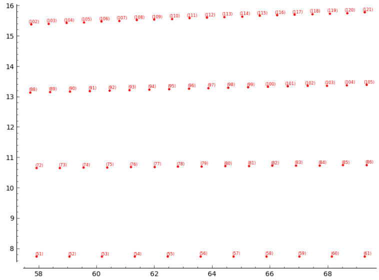
University of North Carolina at Greensboro

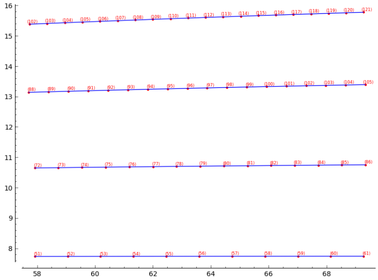
18 May 2015

Fractional Derivative of Hurwitz Zeta Functions

Let $s = \sigma + ti$ where $\sigma > 1$, $0 < a \leq 1$, and $\alpha > 0$

$$\zeta^{(\alpha)}(s, a) = (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n + a)}{(n + a)^{s}}.$$



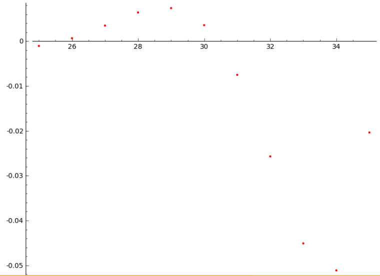


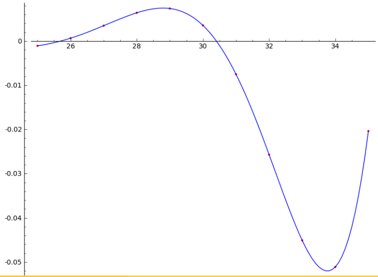
Generalized Non-Integer Stieltjes Constants

Definition

The non-integral generalized Stieltjes Constants is the sequence of numbers $\{\gamma_{\alpha+n}(a)\}_{n=0}^{\infty}$ with the property

$$\sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^{s}} = \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^{n}\gamma_{\alpha+n}(a)}{n!}(s-1)^{n}, \quad s \neq 1$$





Self-Introduction and Research Interests

Zhenchao Ge

Advisor: Micah B. Milinovich

Univeristy of Mississippi

UNCG Summer School
May 18 2015

Research Interests

Analytic Number theory, Riemann zeta-function, Dedekind zeta-function and L-functions.

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Current Research Focus: Analytic theory of algebraic numbers

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- (1966) Yu.V.Linnik and A.I.Vinogradov proved that is $O(p^{\frac{1}{4}+\varepsilon})$.

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- I.M. Vinogradov conjectured the least prime quadratic residue module p is $O_\varepsilon(p^\varepsilon)$.
- (1966) Yu.V.Linnik and A.I.Vinogradov proved that is $O(p^{\frac{1}{4}+\varepsilon})$.
- (2014) P.Pollack generalized this in abelian number fields. The least prime that splits completely in an abelian number field is $O(|D|^{\frac{1}{4}+\varepsilon})$. (Burgess's bound)

The least prime that splits completely in a number field

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Sample Result

The least prime that splits completely in a S_3-sextic extension is $O(|D|^{0.499})$.

Distributions of Primes and the Lang-Trotter Conjecture

Luke Giberson

Clemson University

UNCG Summer School

Primes in Quadratic Progressions

Does the quadratic polynomial $f(n) = n^2 + 1$ produce infinitely many primes?

$$1, 5, 10, 17, 26, 37, 50, 65, 82, 101, 122, 145, 170, 197, 226, 257, \ldots$$

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Consider the elliptic curve $E : y^2 = x^3 - x$, and consider the trace of Frobenius for certain primes p.

Prime p	$a_p(E)$
5	-2
17	2
37	-2
101	-2
197	-2
257	2

The Lang-Trotter Conjecture

Conjecture (Hardy-Littlewood F, 1923)

Let $f(n) = an^2 + bn + c$ *be a "reasonable" quadratic progression. Then*

$$\#\{p < x : p = f(n) \text{ for some } n \in \mathbb{N}\} \sim D_f \frac{\sqrt{x}}{\log x},$$

where the constant D_f *is explicitly predicted.*

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Conjecture (Lang-Trotter, 1976)

Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication. Fix an integer $r \neq 0$. Then

$$\#\{p < x : a_p(E) = r\} \sim C_{E,r} \frac{\sqrt{x}}{\log x},$$

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Zeros of ζ as spectral parameters

Who? Kim Klinger-Logan
Advisor: Paul Garrett

From? University of Minnesota

History

1977 Haas attempted to numerically compute solutions to

$$(\Delta - \lambda_s)u = 0$$

with $\lambda_s = s(s-1)$ on $\Gamma\backslash\mathfrak{H}$.
Shortly after, zeros of ζ and $L(s, \chi_{-3})$ were observed on the list of spectral parameters.

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Wait... did you just say zeros of ζ showed up as *garbage*?!
Hejhal observed that what Haas had actually solved was

$$(\Delta - \lambda_s)u = \delta_\omega^{afc}$$

for δ_ω^{afc} the automorphic dirac delta at the corners of the fundamental domain of $\Gamma\backslash\mathfrak{H}$.

What now?

1982-1983 Colin de Verdiere showed how to make a genuinely self-adjoint operator plausibly related to the problem.

The status of Colin de Verdiere's speculation was unclear until recent work of Bombieri and Garrett gave a precise formulation in terms of distributions and Sobolev spaces.

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For more information visit: math.umn.edu/˜garrett
math.umn.edu/˜kling202
or e-mail: kling202@umn.edu
Thank you!

Explicit formula of a generalized Ramanujan sum

Patrick Kühn,
Universität Zürich

UNCG summer school, May 18-22, 2015

Generalize the Ramanujan sum by introducing a parameter $\beta \in \mathbb{N}$:

$$c_q^{(\beta)}(n) = \sum_{(h,q^\beta)_\beta=1} e^{2\pi i h n/q^\beta}.$$

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In a recent paper, we proved

$$\mathfrak{C}^{(\beta)}(n,x) = \sum_{q \le x}{}' c_q^{(\beta)}(n) = -2\sigma_1^{(\beta)}(n) + \lim_{\nu \to \infty} \sum_{|\gamma| \le T_\nu} \frac{\sigma_{1-\rho/\beta}^{(\beta)}(n)}{\zeta'(\rho)} \frac{x^\rho}{\rho} + \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi/x)^{2k}}{(2k)! k \zeta(2k+1)} \sigma_{1+2k/\beta}^{(\beta)}(n) \qquad (1)$$

where $\sigma_z^{(\beta)}(n) = \sum_{d^\beta | n} d^{\beta z}$.

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where $\sigma_z^{(\beta)}(n) = \sum_{d^\beta | n} d^{\beta z}$. Note that this formula generalizes Titchmarsh explicit formula for $n = 1$ since $c_q^{(\beta)}(1) = \mu(q)$:

$$M'(x) = -2 + \lim_{\nu\to\infty} \sum_{|\gamma| \le T_\nu} \frac{x^\rho}{\rho \zeta'(\rho)} + \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi/x)^{2k}}{(2k)! k \zeta(2k+1)}$$

Compare L.H.S. with R.H.S. for $\beta = 2$:

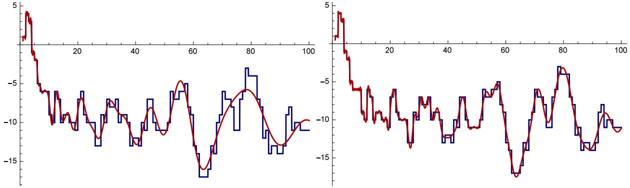

Figure: In blue: $\mathfrak{C}^{(2)}(24, x)$, in red: R.H.S. of (1) with 5 and 25 pairs of zeros and $1 \leq x \leq 100$.

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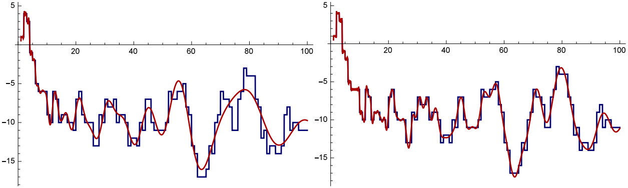

Figure: In blue: $\mathfrak{C}^{(2)}(24, x)$, in red: R.H.S. of (1) with 5 and 25 pairs of zeros and $1 \leq x \leq 100$.

Moreover, RH is connected to the rate of growth of $\mathfrak{C}^{(\beta)}(n, x)$:

Theorem (RH equivalence)

RH is true if and only if $\mathfrak{C}^{(\beta)}(n, x) \ll_{\beta,n} x^{1/2+\varepsilon}$

Liftings of Modular Forms

Huixi Li, Clemson University

UNCG Summer School

May 18, 2015

Fermat's Last Theorem

Conjecture (Fermat, 1637)

No three positive integers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two.

Theorem (Frey's Equation/ Ribet's Theorem, 1984)

If Fermat's equation had any solution (a, b, c) for exponent $p > 2$, then it could be shown that the elliptic curve (now known as a Frey curve)

$$y^2 = x(x - a^p)(x + b^p)$$

is not modular.

Theorem (Modularity Theorem/ Taniyama-Shimura-Weil conjecture)

Any elliptic curve over $\mathbb{Q}$ can be obtained via a rational map with integer coefficients from the classical modular curve $X_0(N)$ for some integer N.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \frac{\zeta(2n)}{(2\pi)^{2n}} = \frac{(-1)^{n+1}B_{2n}}{2(2n)!} \in \mathbb{Q}.$$

Theorem (Shimura, 1976)

Let $f = \sum_{n=-m}^{\infty} a_n e^{2\pi i n z} \in S_k(N, \chi)$ be primitive, then there exist two complex numbers $c^+(f)$ and $c^-(f)$, such that for $1 \leq m \leq k-1$,

$$\frac{L(m,f)}{(2\pi)^m c^{\pm}(f)} \in \mathbb{Q}(a_f) \subseteq \overline{\mathbb{Q}}.$$

Theorem (Saito-Kurokawa Lift)

The Saito-Kurokawa lifting σ_k takes level 1 modular forms f of weight $2k-2$ to level 1 Siegel modular forms of degree 2 and weight k,

$$S_{2k-2}(SL_2(Z)) \cong S^+_{k-\frac{1}{2}}(\Gamma_0(4)) \cong J^c_{k,1}(SL_2(\mathbb{Z})) \cong S_k(Sp_4(\mathbb{Z})).$$

Approximations of L-functions

Junxian Li

University of Illinois at Urbana-Champaign

UNCG Summer School
May 18, 2015

I. Approximations by Dirichlet series

Definition 1 $F_N(s) = \sum_{n=1}^{X} \frac{1}{n^s}$.

Knopp Every point of $\sigma = 1$, the line of convergence for $\zeta(s)$, is an accumulation point of the zeros of the partial sum $F_N(s)$.

Turán Riemann Hypothesis is valid if there are positive numbers n_0 and K such that for $N > n_0$ the truncated zeta function $F_N(s)$ does not vanish in the half-plane $\sigma \geq 1 + \frac{K}{\sqrt{N}}$.

Montgomery For given $0 < c < 4/\pi - 1$, and N large enough, $F_N(s)$ always has a zero in the half plance $\sigma > 1 + c\frac{\log\log N}{\log N}$.

Definition 2 $\zeta_N(s) = F_N(s) + \chi(s)F_N(1-s)$.

Spira For $N = 1$ and 2, all the zeros of $\zeta_N(s)$ lie on the critical line. For $N \geq 3$, there may be infinite many zeros off the line.

II. Approximations by finite Euler Product

Definition 3 $\zeta_X(s) = P_X(s) + \chi(s)P_X(\bar{s})$

Gonek $\zeta(s) = P_X(s)Z_X(s)\left(1 + O\left(\frac{X^{-\sigma-2}}{\tau^2 \log^2 X}\right)\right)$

Gonek There is a positive absolute constant C_0 such that if $|\chi(\sigma + it)| = 1$ with $0 \leq \sigma \leq 1$ and $|t| \geq C_0$, then $\sigma = \frac{1}{2}$.

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L-functions: Dirichlet L-functions , Dedekind zeta-functions , Hecke L-functions , Artin L-functions , Automorphic L-functions, the Selberg class.

- Equivalent statements of corresponding conjectures.
- Various approximations and error estimations.
- Zero free regions, pair correlations and density.
- Numerical computations.

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Representations of integral Hermitian forms by sums of norms

Jingbo Liu

Wesleyan University

May 2015

Quadratic forms over $\mathbb{Z}$

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Mordell (1937) $q(x) = \sum_{i=1}^{6} x_i^2 + (\sum_{i=1}^{6} x_i)^2 - 2x_1x_2 - 2x_2x_6$ cannot be represented by a sum of squares.

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Kim and Oh (2005) $\quad g_{\mathbb{Z}}(n) = O(3^{n/2} n \log n)$ when n is large.

Hermitian forms over $\mathfrak{O}_E$

Let $E = \mathbb{Q}(\sqrt{-\ell})$ where ℓ is a square free positive integer and $\mathfrak{O}_E$ be the ring of integers of E. An integral Hermitian form over $\mathfrak{O}_E$ is of the form

$$h(x_1, \ldots, x_n) = \sum_{1 \leq i,j \leq n} a_{ij} x_i \overline{x_j}$$

where $a_{ij} = \overline{a_{ji}} \in \mathfrak{O}_E$.

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Theorem (L-2015)

$$g_E(n) = O\left((p+5)^{n+2}(n \log(n\ell) + \ell B_{3,\chi} + \ell^3)\right)$$

when n is large, where p is the smallest prime that is inert in E.

The Computation of Galois Groups over Local Fields

Jonathan Milstead

May 18, 2015

OM Related Things

1. **Splitting Field (Milstead, Pauli, Sinclair)**
   - Variation of OM Algorithm
   - Brute Force method
2. **Ramification Polygon**: Newton polygon of $\dfrac{\phi(\alpha x + \alpha)}{\alpha^n}$.

Interested in 1 or 2 segment cases (**Grieve, Pauli**).
Blocks ($\Delta_{i,j}$):

.

$$\left\{ \alpha' : \begin{array}{l} \phi(\alpha') = 0 \text{ and either} \\ \\ v_L(\alpha' - \alpha_1) > m_i + 1 \ \underline{\text{or}} \\ v_L(\alpha' - \alpha_1) = m_i + 1 \text{ and } \underline{a_j}\left(\dfrac{(-1 + \frac{\alpha'}{\alpha_1})^{e_i}}{\alpha_1^{h_i}}\right) = 0 \end{array} \right\}$$

Polynomials over $\mathbb{Q}$

3. **Stauduhar's method (1973):**
   - Key Challenge: finding a G-relative H-invariant $F \in \mathbb{Z}[X_1, ..., X_n]$, i.e., F so that $\text{Stab}_G F := \{\sigma \in G \mid F^\sigma = F\} = H$ where $H < G \leq S_n$
   - Construct resolvent $R_F := \prod_{\sigma \in G//H} (T - F^\sigma(\alpha_1, ..., \alpha_n))$ to see if $\text{Gal}(f) \leq H^g$.
4. **Fieker, Klüners**
   - The "first" practical degree independent algorithm.
   - Special Invariants: invariants in terms of invariants of smaller degree (Blocks).
   - Generic Invariants: Double Coset Decomposition (ladder) and Orbit Sums. Basis or probabilistic.

Some Additive Combinatorics

Hans Parshall

University of Georgia

May 18, 2015 at UNCG

Additive Structure in the Integers

If $A \subseteq \mathbf{Z}$ is "additively structured", then it should contain long arithmetic progressions, $\{n + D, n + 2D, \ldots, n + \ell D\}$.

- Subgroups of $\mathbf{Z}$ are long arithmetic progressions.
- $|A + A| < 2|A| \Leftrightarrow A = \{n + D, n + 2D, \ldots, n + \ell D\}$

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(Szemerédi, 1975) If A has positive density in $\mathbf{Z}$, then A contains arbitrarily long arithmetic progressions.

(Bourgain, 1990) Even if A is fairly sparse, $A + A$ contains fairly long arithmetic progressions.

Additive Structure in the Primes

(Green-Tao, 2004) For any $\ell \in \mathbf{N}$, there exist infinitely many $n, D \in \mathbf{N}$ for which the following are all prime:

$$n + D, n + 2D, \ldots, n + \ell D$$

(Maynard, Tao, 2013) For any $m \in \mathbf{N}$, there exist $d_1, \ldots, d_m \in \mathbf{N}$ such that there exist infinitely many primes p for which the following are all prime:

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Adding ideas of Pintz, one can find prime configurations:

$$\begin{array}{cccc} n + D & n + 2D & \ldots & n + \ell D \\ n + D + d_1 & n + 2D + d_1 & \ldots & n + \ell D + d_1 \\ \vdots & \vdots & \ddots & \vdots \\ n + D + d_m & n + 2D + d_m & \ldots & n + \ell D + d_m \end{array}$$

and there is a satisfying $\mathbf{F}_q[t]$ analogue (P, 2015).

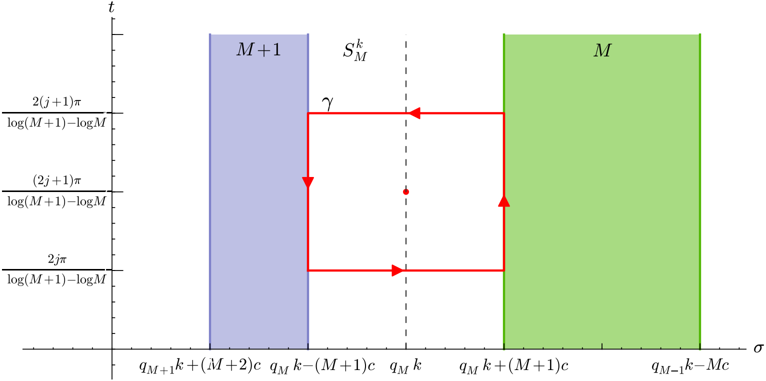
Research Interest:

Algorithms for Local Fields and Zeros of Derivatives of Zeta

Sebastian Pauli

University of North Carolina at Greensboro

Zeros of Derivatives of ζ – Right Half Plane

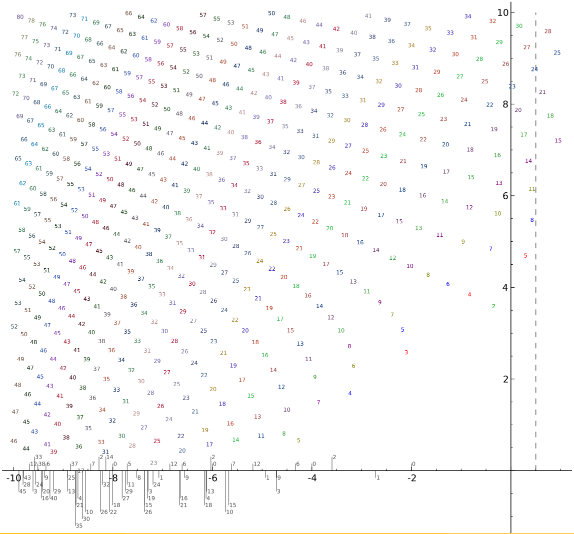


Theorem (Binder, P., Saidak)

For $M \geq 2$ and $q_M = \dfrac{\log\left(\frac{\log M}{\log M+1}\right)}{\log\left(\frac{M}{M+1}\right)}$ the red box contains one zero of $\zeta^{(k)}(s)$.

Zeros of Derivatives of ζ – Left Half Plane

with Ricky Farr



James Rudzinski
UNCG

May 18, 2015

UNCG Number Theory Summer School

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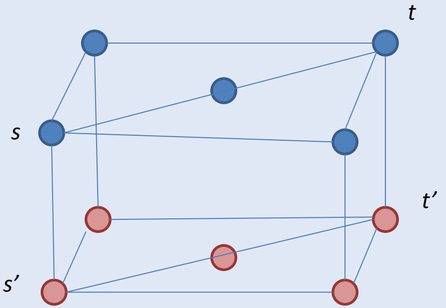
Some Areas of Interest

- Graph Theory
  - Graph Coloring
  - Pebbling
  - Bunk Bed Graphs
    - Percolation, Connectivity
- Game Theory
  - Combinatorial Games
    - Nim, Dynamic Nim

Current Research

Bunk Bed Conjecture – Given a bunk bed graph with a probability function defined on the edges, the probability that *s* is connected to *t* is greater than the probability that *s* is connected to *t'*.



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What is the Maximum Amount of Information on Field Arithmetic that We Can Get out of L-Functions?

Jonathan W. Sands

University of Vermont, visiting UNCG

UNCG Workshop, May 2015

Zeta Functions and Number Fields

Let

- F be a number field, a finite degree extension of the rationals.
- $\mathcal{O}_F$ be the subring of elements satisfying a monic polynomial over $\mathbb{Z}$.
- h_F be the finite number of isomorphism classes of nonzero ideals of $\mathcal{O}(F)$.
- w_F be the number of roots of unity in F.
- R_F be the regulator determinant obtained from the units of $\mathcal{O}_F$.
- $\zeta_F(s) = \sum_I |\mathcal{O}_F/I|^{-s}$, which converges to an analytic function when the real part of s is greater than 1, and has an analytic continuation to the entire complex plane except for a pole at s=0.

Connections between Arithmetic and L-Values

Fundamental Theorem: The leading Taylor-McLaurin coefficient of $\zeta_F(s)$ is $\zeta_F^*(0) = -h_F R_F / w_F$

Questions:

- What similar results exist for the values at negative integers? (Lichtenbaum's Conjecture for the algebraic K-groups of $\mathcal{O}_F$).
- Can we decompose both sides of the equation into matching pieces using characters of a group of automorphisms of F? (Stark's conjecture for Artin L-functions.)
- Can we get information on how such automorphisms act on the ideal classes of F from the L-functions? (Generalized Brumer-Stark conjecture.)
- What if F is not a commutative field, but a skew field? (Generalized Eichler Mass-formula?)
- Mix and match all of the above! Refine! (Generalized Coates-Sinnott conjecture for K-groups, Conjectures of Snaith, Buckingham, Burns, Gross, Rubin, Popescu, Emmons-Popescu ...)

Enumerating Invariants and Extensions of p-adic Fields

Brian Sinclair

University of North Carolina at Greensboro

18 May 2015

Additional Invariants

Beyond considering degree and discriminant:

- Ramification Polygons: Newton Polygon of $\frac{\varphi(\alpha x + \alpha)}{\alpha^n}$
- Residual Polynomials of the Ramification Polygon ($\mathcal{A}$)

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- Finding $\mathcal{A}$ given a ramification polygon.

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Algorithms for enumerating invariants

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- Finding $\mathcal{A}$ given a ramification polygon.

Algorithms for enumerating extensions with additional invariants

- A specialization of the algorithm of Pauli and Roblot.
- Generating polynomials are *reduced Eisenstein* (Monge 2011).

Additional Invariants

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- Residual Polynomials of the Ramification Polygon ($\mathcal{A}$)

Algorithms for enumerating invariants

- Finding ramification polygons given degree and discriminant.
- Finding $\mathcal{A}$ given a ramification polygon.

Algorithms for enumerating extensions with additional invariants

- A specialization of the algorithm of Pauli and Roblot.
- Generating polynomials are *reduced Eisenstein* (Monge 2011).

Algorithms for counting extensions with additional invariants

- Specializations of Krasner's mass formula given ramification polygon or both the polygon and its residual polynomials.

Example: Totally ramified over Q_3 with $n = 9, disc = 3^{15}$

We can enumerate extensions by considering reduced (in the Monge sense) Eisenstein polynomials having given ramification polygon and residual polynomials.

<table>
<tr><th>Ramification Polygon</th><th>Rep. of $\mathcal{A}$</th><th>Polys.</th><th colspan="3">Extensions</th></tr>
<tr><td>$\{(1,7),(9,0)\}$</td><td>$(z+1)$</td><td>6</td><td>54</td><td>54</td><td rowspan="3">162</td></tr>
<tr><td rowspan="2">$\{(1,7),(3,3),(9,0)\}$</td><td>(z^2+1, z^3+1)</td><td>6</td><td>54</td><td rowspan="2">108</td></tr>
<tr><td>$(2z^2+1, z^3+2)$</td><td>18</td><td>54</td></tr>
</table>

Compare Pauli-Roblot:
$3^{12}2^2 = 2\ 125\ 764$ polynomials to generate 162 extensions.
(13 122 per extension)

Stark's Conjecture as it relates to Hilbert's 12th Problem

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May 18, 2015

Let F be a real quadratic field, $\mathcal{O}_\mathsf{F}$ the ring of integers in F, and $\mathfrak{m}$ an integral ideal in $\mathcal{O}_\mathsf{F}$ with $\mathfrak{m} \neq (1)$. There are two infinite primes associated to the two distinct embeddings of F into $\mathbb{R}$, denoted by $\mathfrak{p}_\infty^{(1)}$ and $\mathfrak{p}_\infty^{(2)}$. Let $\mathcal{H}_2 := H(\mathfrak{m}\mathfrak{p}_\infty^{(2)})$ denote the ray class group modulo $\mathfrak{m}\mathfrak{p}_\infty^{(2)}$, which is a finite abelian group.

Given a class $\mathcal{C} \in \mathcal{H}_2$, there is an associated partial zeta function $\zeta(s, \mathcal{C}) = \sum \mathsf{N}\mathfrak{a}^{-s}$, where the sum runs over all integral ideals (necessarily rel. prime to $\mathfrak{m}$) lying within the class $\mathcal{C}$. The function $\zeta(s, \mathcal{C})$ has a meromorphic continuation to $\mathbb{C}$ with exactly one (simple) pole at $s = 1$. We have $\zeta(0, \mathcal{C}) = 0$ for all $\mathcal{C} \in \mathcal{H}_2$, but $\zeta'(0, \mathcal{C}) \neq 0$ (if certain conditions are met).

First crude statement of Stark's conjecture: $e^{-2\zeta'(0,\mathcal{C})}$ is an algebraic integer, indeed this real number is conjectured to be a root of a palindromic monic polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_2 x^2 + a_1 x + 1 \in \mathbb{Z}[x].$$

For this reason, $e^{-2\zeta'(0,\mathcal{C})}$ is called a "Stark unit". By class field theory, there exists a ray class field $\mathsf{F}_2 := \mathsf{F}(\mathfrak{m}\mathfrak{p}_\infty^{(2)})$ with the following special property: F_2 is an abelian extension of F with $\mathsf{Gal}(\mathsf{F}_2/\mathsf{F}) \cong \mathcal{H}_2$. Stark's conjecture states more precisely that $e^{-2\zeta'(0,\mathcal{C})} \in \mathsf{F}_2$ for all $\mathcal{C} \in \mathcal{H}_2$.
This fits the general theme of Hilbert's 12th problem: Construct analytic functions which when evaluated at "special" points produce algebraic numbers which generate abelian extensions over a given base field.

My Research Interest

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May 18, 2015

Automorphic for $\Gamma = SL(2,\mathbb{Z})$

- smooth $\quad f : \mathcal{H}^2 \to \mathbb{C}$
- periodic $\quad f(\gamma z) = f(z) \quad \gamma \in SL(2,\mathbb{Z})$
- eigenform $\quad \Delta f = \nu(1-\nu) f \qquad \Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
- polynomial growth $f(iy) = O(y^N)$

Fourier expansion

$$f(z) = \sum_{m\in\mathbb{Z}} \underbrace{A_m(y) e^{2\pi i m x}}_{W_m(z)}$$

A Whittaker function of type ν

- smooth
- eigenform $\quad \Delta W(z) = \nu(1-\nu) W(z)$
- $W\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot z\right) = W(z) e^{2\pi i m u}$

Example $\quad W(z,\nu,\psi) := \int_{-\infty}^{\infty} I_\nu\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot z\right) e^{2\pi i m u}\, du \qquad I_\nu(z) = y^\nu$

Theorem 3.4.8 (Multiplicity one) *Let* $\Psi(z)$ *be an* $SL(2,\mathbb{Z})$*–Whittaker function of type* $\nu \neq 0, 1$, *associated to an additive character* ψ, *which has rapid decay at* ∞. *Then* $\quad \Psi(z) = aW(z,\nu,\psi)$

Automorphic forms for $\Gamma = SL(3,\mathbb{Z})$

- smooth $f : \mathcal{H}^3 \to \mathbb{C}$
- periodic $f(\gamma z) = f(z)$
- eigenform $\Delta_1 f = \lambda_1 f \quad \Delta_2 f = \lambda_2 f$

***A* Whittaker function of type ν**

- smooth
- eigenform $\Delta_1 W = \lambda W \qquad \Delta_2 W = \mu W$
- $W(uz) = W(z)e^{2\pi i(u_1+u_2)}$

Example Jacquet Whittaker function for $SL(3,\mathbb{Z})$

$$W_{\text{Jacquet}}(z,\nu,\psi_m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_\nu(w_3 \cdot u \cdot z)\overline{\psi_m(u)}\, du_{1,2}du_{1,3}du_{2,3} \qquad I_\nu(z) = y_1^{\nu_1+2\nu_2}y_2^{2\nu_1+\nu_2}$$

Theorem (Diaconu and Goldfeld) *Fix* $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$. *Let* $\Psi_\nu(z)$ *be an* $SL(3,\mathbb{Z})$ *Whittaker func*
Assume that $\Psi_\nu(z)$ *has sufficient decay in* y_1, y_2 *so that*

$$\int_0^\infty\int_0^\infty y_1^{\sigma_1}y_2^{\sigma_2}\,|\Psi_\nu(y)|\,\frac{dy_1dy_2}{y_1y_2}$$ *converges for sufficiently large* σ_1,σ_2. *Then* $\Psi_\nu(z) = c\cdot W_{\text{Jacquet}}(z,\nu,\psi)$

Applications of Reduction Theory to Automorphic Forms

Dan Yasaki

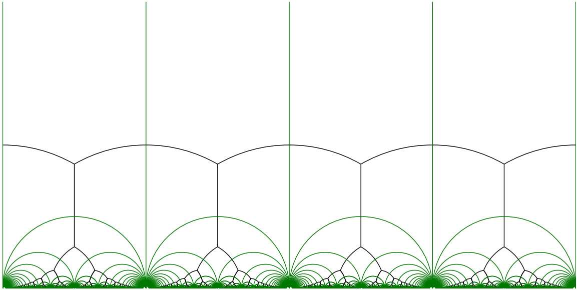
The University of North Carolina Greensboro

May 18–22, 2015
UNCG Summer School 2015
Zeta Functions – New Theory and Computations

“Modular forms” ($f(z) = \sum a_n q^n$)

Manin:

$$H_1(X_0(N), \partial X_0(N); \mathbb{Q}) \simeq \mathcal{M}_2(\Gamma_0(N))$$

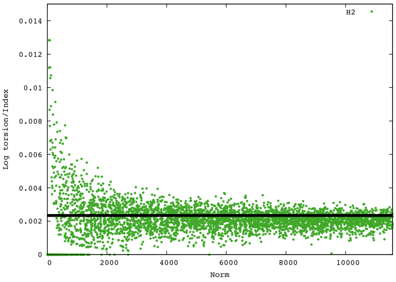


Generalize: F a number field of degree $r + 2s$

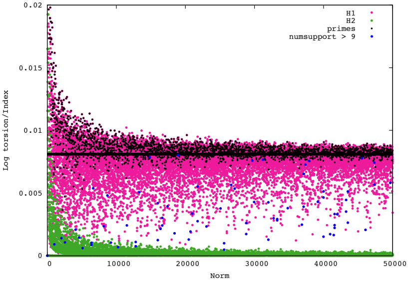
$\mathrm{GL}_2 / \mathbb{Q}$	GL_2 / F
$\Gamma_0(N) \subset \mathrm{GL}_2(\mathbb{Z})$	$\Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_2(\mathcal{O}_F)$
$\mathfrak{h}$	$\mathfrak{h}^r \times \mathfrak{h}_3^s \times \mathbb{R}^{r+s-1}$
modular symbols	sharblies

UNCG

Torsion plots: $\delta = 1$ examples



F = cubic fld of disc -23
$X = \mathfrak{h} \times \mathfrak{h}_3 \times \mathbb{R}$



$F = \mathbb{Q}(i)$
$X = \mathfrak{h}_3$

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Research Interest: Gaps between the zeros of the Riemann zeta function

Kam-hung Yau

University of Auckland

May 12, 2015

Things I like to think about

Let $N(T)$ be the number of zeros of $\zeta(s)$, then one can deduce

$$N(T) := \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T),$$

with $s = \sigma + it$ in the rectangle $0 \le \sigma \le 1$, $0 \le t \le T$. Define $\lambda := \lim sup(\gamma' - \gamma)\frac{\log\gamma}{2\pi}$ and $\mu := \lim inf(\gamma' - \gamma)\frac{\log\gamma}{2\pi}$, where γ runs over all the ordinates of the zeros of the $\zeta(s)$.

Natural question to ask:

Is $\mu = 0$ and $\lambda = +\infty$?

Progress toward gaps:

Montgomery and Odlyzko (1984): $\mu < 0.5179$.
Conery, Ghosh, and Goneck (1984): $\mu < 0.5172$.
Bui, Milinovich, and Ng (2010): $\mu < 0.5155$.
Feng and Wu (2012): $\mu < 0.5154$.

How to detect gaps?

Define

$$A(t) := \sum_{k \le K} a_k k^{-it}$$

with T being large and $K := T(\log T)^{-2}$.

$$M_1 := \int_{T/2}^{2T} |A(t)|^2 dt \text{ and } M_2(c) := \int_{-\pi c/\log T}^{\pi c/\log T} \sum_{T/2 \le \gamma \le 2T} |A(\gamma + \alpha)|^2 d\alpha.$$

Then $M_2(c)$ is monotonically increasing and $M_2(\mu) \le M_1 \le M_2(\lambda)$. If $M_2(c) > M_1$ for some c and $A(t)$ then $\mu < c$. With a little working one can show $M_2(c)/M_1 = h(c) + o(1)$, where

$$h(c) = c - \frac{\Re(\sum_{nk \le K} a_k \bar{a}_{nk} g_c(n) \Lambda(n) n^{-1/2})}{\sum_{k \le K} |a_k|^2}, \text{ and } g_c(n) = \frac{2 \sin(\pi c \frac{\log n}{\log T})}{\pi \log(n)}.$$

UNCG Summer School in
Computational Number Theory

UNCG

Zeta Functions – New Theory and Computations

May 18 to May 22, 2015

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

Speakers

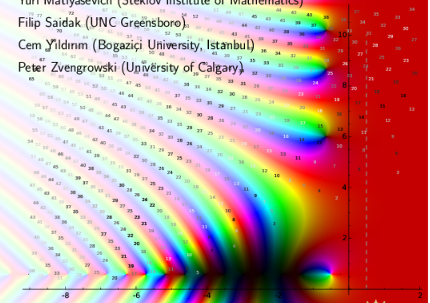
Fredrik Johansson (INRIA Bordeaux-Sud-Ouest)

Yuri Matiyasevich (Steklov Institute of Mathematics)

Filip Saidak (UNC Greensboro)

Cem Yıldırım (Bogaziçi University, Istanbul)

Peter Zvengrowski (University of Calgary)



Organizers: Sebastian Pauli, Filip Saidak, Brett Tangedal, Dan Yasaki

www.uncg.edu/mat/numbertheory/summerschool