EXERCISES FOR UNITS

MICHAEL E. POHST

1. Multiple choice

1. An integer α of a number field F is a root of unity precisely if $T_2(\alpha) = n$.
a) yes, b) no.
2. Number fields F with $r_1 > 0$ satisfy $TU_F = \{\pm 1\}$.
a) yes, b) no.
3. Let F be an imaginary quadratic Field, i.e. $2 = n = 2r_2$. Then TU_F has order 2.
a) yes, b) no.
4. Let F be an algebraic number field of order $n = r_1 + 2r_2$. The rank r of the unit group is 1 for
a) totally complex quartic fields ($r_2 = 2$), b) cyclic (=Galois) cubic fields, c) imaginary quadratic fields ($d_F < 0$).
5. Let $F = \mathbb{Q}(\sqrt{m})$ be a real quadratic number field for some square–free positive integer $m \equiv 0 \bmod 3$. Then the norm of the fundamental unit of F is negative.
a) yes, b) no.

2. Computations

1. Caculate a lower regulator bound for $F = \mathbb{Q}(\sqrt{m})$ with $m \in \{9930, 9931, 9933\}$ which is larger than one fifth of the actual regulator, $28.9, 189.1, 5.0$ respectively. How large do you need to choose K?
2. Let $F = \mathbb{Q}(\rho)$, ρ a zero of $x^4+x^3-3x^2-x+1$. Compute a system of fundamental units for o_F. What is the regulator of F?
3. Let $F = \mathbb{Q}(\rho)$ for a zero of $x^3 - x - 1$. Compute a full set of non–associate solutions of $N(\alpha) = 5^2 7^2 11^2$.

3. Proofs

1. Let p be an odd prime number . Set $\zeta = e^{2\pi i/p}$ and $R = \mathbb{Z}[\zeta]$. Let $\varepsilon \in U(R)$. Prove:
a) $\varepsilon/\bar{\varepsilon}$ belongs to $U(R)$. (Overlining denotes complex conjugation.)
b) $\varepsilon/\bar{\varepsilon}$ equals ζ^k for an exponent $k \in \{1, ..., p\}$.
c) Every element of $U(R)$ is the product of an element of $U(R) \cap \mathbb{R}$ and a root of unity.
d) $U(R) = U(\mathbb{Z}[\zeta + \zeta^{-1}]) \times \langle \zeta \rangle$.
2. Let ζ be a zero of $x^4 + 1$ and $F = \mathbb{Q}(\zeta)$. Determine the unit group and the regulator of F. (Hint: What is the unique real quadratic subfield of F?)