EXERCISES FOR CLASS GROUPS

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1. Multiple choice

1. Let $\mathbf{a} = aO_F + \alpha O_F$ be an integral ideal of a number field F. Then $\mathbf{a}^2 = a^2O_F + \alpha^2 O_F$.
a) yes, b) no.
2. In $R = \mathbb{Z}[\sqrt{5}]$ the ideal norm is multiplicative.
a) yes, b) no.
3. The imaginary quadratic field $F = \mathbb{Q}(\sqrt{-5})$ has class number 1.
a) yes, b) no.
4. Let $\mathbf{p}$ be a non–zero prime ideal in O_F. Then $\mathbf{p} \cap \mathbb{N} = \{p\}$ for some prime number p.
a)yes, no.
5. There exist prime ideals in O_F containing more than one prime number.
a) yes, no.

2. Computations

1. Carry out the details of the computation of the class group of $\mathbb{Q}(\sqrt{-814})$ (compare lecture).
2. Compute class groups and unit groups of number fields with Magma (conditional/unconditional).
a) $F = \mathbb{Q}(\rho)$, ρ a zero of $x^3 - 7823$;
b) $F = \mathbb{Q}(\rho)$, ρ a zero of $x^4 - x^3 - 6x^2 - 2x + 4$;
c) $F = \mathbb{Q}(\rho)$, ρ a zero of $x^6 - 114x^4 + 48x^3 + 3249x^2 - 2736x - 19456$.
3. Let $F = \mathbb{Q}(\rho)$ for a zero of $x^3 - x - 1$. Compute a full set of non–associate solutions $\alpha \in O_F$ of $N(\alpha) = 5^2 7^2 11^2$.
4. Let $F = \mathbb{Q}(\rho)$ for a zero of $x^6 + x^3 + 1$. Use Magma to compute a full set of non–associate solutions $\alpha \in O_F$ of $|N(\alpha)| \in \{57, 58\}$.
5. Develop methods for deciding whether two given (integral) ideals
a) are equal;
b) belong to the same ideal class.

3. Proofs

1. Prove several of the statements on fractional ideals.
2. Introduce 2–element normal presentations?
3. Use the fact that the class number of $F = \mathbb{Q}(\sqrt{-5})$ is 2 to prove that $y^2 = 2x^3 - 5$ has exactly 2 integral solutions $(x, y) = (3, \pm 7)$.