Special Values of Dirichlet L-functions — UNCG Summer School

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OR

Why Algebraists Love L-functions Too!

A famous theorem in Number Theory, first proved by Dirichlet ca. 1840, states that if a and m are positive integers that are relatively prime then the arithmetic sequence $\{a + km \mid k \in \mathbb{Z}^{\geq 0}\}$ contains an infinite number of prime numbers. Dirichlet's proof uses analysis in a crucial way and certain number-theoretic functions called "Dirichlet characters" that are interesting in their own right.

Definition: A complex-valued function $\chi : \mathbb{Z}^+ \to \mathbb{C}$ is said to be a "Dirichlet character modulo m" (m is a fixed positive integer; a and b below are arbitrary positive integers) if

a) $$\chi(a) \begin{cases} = 0 & \text{if } \gcd(a,m) \neq 1 \\ \neq 0 & \text{if } \gcd(a,m) = 1. \end{cases}$$

b) If $a \equiv b \pmod{m}$, then $\chi(a) = \chi(b)$.

c) $\chi(ab) = \chi(a)\chi(b)$.

Four simple examples, when $m = 5$, are the following

a	1	2	3	4	5	parity	conductor
$\chi_0(a) =$	1	1	1	1	0	even	1
$\chi(a) =$	1	i	$-i$	-1	0	odd	5
$\chi^2(a) =$	1	-1	-1	1	0	even	5
$\chi^3(a) =$	1	$-i$	i	-1	0	odd	5

χ_0 is called the "trivial character modulo 5". If χ takes on a single value other than 0 and 1, we say that it is a "nontrivial character".

Special Values

A "partial ζ-function"

$m \in \mathbb{Z}^+$ fixed $\qquad 1 \le a \le m$

$$\zeta(s, a, m) = \sum_{\substack{n \equiv a \pmod{m} \\ n \in \mathbb{Z}^+}} \frac{1}{n^s}$$

order $= \varphi(m)$

$\downarrow$

$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$

group homomorphism

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

$$= \sum_{\substack{a=1 \\ \gcd(a,m)=1}}^{m} \chi(a)\, \zeta(s, a, m)$$

By orthogonality,

$$\zeta(s, a, m) = \frac{1}{\varphi(m)} \sum_{\chi} \bar{\chi}(a)\, L(s, \chi)$$

Can isolate a single congruence class with the right combo of L-fcts.

End game:

$$\lim_{s \to 1^+} \sum_{\substack{p \equiv a(m) \\ \gcd(a,m)=1}} \frac{\log p}{p^s} \to \infty$$

which can only happen if $\exists$ an ∞ # of such primes.

Note: $\displaystyle\lim_{s \to 1^+} \zeta(s, a, m) = \infty$.

Special Values

Basic facts and conventions:

1) $\chi(1) = 1 \quad \forall \chi$; if $\chi(a) \neq 0$, then $\chi(a)$ is a root of unity.
2) $\exists$ exactly $\varphi(m)$ distinct Dirichlet characters defined modulo m.
3) The Dirichlet chars. mod m form a group under multiplication $\cong (\mathbb{Z}/m\mathbb{Z})^\times$.
4) If $m \geq 2$, we say that χ is $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ if $\chi(m-1) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$.
   If $m = 1$, the trivial character mod 1 : $\chi_0(a) = 1 \quad \forall a \in \mathbb{Z}^+$, is defined to be even by default.

Definition of an Induced Modulus: Let χ be a Dirichlet character mod m and let d be any positive divisor of m. The number d is called an "induced modulus for χ" if we have $\chi(a) = 1$ whenever $(a, m) = 1$ and $a \equiv 1 \pmod{d}$.

Example: $m = 6$

a	1	2	3	4	5	6	parity	conductor
$\chi(a) =$	1	0	0	0	-1	0	odd	3

$d = 1$ and $d = 2$ are not induced moduli for χ but $d = 3$ is an induced modulus for χ. "Primitive" version of χ:

$m = 3 \quad \chi(a) = 1 \quad -1 \quad 0 \;\big|\; 1 \quad -1 \quad 0$

Definition: Let χ be a Dirichlet character mod m. The smallest induced modulus d for χ is called the "conductor of χ," and is denoted by f_χ.

Definition: If χ is a Dirichlet char. mod m and $f_\chi = m$, then we say that we are working with the "primitive version of χ."

Example: If χ is the trivial character mod $m \geq 1$, then $f_\chi = 1$ and its primitive version is given by $\chi_0(a) = 1 \quad \forall a \in \mathbb{Z}^+$.

Special Values

Blanket assumption from now on: We always work with the primitive version of a Dirichlet character χ.

Note: χ is non-trivial, i.e. $\chi \neq \chi_0$, iff $f_\chi > 1$.

Definition of a Dirichlet L-function: Given a Dirichlet character χ mod m, the corresponding L-function is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \text{for } \operatorname{Re}(s) > 1.$$

Note: Since $L(s,\chi_0) = \zeta(s)$, we see that the Riemann ζ-function is just one function among an infinite class of related functions defined by the same means.

Euler product: $L(s,\chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$ for $\operatorname{Re}(s) > 1$.

Analytic continuation: If $\chi \neq \chi_0$, there exists a uniquely defined entire function (i.e. holomorphic on all of $\mathbb{C}$) $\tilde{L}(s,\chi)$ such that $\tilde{L}(s,\chi) = L(s,\chi)$ $\forall s$ with $\operatorname{Re}(s) > 1$. When $\chi = \chi_0$, the extended function $\tilde{L}(s,\chi_0)$ is holomorphic everywhere except at $s=1$ where it has a simple pole with residue $= 1$. We'll drop the tilde from now on!

When we speak of the "special values of Dirichlet L-functions", we are referring to the values $L(n,\chi)$ for $n \in \mathbb{Z}$. The functional equation of a given $L(s,\chi)$ relates the values at s to those at $1-s$ so that the values

$$\begin{array}{cccc} L(0,\chi), & L(-1,\chi), & L(-2,\chi), & \ldots \text{ are directly related to} \\ \updownarrow & \updownarrow & \updownarrow & \\ L(1,\chi), & L(2,\chi), & L(3,\chi), & \text{etc.} \end{array}$$

It is very interesting that the formulas for the first row of values are much nicer than those of the second row!

Special Values

Before stating the functional equation, we define

$$a_\chi = \begin{cases} 0 & \text{if } \chi \text{ is even} \\ 1 & \text{if } \chi \text{ is odd.} \end{cases}$$

Given χ, the corresponding "completed" L-function is defined by

$$\Lambda(s,\chi) = \underbrace{\left(\frac{f_\chi}{\pi}\right)^{(s+a_\chi)/2} \Gamma\left(\frac{s+a_\chi}{2}\right)} L(s,\chi)$$

(Sophisticated) Comment: Valuation theory tells us there is an infinite prime p_∞ that should be considered on an equal footing with the usual (finite) primes 2, 3, 5, 7, ... To obtain $L(s,\chi)$, we multiplied together all Euler factors associated to the finite primes. The underlined expression above is the Euler factor associated to p_∞. It differs between even χ and odd χ because p_∞ does not appear in the conductor $\breve{f}_\chi$ for even χ but does appear for odd χ:

$$\breve{f}_\chi = \begin{cases} f_\chi & \text{if } \chi \text{ is even} \\ p_\infty f_\chi & \text{if } \chi \text{ is odd.} \end{cases}$$

The completed L-function is preferable since all primes are accounted for!

Functional Eq: $\Lambda(1-s,\bar{\chi}) = W(\chi)\Lambda(s,\chi)$, where $|W(\chi)| = 1$. The number $W(\chi) \in \mathbb{C}$ is known as the "Artin root number" and it may be written explicitly as a (normalized) Gauss sum.

(Sophisticated) Comment II: There are various ways to derive the functional equation. The most elegant and profound method involves θ-functions (Riemann's 2nd proof involved such functions). A special inversion formula for θ-functions translates directly into the functional equation above!

Special Values

In 1735, Euler gave an exact formula for $\zeta(2)$. Later, he evaluated $\zeta(2j)$ for $j = 1, 2, 3, \ldots$ in one fell swoop! We record the first two formulas:

$$\zeta(2) = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \frac{\pi^4}{90}.$$

Nice formulas for $\zeta(3)$, $\zeta(5)$, ... slipped through Euler's net and to this day these values have an air of mystery about them. In order to give a nice description of Euler's evaluation of $\zeta(2j)$, $j \in \mathbb{Z}^+$, we need to introduce the Bernoulli numbers. They appear in the power series expansion around $x = 0$ of the function

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \left(\frac{x^j}{j!} \right),$$

where B_j is called the "jth Bernoulli number." These numbers first appeared in 1713 in a work of Jacob Bernoulli entitled "Ars Conjectandi."

Recall the following formula for the geometric series: If $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ (a is a fixed constant).

For a fixed $j \in \mathbb{Z}^+$,

$$\frac{2}{x^2 - j^2} = -\frac{2}{j^2 - x^2} = \frac{-2/j^2}{1 - x^2/j^2} = -\frac{2}{j^2} \sum_{n=0}^{\infty} \left(\frac{x^2}{j^2} \right)^n \quad \text{for small enough } x$$

or

$$f_j(x) = \frac{2x}{x^2 - j^2} = \sum_{n=0}^{\infty} -\frac{2}{j^{2n+2}} x^{2n+1}.$$

The sequence of Bernoulli numbers starts as follows:

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$,

$B_6 = \frac{1}{42}$, $B_7 = 0$, $B_8 = -\frac{1}{30}$, $B_9 = 0$, $B_{10} = \frac{5}{66}$, $B_{11} = 0$,

$B_{12} = -\frac{691}{2730}$, $B_{13} = 0$, $B_{14} = \frac{7}{6}$, ...

Comments i) They are all elements of the set of rational numbers.

ii) $B_3 = B_5 = B_7 = \cdots = 0$.

iii) $B_2, B_4, B_6, \ldots$ are all nonzero and they alternate in sign.

iv) The denominators are well understood in terms of a theorem due to T. Clausen and C. von Staudt. The numerators are far less well understood.

We will also later need the Bernoulli polynomials which may be defined as follows:

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}$$

The first few are: $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$,

$B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$,

$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$,

$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$,

$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$

Note that $B_n(0) = B_n \quad \forall n \in \mathbb{Z}^{\geq 0}$ and we also have

$$B_n(0) = B_n(1) \quad \forall n \in \mathbb{Z}^{\geq 2}.$$

The Laurent series expansion of $\pi \cot(\pi x)$ about $x = 0$ is

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j}}{(2j)!} B_{2j} \pi^{2j} x^{2j-1} \quad \text{for } 0 < |x| < 1$$

where the B_{2j}'s are the even index (nonzero) Bernoulli numbers.

The partial fraction decomposition of $\pi \cot(\pi x)$ about $x = 0$ is

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{j=1}^{\infty} \frac{2x}{x^2 - j^2} \quad \text{for } 0 < |x| < 1.$$

$$= \frac{1}{x} + \left[-\frac{2}{1^2} x - \frac{2}{1^4} x^3 - \frac{2}{1^6} x^5 - \frac{2}{1^8} x^7 - \cdots \right]$$

$$+ \left[-\frac{2}{2^2} x - \frac{2}{2^4} x^3 - \frac{2}{2^6} x^5 - \frac{2}{2^8} x^7 - \cdots \right]$$

$$+ \left[-\frac{2}{3^2} x - \frac{2}{3^4} x^3 - \frac{2}{3^6} x^5 - \frac{2}{3^8} x^7 - \cdots \right]$$

$$+ \\ \vdots$$

Weierstrass Double Series Theorem

$$\pi \cot(\pi x) = \frac{1}{x} - 2\zeta(2) x - 2\zeta(4) x^3 - 2\zeta(6) x^5 - 2\zeta(8) x^7 - \cdots$$

Comparing with the Laurent series expansion gives

$$\zeta(2j) = \frac{(-1)^{j-1} 2^{2j-1}}{(2j)!} \cdot \pi^{2j} \cdot B_{2j} \quad \text{for } j = 1, 2, 3, \ldots$$

$j = 1$: $\zeta(2) = \dfrac{\pi^2}{6}$ since $B_2 = \dfrac{1}{6}$

$j = 2$: $\zeta(4) = \dfrac{(-1) \cdot 2^3}{4!} \cdot \pi^4 \cdot \left(-\dfrac{1}{30}\right) = \dfrac{\pi^4}{90}$ since $B_4 = -\dfrac{1}{30}$

Note that we learn nothing about $\zeta(3), \zeta(5), \zeta(7), \ldots$!

Special Values

We now consider the special values of $\zeta(s)$ at $s = 0, -1, -2, \ldots$

Theorem 1: $\zeta(0) = -\frac{1}{2}$

and in general, $\zeta(1-n) = -\frac{B_n}{n} \quad \forall n \in \mathbb{Z}^{\geq 2}$

Compare to Euler's formula!! This one is much cleaner.

Since $B_3 = B_5 = B_7 = \cdots = 0$, we obtain $\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0$, the so-called "trivial zeros" of $\zeta(s)$. Perhaps this is why we know so little about the values $\zeta(3), \zeta(5), \ldots$?!

For each prime $p \in \mathbb{Z}^{\geq 2}$, $\exists$ a p-adic zeta function $\zeta_p(s)$ defined as a continuous function for $s \in \mathbb{Z}_p \setminus \{1\}$ which interpolates the values of the function

$$(1 - p^{-s})\zeta(s) = \prod_{q \neq p} \frac{1}{1 - q^{-s}}$$

at the negative integers as follows:

$$\zeta_p(1-n) = (1 - p^{n-1})\zeta(1-n) = (p^{n-1} - 1)\frac{B_n}{n} \in \mathbb{Q}$$

$\forall n \in \mathbb{Z}^{\geq 2}$. Note that the negative integers are dense in $\mathbb{Z}_p$! The fact that this interpolation can be carried out in a continuous manner is due to a system of congruences for Bernoulli numbers we owe to Kummer. These congruences were seen as an isolated curiosity until they we understood in the above way 110 years later!

Special Values

We now consider the special values of any given Dirichlet L-function $L(s, \chi)$ at $s = 0, -1, -2, \ldots$ Once we define "generalized Bernoulli numbers", the formula is essentially the same as what we just recorded for $\zeta(s)$.

Definition: Given a Dirichlet character χ of conductor f_χ and a fixed integer $n \geq 1$, we define the generalized Bernoulli number $B_{n,\chi}$ by

$$B_{n,\chi} = f_\chi^{\,n-1} \sum_{a=1}^{f_\chi} \chi(a) B_n\left(\frac{a}{f_\chi}\right), \quad \text{where } B_n(X)$$

is the nth Bernoulli polynomial.

Theorem 2: $L(1-n, \chi) = -\dfrac{B_{n,\chi}}{n} \quad \forall n \in \mathbb{Z}^{\geq 1}$.

Comments: i) $B_{1,\chi_0} = B_1(1) = 1 - \frac{1}{2} = \frac{1}{2}$, so the correct formula $\zeta(0) = -\frac{1}{2}$ is given at $s = 0$ for $n = 1$ unlike in Theorem 1.

ii) For $\chi \neq \chi_0$, we obtain

$$B_{1,\chi} = \frac{1}{f_\chi} \sum_{a=1}^{f_\chi} \chi(a) a$$

and $L(0, \chi) = -B_{1,\chi}$. We note for future reference that if $\chi \neq \chi_0$ is even, then $L(0,\chi) = 0$, and if odd, then $L(0,\chi) \neq 0$.

iii) We may define p-adic L-functions as well based upon generalized Kummer congruences for generalized Bernoulli numbers!

Special Values

In order to clinch his proof, Dirichlet needed to show exactly this:
$$\boxed{L(1,\chi) \neq 0 \quad \text{for } \chi \neq \chi_0}$$

One class of characters was particularly troublesome, namely, those which take on only the values 0, 1, and −1, i.e. the quadratic characters: $\chi \neq \chi_0$, but $\chi^2 = \chi_0$. The Kronecker symbol is the classic example of a quadratic Dirichlet character.

Kronecker symbol: Let d be the discriminant of a quadratic number field. For example, $d = 40$ is the discriminant of $\mathbb{Q}(\sqrt{10}) = \{a + b\sqrt{10} \mid a, b \in \mathbb{Q}\}$.

Define: i) $\chi_d(1) = 1$

ii) p an odd prime: $\chi_d(p) = \left(\frac{d}{p}\right) \leftarrow$ Legendre symbol $= 0$ if $p \mid d$.

iii)
$$\chi_d(2) = \begin{cases} 0 & \text{if } d \text{ is even} \\ 1 & \text{if } d \equiv 1 \pmod 8 \\ -1 & \text{if } d \equiv 5 \pmod 8. \end{cases} \qquad \chi_{40}(2) = 0$$

iv) For every other $n \in \mathbb{Z}^+$ use Fund. Thm. of Arith. and extend multiplicatively, i.e. $\chi_d(6) = \chi_d(2)\chi_d(3)$.

Using the Law of Quadratic Reciprocity, we may prove that χ_d is a Dirichlet character of conductor $= |d|$ and that
$$\chi_d \text{ is } \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix} \text{ if } \begin{Bmatrix} d > 0 \\ d < 0 \end{Bmatrix}.$$

Conversely, we may prove that if χ is a (primitive) quadratic Dirichlet character, then $\chi = \chi_d$, where χ_d is the Kronecker symbol attached to a quadratic field F whose discriminant d_F is equal to d (when I say χ_d is "attached" to F, I mean that χ_d tells us precisely how the rational primes factor in $\mathcal{O}_F$).

Special Values

In order to prove that $L(1,\chi_d) \neq 0$ for quadratic χ_d Dirichlet related $L(s,\chi_d)$ to the Dedekind zeta-function $\zeta_F(s)$ of the corresponding quadratic field F with $d_F = d$.

Definition:

$$\zeta_F(s) = \sum_{\substack{\text{all nonzero}\\ \text{ideals } A \subseteq \mathcal{O}_F}} \frac{1}{NA^s}, \quad Re(s) > 1, \quad \text{where } NA = [\mathcal{O}_F : A].$$

Note: $\zeta_{\mathbb{Q}}(s) = \zeta(s)$

Theorem 3: $\zeta_F(s)$ has an Euler product rep. in terms of prime ideals, a meromorphic continuation to $\mathbb{C} \setminus \{1\}$, a functional equation of standard type, etc...
Of particular interest is the fact that $\zeta_F(s)$ has a simple pole at $s=1$ and the residue involves all of the basic invariants of F (class number h_F, d_F, regulator R_F, ...)!!

Theorem 4: If F is quadratic and χ_d is the corresponding Kronecker symbol, then

$$\zeta_F(s) = \zeta(s) L(s,\chi_d)$$

At $s=1$:

$$\left(\frac{res(F)}{s-1} + a_0 + \cdots\right) = \left(\frac{1}{s-1} + \gamma + \cdots\right)\left(L(1,\chi_d) + L'(1,\chi)(s-1) + \cdots\right)$$

If $L(1,\chi_d) = 0$, then $\zeta_F(s)$ would have a removable singularity at $s=1$. Contradiction!

Not only is $L(1,\chi_d) \neq 0$, but $L(1,\chi_d) = res(F)$, which gives a direct connection between $L(1,\chi_d)$ and the invariants of the field F! Based upon this connection, Dirichlet was able to derive his famous class number formula for quadratic fields.

Special Values

Dirichlet's formula is cleaner at $s=0$, so we write it at $s=0$ instead of at $s=1$.

Complex quadratic: $d<0$, χ_d is odd, $L(0,\chi_d)\neq 0$.
We assume $d<-4$ to avoid the Gaussian and Eisenstein fields.

$$\zeta_F(0) = -\frac{h_F}{2}, \quad \text{where } d = d_F.$$

Note: This generalizes the special value $\zeta(0) = -\frac{1}{2}$.
We have (recall that $B_{1,\chi} = \frac{1}{f_\chi}\sum_{a=1}^{f_\chi}\chi(a)a$)

$$-\frac{h_F}{2} = \zeta_F(0) = \zeta(0)L(0,\chi_d) = -\frac{1}{2}\cdot\left(-B_{1,\chi_d}\right)$$

or $h_F = -B_{1,\chi_d}$. Landau writes this as follows:

$$\boxed{h(d) = \frac{1}{|d|}\left(\Sigma t - \Sigma r\right)},$$

where r runs through the numbers in the interval $1\le r<|d|$ for which $\chi_d(r)=1$ and t through those numbers in the same interval for which $\chi_d(t)=-1$.

Real quadratic: $d>0$, χ_d is even, $L(0,\chi_d)=0$.

$$\zeta_F(0) = 0, \quad \text{but } \zeta_F'(0) = -\frac{h_F R_F}{2}, \quad \text{where } R_F = \log(\varepsilon_F)$$

and $\varepsilon_F > 1$ is the fundamental unit of $\mathcal{O}_F^\times$. We have

$$L'(0,\chi_d) = \frac{1}{2}\log\left(\frac{\prod_t \sin\left(\frac{\pi t}{d}\right)}{\prod_r \sin\left(\frac{\pi r}{d}\right)}\right)$$

with t and r having the same meaning as above.

Since $\zeta_F'(0) = \zeta(0)L'(0,\chi_d)$, we deduce that

$$\boxed{\varepsilon_F^{2h(d)} = \frac{\prod_t \sin\left(\frac{\pi t}{d}\right)}{\prod_r \sin\left(\frac{\pi r}{d}\right)}}$$