Distribution of the Zeros of the Derivatives of the Riemann Zeta Function

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Outline

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The Riemann zeta function

Definition

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{where } s = \sigma + it \text{ with } \sigma > 1$$

Functional equation for $\zeta(s)$ where $s \in \mathbb{C} \setminus \{1\}$

$$\zeta(1-s) = 2\Gamma(s)\zeta(s)(2\pi)^{-s}\cos\frac{\pi s}{2}$$

$\zeta(s)$ has a simple pole at $s = 1$

Trivial zeros

$\zeta(-2j) = 0$ for $j \in \mathbb{N}$

The Riemann hypothesis and the derivatives of ζ

Prime number theorem

All non-trivial zeros of ζ are in the critical strip $0 < \sigma < 1$.

Riemann hypothesis

All non-trivial zeros of ζ are of the form $\frac{1}{2} + it$.

Speiser 1934

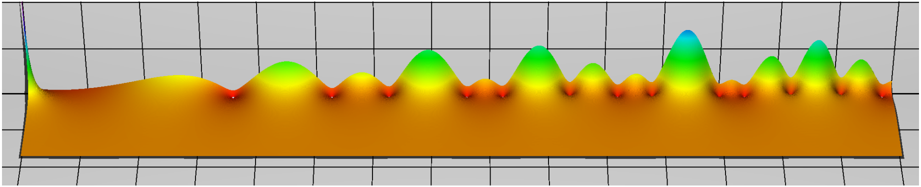
Riemann hypothesis $\iff \zeta'(\sigma + it)$ has no zeros for $0 < \sigma < \frac{1}{2}$

Yildirim 1996

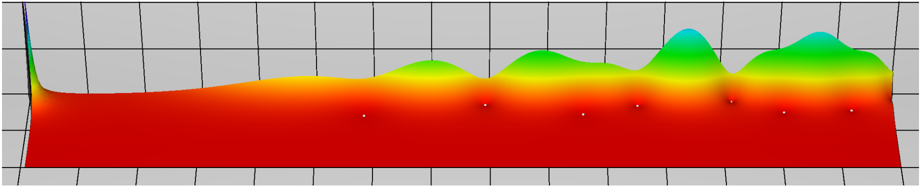
The Riemann hypothesis implies

- ζ'' has no zeros in the strip $0 \leq \sigma < \frac{1}{2}$
- ζ''' has no zeros in the strip $0 \leq \sigma < \frac{1}{2}$

Plots of $|\zeta|$ and $|\zeta'|$

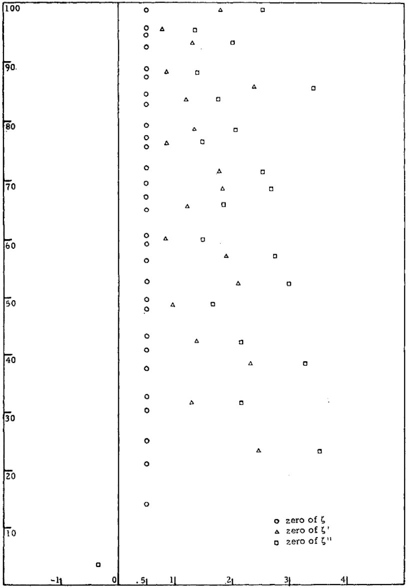


$|\zeta(\sigma + it)|$ for $0 \leq \sigma \leq 8$ and $0.1 \leq t \leq 60$

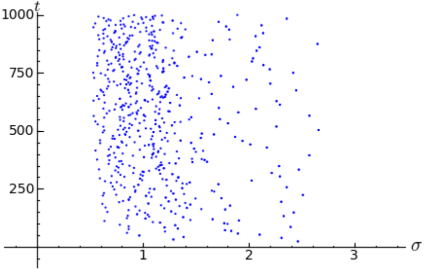


$|\zeta'(\sigma + it)|$ for $0 \leq \sigma \leq 8$ and $0.2 \leq t \leq 60$

Zeros of ζ, ζ', and ζ'' (Spira 1965)



Zeros of ζ', $\zeta^{(34)}$,$\zeta^{(67)}$, $\zeta^{(100)}$



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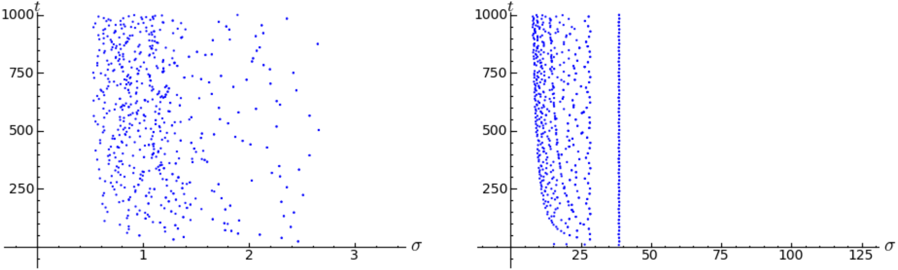

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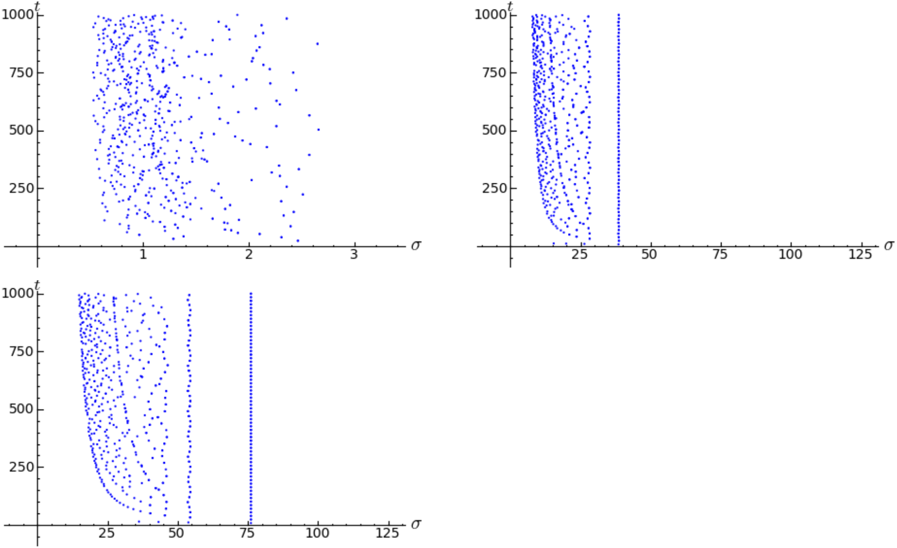

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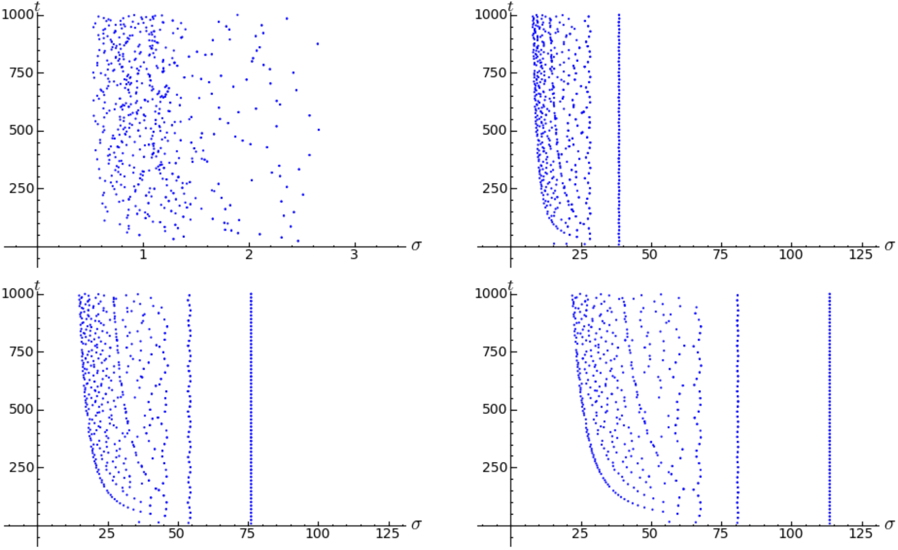


Outline

Known zero-free regions

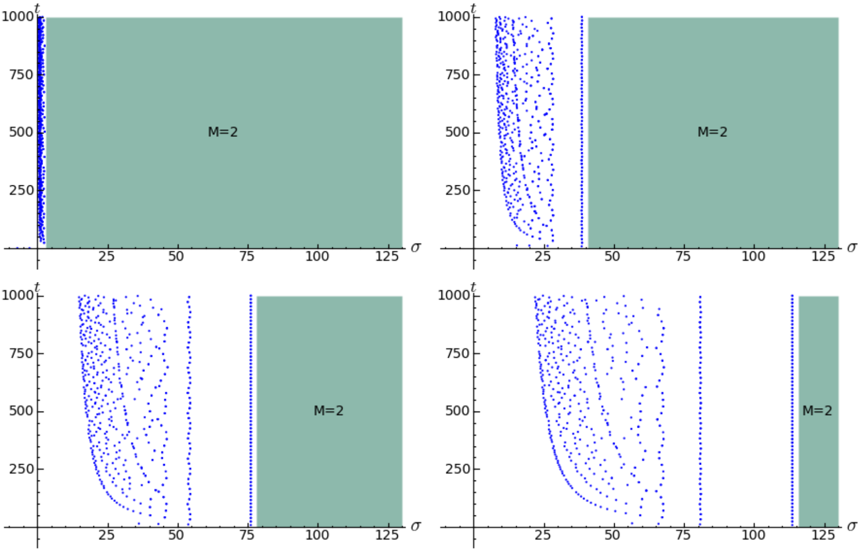
If $\sigma \geq \dots$, then the function has no zero for $s = \sigma + it$ for all $t \in \mathbb{R}$.

	ζ	ζ'	ζ''	$\zeta^{(k)}$ for $k \geq 3$
Hadamard and de la Vallée-Poussin 1896	1			
Spira 1965				$\frac{7}{4}k + 2$
Verma and Kaur 1982				$1.13588k + 2$
Skorokhodov 2003		2.93938	4.02853	

Note that

$$q_2 := \frac{\log \frac{\log 2}{\log 3}}{\log \frac{2}{3}} = 1.13588\dots$$

Zeros of ζ', $\zeta^{(34)}$, $\zeta^{(67)}$, $\zeta^{(100)}$



Finding zero free regions

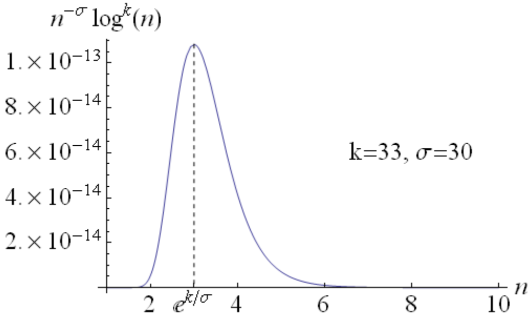
We find $s = \sigma + it$ such that

$$\begin{aligned} \left|\zeta^{(k)}(s)\right| &= \left|\sum_{n=2}^{\infty} \frac{\log^k n}{n^s}\right| \\ &\geq \left|\frac{\log^k N}{N^s}\right| - \left|\sum_{n=2}^{N-1} \frac{\log^k n}{n^s} + \sum_{n=N+1}^{\infty} \frac{\log^k n}{n^s}\right| \\ &= \frac{\log^k N}{N^\sigma} - \sum_{n=2}^{N-1} \frac{\log^k n}{n^\sigma} - \sum_{n=N+1}^{\infty} \frac{\log^k n}{n^\sigma} > 0. \end{aligned}$$

That is, for $N \in \mathbb{N}^{>1}$ we find the regions in $\mathbb{C}$ where $\zeta^{(k)}(s)$ is dominated by $\frac{\log^k N}{N^s}$.

Dominant Term

$$\frac{\log^k N}{N^s} \text{ can dominate } \zeta^{(k)}(s) = \sum_{n=2}^{\infty} \frac{\log^k n}{n^s} \text{ if } N \approx e^{\frac{k}{\sigma}}.$$



q_N

There is no dominant term if

$$\frac{\log^k N}{N^\sigma} = \left|\frac{\log^k N}{N^s}\right| = \left|\frac{\log^k (N+1)}{(N+1)^s}\right| = \frac{\log^k (N+1)}{(N+1)^\sigma}.$$

This is the case when $\sigma = k \cdot q_N$ where

$$q_N = \frac{\log \frac{\log(N+1)}{\log N}}{\log \frac{N+1}{N}}.$$

In particular

$$q_2 \approx 1.13588, \quad q_3 \approx 0.808484, \quad q_4 \approx 0.668855.$$

The head $H_M^k(\sigma)$ and the tail $T_M^k(\sigma)$

Let

$$H_M^k(s) := \sum_{n=2}^{M-1} Q_n^k(s) = \sum_{n=2}^{M-1} \frac{\log^k n}{n^s}$$

and

$$T_M^k(s) := \sum_{n=M+1}^{\infty} Q_n^k(s) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^s}.$$

Our goal will be to show that

$$|\zeta^{(k)}(s)| \geq Q_M^k(\sigma) - H_M^k(\sigma) - T_M^k(\sigma) = Q_M^k(\sigma)\left(1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{T_M^k}{Q_M^k}(\sigma)\right) > 0$$

The head $H_M^k(\sigma)$

$$H_M^k(\sigma) = \sum_{n=2}^{M-1} \frac{\log^k n}{n^\sigma} = \sum_{n=2}^{M-1} Q_n^k(\sigma) = Q_M^k(\sigma)\left(\frac{Q_{M-1}^k}{Q_M^k}(\sigma) + \cdots + \frac{Q_2^k}{Q_M^k}(\sigma)\right)$$

$$= Q_M^k(\sigma)\left(\frac{Q_{M-1}^k}{Q_M^k}(\sigma)\left(1+\frac{Q_{M-2}^k}{Q_{M-1}^k}(\sigma)\left(1+\ldots\left(1+\frac{Q_2^k}{Q_3^k}(\sigma)\right)\ldots\right)\right)\right)$$

For $2 \leq n \leq M$ and $\sigma \leq q_{M-1}k - cM$ where $c \in \mathbb{R}^{>0}$ a solution of $1 - \frac{1}{e^c-1} - \frac{1}{e^c}(1+\frac{1}{c}) \geq 0$ we have

$$\frac{Q_{n-1}^k}{Q_n^k}(\sigma) \leq \left(\frac{n}{n-1}\right)^{-cM} \leq \left(\frac{M}{M-1}\right)^{-cM} \leq \frac{1}{e^c}.$$

It follows that

$$\frac{H_M^k}{Q_M^k}(\sigma) \leq \sum_{n=1}^{\infty} \frac{1}{(e^c)^n} = \frac{1}{1-\frac{1}{e^c}} - 1 = \frac{1}{e^c-1}.$$

The tail $T_M^k(\sigma)$

For $\sigma \geq q_M k + c(M+1)$ we have

$$T_M^k(\sigma) = \sum_{n=M+1}^{\infty} Q_n^j(\sigma) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^\sigma} \leq \int_M^\infty \frac{\log^k x}{x^\sigma} dx$$
$$< \frac{\log^k M}{M^\sigma} \frac{M}{\sigma - 1} \left(1 + \frac{k}{(\sigma-1)\log M - k + 1}\right)$$

With $k \geq k_M = \frac{(2M+1)c}{q_{M-1} - q_m}$ this gives

$$R_{M+1}^k(\sigma) \leq R_{M+1}^{k_M}(q_M k_M + c(M+1)) < \frac{1}{c}.$$

The head $H_M^k(\sigma)$ and the tail $T_M^k(\sigma)$

Now

$$
\begin{aligned}
|\zeta^{(k)}(s)| &\geq Q_M^k(\sigma) - H_M^k(\sigma) - T_M^k(\sigma) \\
&= Q_M^k(\sigma)\left(1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{T_M^k}{Q_M^k}(\sigma)\right) \\
&> Q_M^k(\sigma)\left(1 - \frac{1}{e^c - 1} - \frac{1}{e^c}\left(1 + \frac{1}{c}\right)\right) \\
&\geq 0,
\end{aligned}
$$

Zero Free Regions

Let $k \in \mathbb{N}$ and $c \in \mathbb{R}^{>0}$ a solution of $1 - \frac{1}{e^c-1} - \frac{1}{e^c}(1 + \frac{1}{c}) \geq 0$.

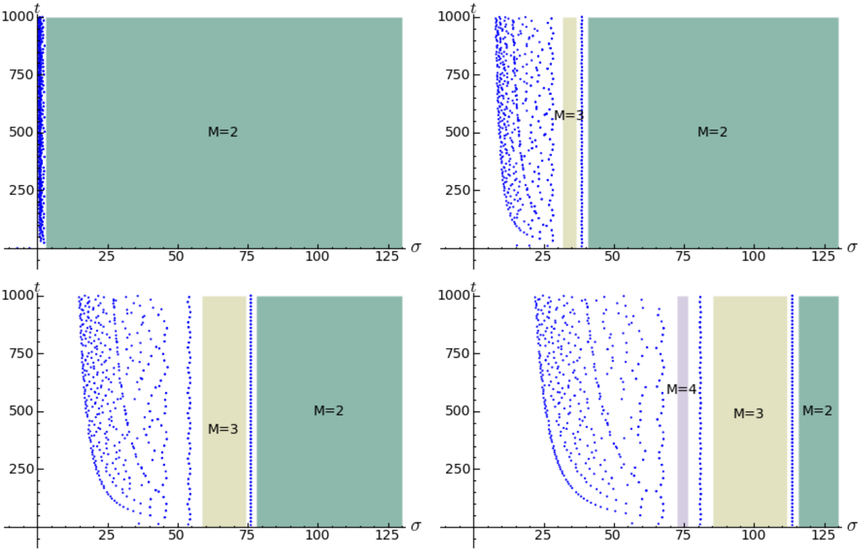
If $M \in \mathbb{N}$, $M > 3$ and

$$q_M k + (M+1)c \leq q_{M-1} k - Mc$$

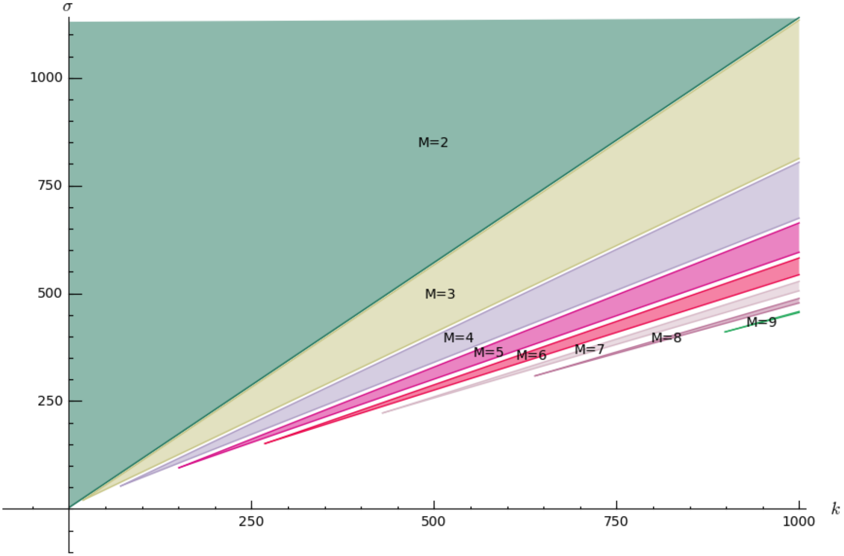
then $\zeta^{(k)}(s) \neq 0$ for

$$q_M k + (M+1)c \leq \sigma \leq q_{M-1} k - Mc.$$

Zeros of ζ', $\zeta^{(34)}$, $\zeta^{(67)}$, $\zeta^{(100)}$

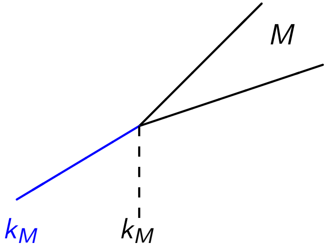


Zero-free regions of $\zeta^{(k)}$ in the k-σ-plane



Extending the wedges

The tips of the wedges are at $k_M = \frac{1}{2}\left((q_M + q_{M-1})k + c\right)$



M	3	4	5	6	7	8	9	10
k_M at tip of wedge	20	77	163	291	465	691	971	1313
k_M at tip of line	19	58	123	220	354	529	748	1014

Outline

Vertical distribution of non-real zeros

Riemann, van Mangoldt 1905

The number of zeros of $\zeta(\sigma + it)$ with $0 < t < T$ is

$$N(T) = T\frac{\log T - 1 - \log 2\pi}{2\pi} + O(\log T)$$

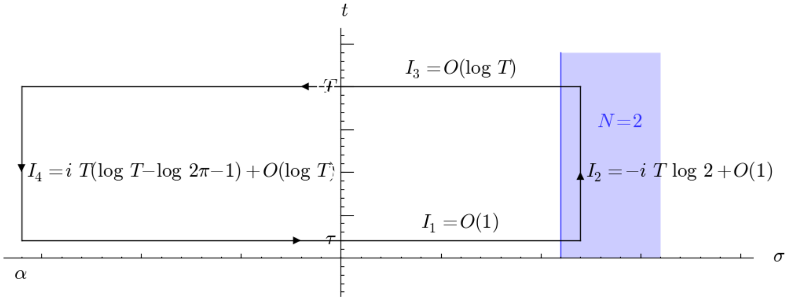
Berndt 1970

The number of zeros of $\zeta^{(k)}(\sigma + it)$ with $0 < t < T$ is

$$N^k(T) = T\frac{\log T - 1 - \log 4\pi}{2\pi} + O(\log T)$$

Berndt's proof

There is $\alpha \in \mathbb{R}$ such that $\zeta^{(k)}(\sigma + it) \neq 0$ for $\sigma < \alpha$. (Spira 1970).
Let $\tau > 0$ such that $\zeta^{(k)}(\sigma + it) \neq 0$ for $0 < t < \tau$.



The number of zeros of $\zeta^{(k)}(\sigma + it)$ with $0 < t < T$ is

$$N^k(T) = \frac{1}{2\pi i}\int_C \frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)}ds = \frac{I_1 + I_2 + I_3 + I_4}{2\pi i} = T\,\frac{\log T - 1 - \log 4\pi}{2\pi} + O(\log T)$$

Some zero-free points

If $\frac{\log^k N}{N^s}$ and $\frac{\log^k (N+1)}{(N+1)^s}$ dominate $\zeta^{(k)}$ and

$$\frac{\log^k N}{N^s} = \frac{\log^k (N+1)}{(N+1)^s}$$

then $\zeta^{(k)}(s) \neq 0$.

Absolute value: $\frac{\log^k N}{N^\sigma} = \frac{\log^k (N+1)}{(N+1)^\sigma}$,

hence $\sigma = k \cdot q_N$

Real part: $\cos(t \cdot \log N) = \cos(t \cdot \log(N+1))$

Imaginary part: $\sin(t \cdot \log N) = \sin(t \cdot \log(N+1))$

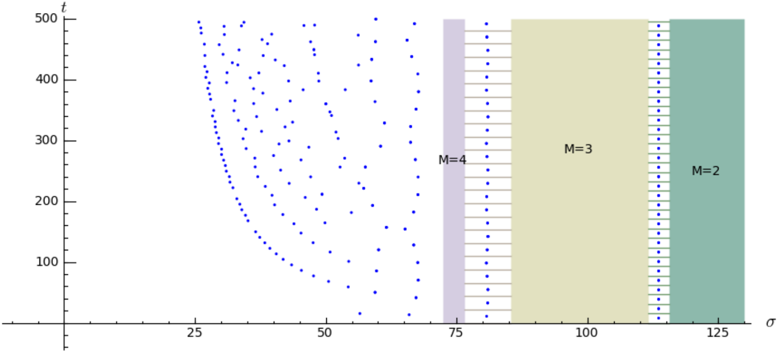
hence $t = \frac{2m\pi}{\log(N+1) - \log(N)}$ for $m \in \mathbb{Z}$

Zero Free Horizontal Line Segements

Lemma

If $q_M k + (M+1)\log 3 \leq \sigma \leq q_{M-1} k - M \log 3$, then $\zeta^{(k)}(s) \neq 0$ for

$$s = \sigma + i \cdot \frac{2\pi j}{\log(M+1) - \log M}.$$



Locations of zeros of $\zeta^{(k)}$

If $\frac{\log^k N}{N^s}$ and $\frac{\log^k (N+1)}{(N+1)^s}$ dominate $\zeta^{(k)}$ and

$$\frac{\log^k N}{N^s} = -\frac{\log^k (N+1)}{(N+1)^s}$$

there might be a zero of $\zeta^{(k)}$ close to s.

Absolute value: $$\frac{\log^k N}{N^\sigma} = \frac{\log^k (N+1)}{(N+1)^\sigma},$$

hence $\sigma = k \cdot q_N$

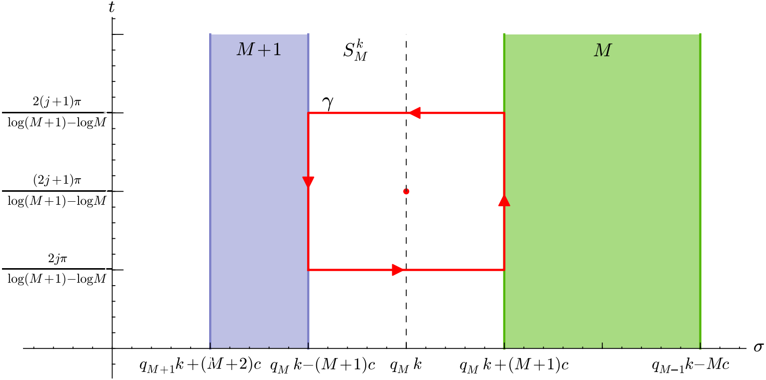
Real part: $\cos(t \cdot \log N) = -\cos(t \cdot \log(N+1))$

Imaginary part: $\sin(t \cdot \log N) = -\sin(t \cdot \log(N+1))$

$$\text{hence } t = \frac{(2m+1)\pi}{\log(N+1) - \log(N)} \text{ for } m \in \mathbb{Z}$$

Locations of zeros of $\zeta^{(k)}$

The point $\color{red}\bullet$ is the only zero of $\frac{\log^k M}{M^s} + \frac{\log^k M+1}{M+1^s}$ inside the curve γ $\color{red}\square$.



We have $\left|\frac{\log^k M}{M^s} + \frac{\log^k M+1}{M+1^s} - \zeta^{(k)}(s)\right| \le \left|\frac{\log^k M}{M^s} + \frac{\log^k M+1}{M+1^s}\right|$.

By Rouché's Theorem $\zeta^{(k)}(s)$ has exactly one simple zero inside γ.

Number of Zeros Between Zero Free Regions

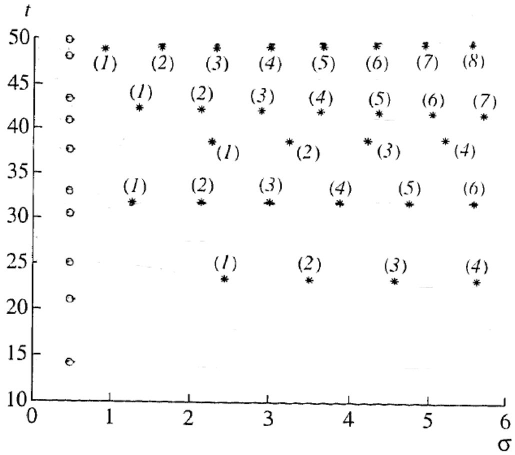
Corollary

Let $N_M^k(T)$ denote the number of zeros ρ of $\zeta^{(k)}(s)$ with $\Im(\rho) \leq T$ and $q_M k + (M+1)\log 3 \leq \Re(\rho) \leq q_{M-1}k - M\log 3$. Then, for all $j \geq 1$,

$$N_M^k\left(\frac{2\pi j}{\log(M+1)-\log(M)}\right) = j.$$

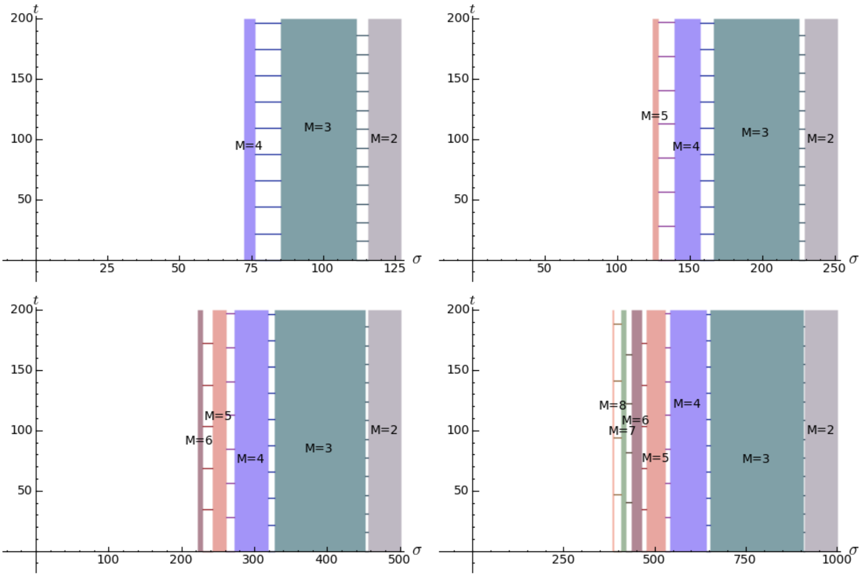
Outline

Zeros of derivatives of ζ (Skorokhodov 2003)



$\circ$	zero of ζ
$*^{(n)}$	zero of $\zeta^{(n)}$

Zero Free Regions for $\zeta^{(100)}$, $\zeta^{(200)}$, $\zeta^{(400)}$, $\zeta^{(800)}$



Chains for large k

For $M \in \mathbb{N}$, $M \geq 2$ there is $K \in \mathbb{N}$ such that

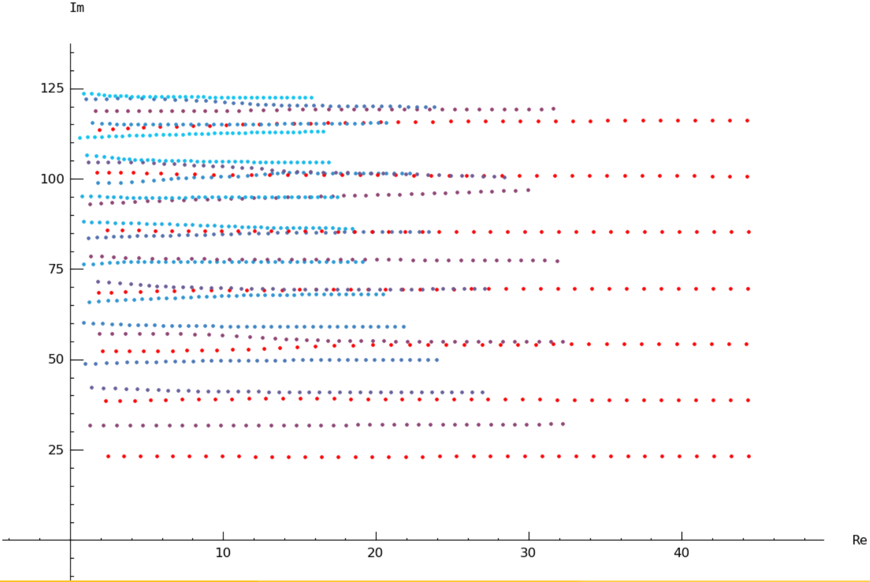
$$q_{M+1}k + (M+2)c \leq q_M k - (M+1)c \text{ for all } k \geq K.$$

For each $k \geq K$ and each $j \in \mathbb{Z}$ there is exactly one zero in a rectangular region given by M, k, and j.

There exists a unique corresponding zero of $\zeta^{(k+1)}(s)$ in the rectangular region given by M, $k+1$, and j.
Thus there is a chain of zeros

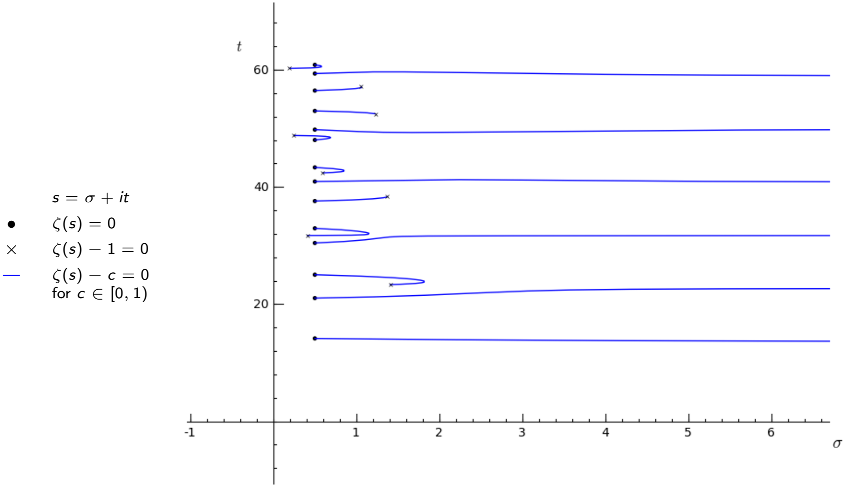
$$\zeta^{(K)}(s), \zeta^{(K+1)}(s), \zeta^{(K+2)}(s), \ldots.$$

Zeros of the 1^{st} to 40^{th} derivatives of ζ



Outline

The curves $s(c)$ given by $\zeta(s(c)) - c = 0$ for $c \in [0, 1)$

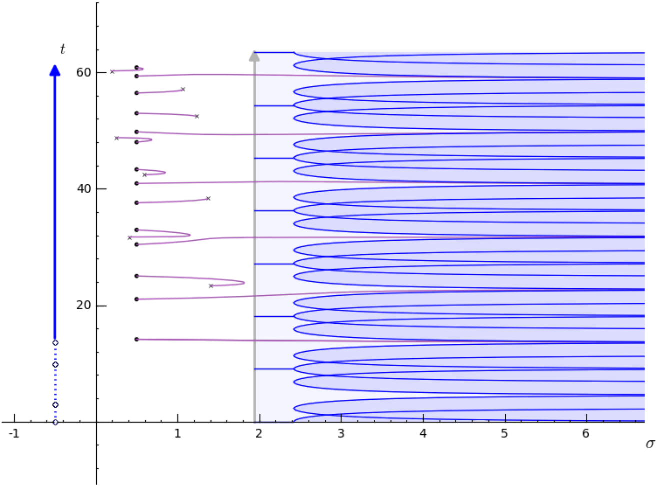


The horizontal asymptotes are $t = \frac{(2m+1)\pi}{\log 2}$ for $m \in \mathbb{N}$.

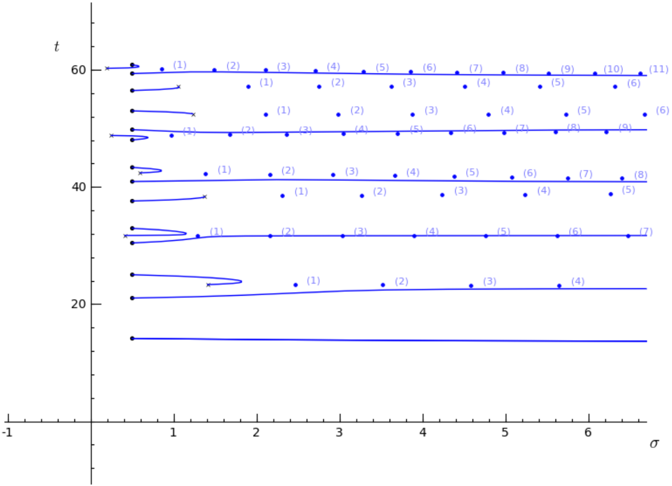
Zero free regions for $\zeta(s) - c$

$s = \sigma + it$

- • $\zeta(s) = 0$
- × $\zeta(s) - 1 = 0$
- — $\zeta(s) - c = 0$ for $c \in [0, 1)$
- — $\zeta(s) - c \neq 0$ for $c \in [0, 1)$
- — $\zeta(s) - 1 \neq 0$



Zeros of ζ and zeros of $\zeta^{(k)}$



$s = \sigma + it$

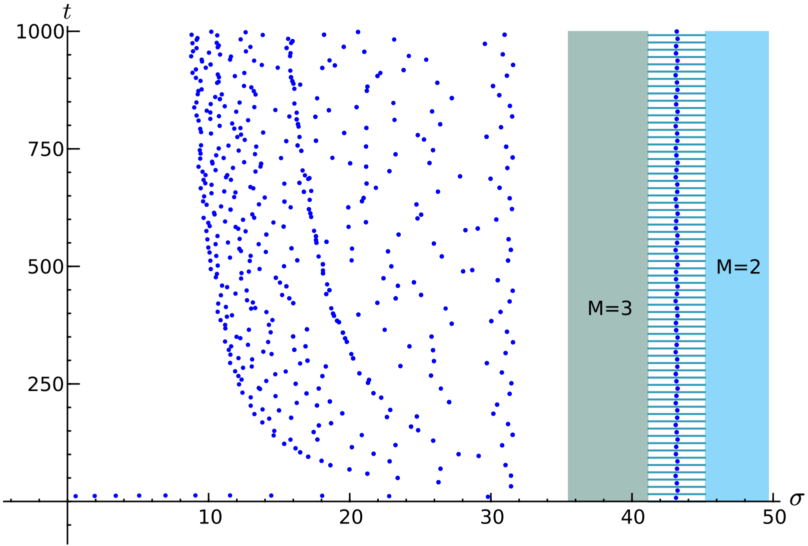
• $\zeta(s) = 0$

× $\zeta(s) - 1 = 0$

— $\zeta(s) - c = 0$ for $c \in [0, 1)$

$\bullet^{(k)}$ $\zeta^{(k)}(s) = 0$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $\zeta(s) - 1 = \sum_{n=2}^{\infty} \frac{1}{n^s}$ $\zeta^{(k)}(s) = \sum_{n=2}^{\infty} \frac{\log^k n}{n^s}$



Thank You.