Convergence of Dirichlet Series and Euler Products

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1 Introduction

These notes are based on lectures given by the author in 2014 at the University of Calgary and in 2015 at the University of N. Carolina Greensboro. The general theme is convergence, in Section 2 this is studied for Dirichlet series and in Sections 3-4 for Euler products. Section 5 gives some examples and concludes with a few questions.

2 Dirichlet Series

By a Dirichlet series we mean an infinite series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n n^{-s}, \quad a_n \in \mathbb{C}.$$

As usual we write $s = \sigma + it$ for the complex variable s. A very familiar example is the case $a_n = 1$, f is then the Riemann zeta function ζ. For $t = 0$, i.e. for $s = \sigma \in \mathbb{R}$, it is proved in elementary calculus that $\zeta(\sigma)$ diverges for $\sigma = 1$ and is absolutely convergent for $\sigma > 1$. This is called the "p-test" (where $p = \sigma$) but should really be called the "ζ-test." Another familiar example (again when $s = \sigma \in \mathbb{R}$) is the alternating zeta series $\eta(s) = \sum (-1)^{n-1} n^{-s}$, known as the Euler-Dedekind function. It is proved in elementary calculus that this series converges for $\sigma > 0$, where the convergence is conditional for $0 < \sigma \leq 1$ and absolute for $1 < \sigma$. In this section we shall prove that very similar results hold, with appropriate hypotheses on the coefficients a_n, for $s \in \mathbb{C}$, i.e. dropping the condition $t = 0$.

From elementary complex analysis, for any $x \in \mathbb{R}^+$, one has $|x^s| = x^\sigma$. In particular $|n^{-s}| = n^{-\sigma}$. Using this together with the zeta-test gives the next result immediately.

1.1 Proposition : If $|a_n|$ is bounded then the Dirichlet series $\sum a_n n^{-s}$ is absolutely convergent for $\sigma > 1$.

In particular this holds for $\zeta(s), \eta(s)$ and all L-functions $L(s, \chi)$ for any Dirichlet character χ, indeed $|a_n| = 1$ for these functions.

1.2 Examples : The mod 3 character χ_2^3 is defined by $\chi(n) = 0, 1, -1$ for n congruent respectively to $0, 1, 2$ modulo 3. One has $L(s, \chi_2^3) = 1 - 2^{-s} + 4^{-s} - 5^{-s} + 7^{-s} - 8^{-s} \ldots$. By the Leibniz alternating series test we see that both $\eta(s), L(s, \chi_2^3)$ converge (conditionally) along the real line $s = \sigma$ for $0 < \sigma \leq 1$. The first objective of this section is to show that this remains true for all t, i.e. Dirichlet series such as in these two examples are convergent for $\sigma > 0$, for all t. The treatment is very close to that of [4].

1.3 Lemma : Let $\alpha, \beta, \sigma \in \mathbb{R},\ 0 < \sigma,\ 0 < \alpha < \beta$. Then

$$|e^{-\alpha s} - e^{-\beta s}| \leq \frac{|s|}{\sigma}(e^{-\alpha\sigma} - e^{-\beta\sigma}) \ .$$

Proof: We have $\ e^{-\alpha s} - e^{-\beta s} = s \int_{\alpha}^{\beta} e^{-us} du,\ $ hence

$$|e^{-\alpha s} - e^{-\beta s}| \leq |s| \int_{\alpha}^{\beta} |e^{-us}| du = |s| \int_{\alpha}^{\beta} e^{-u\sigma} d\sigma = \frac{|s|}{\sigma}(e^{-\alpha\sigma} - e^{-\beta\sigma}) \ . \ \square$$

1.4 Corollary : Set $\alpha = \log(m),\ \beta = \log(n)\ ,\ 0 < m < n,\ \sigma > 0$, then

$$|m^{-s} - n^{-s}| \leq \frac{|s|}{\sigma}(m^{-\sigma} - n^{-\sigma}) \ .$$

1.5 Lemma (Abel's summation formula) : Let $a_k, b_k \in \mathbb{C},\ n \geq 1,$ and set $A_n = a_1 + \ldots + a_n$. Then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k) \ .$$

Proof: Let $A_0 = 0$. Then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1}$$

which is the same as the right hand side of Abel's formula. $\square$

1.6 Corollary : The sum $\sum_{k=1}^{\infty} a_k b_k$ converges if both $\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k)$ and $\{A_n b_{n+1}\}$ are convergent.

We remark that Abel's summation formula can be thought of as a discrete version of the familiar integration by parts formula from calculus, this should be clear by writing them side by side as

$$\sum_{k=1}^{n} b_k a_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k) \ , \int u dv = vu - \int v du \ .$$

Before turning to the main theorem of this section, we recall some standard facts about convergence of an infinite series of complex numbers z_n. The partial sums are written $\ S_n := \sum_{k=1}^{n} z_k$, and one says that $\ \sum_{k=1}^{\infty} z_k = S$ if and only if $\ \lim_{n\to\infty} S_n\ $ exists and equals S, in this case the series is said to be convergent. A necessary condition for convergence is $\ z_n \to 0$. A necessary and sufficient condition, the Cauchy convergence criterion, is that for any given real number $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N,\ \ |S_n - S_m| < \varepsilon$. As before, $\ A_n = a_1 + \ldots + a_n$.

1.7 Theorem : Consider $\sum_{n=1}^{\infty} a_n n^{-s},\ a_n \in \mathbb{C}$. If $\{|A_n|\}$ is bounded then the series converges for $\sigma > 0$.

Proof: We have $|A_n| \leq C$, for some $C > 0$ and for all n. We shall use Corollary 1.6, with $a_n = a_n$ and $b_n = n^{-s}$. Then $|A_n b_{n+1}| = |A_n| \cdot |b_{n+1}| \leq C \cdot (n+1)^{-\sigma} \to 0$ as $n \to \infty$. Hence the second condition of Corollary 1.7, $\{A_n b_{n+1}\}$ converges (in this case to 0), is satisfied.

For the first condition, we apply the Cauchy convergence criterion to $\sum_{k=1}^{\infty} A_k((k+1)^{-s} - k^{-s})$. Given $\varepsilon > 0$ and using Corollary 1.4 we have

$$|S_n - S_m| = |\sum_{k=m+1}^{n} A_k((k+1)^{-s} - k^{-s})| \leq C \cdot \sum_{k=m+1}^{n} |(k+1)^{-s} - k^{-s}|$$

$$\leq \frac{C|s|}{\sigma} \sum_{k=m+1}^{n} (\frac{1}{k^\sigma} - \frac{1}{(k+1)^\sigma}) = \frac{C|s|}{\sigma}(\frac{1}{(m+1)^\sigma} - \frac{1}{(n+1)^\sigma}) \leq \frac{C|s|}{\sigma(m+1)^\sigma} < \varepsilon$$

for m sufficiently large. □

The first objective of this section is thus accomplished. We give a corollary. Recall that the trivial (also called principal) Dirichlet character modulo q is given by $\chi(n) = 0$ for all n such that $\gcd(q,n) > 1$, and $\chi(n) = 1$ when $\gcd(q,n) = 1$.

1.8 Corollary : For $\eta(s)$ or for $L(s,\chi)$ with χ any non-trivial Dirichlet character χ modulo q, the Dirichlet series converges for $\sigma > 0$.

Proof: For η, $A_n \in \{0,1\}$ is bounded. For any non-trivial character χ modulo q one has $A_q = \chi(1) + \ldots + \chi(q) = 0$ so $\{A_n\}$ is periodic modulo q, hence finite and bounded. □

We remark that Theorem 1.7 is proved in [4], but the proof is a little less direct than the one given above, and is restricted to the case $a_n \in \mathbb{R}$ (for no apparent reason).

The second objective of this section is to consider possible strengthening of the above results, in particular 1.3, 1.7, and their corollaries. It will be seen in Sections 4-5 that such strengthening could be very useful. First consider Corollary 1.4. Another obvious (second) upper bound is $|m^{-s} - n^{-s}| \leq |m^{-s}| + |n^{-s}| = m^{-\sigma} + n^{-\sigma}$. It can be seen that for each fixed values for m, n, σ there is a t_* such that the first upper bound (from 1.4) is better for $t < t_*$ whereas the second, which is simply a constant, is better for $t > t_*$. Indeed the second becomes better and better as t increases. Whether this can be used in some way to strengthen Theorem 1.7 is presently not known. It may also be possible to find a third upper bound that improves both the first and second, of course their minimum will be one such.

It is in fact possible to strengthen Theorem 1.7 using Corollary 1.4 as it stands, and the next theorem is an example.

1.9 Theorem : Consider $\sum_{n=1}^{\infty} a_n n^{-s}$, $a_n \in \mathbb{C}$. If there exists $C > 0$ such that $|A_n| < C \cdot \log(n)$, $n \geq 2$, then the series converges for $\sigma > 0$.

Proof: As in the proof of Theorem 1.7, the second convergence condition follows since $C \cdot \log(n) \cdot (n+1)^{-\sigma} \to 0$ as $n \to \infty$. For the first convergence condition, proceeding as in 1.7, we have

$$|S_n - S_m| = \sum_{k=m+1}^{n} |A_k| \cdot (k^{-s} - (k+1)^{-s}) .$$

Here $m \geq 1$, $k \geq 2$, hence from both the hypothesis and Corollary 1.4

$$\sum_{k=m+1}^{n} |A_k| \cdot (k^{-s} - (k+1)^{-s}) \leq \frac{C|s|}{\sigma} \sum_{k=m+1}^{n} \log(k) \cdot (k^{-\sigma} - (k+1)^{-\sigma}) \ .$$

For convenience write $C|s|/\sigma = K$ henceforth, then the last expression, after a small rearrangement of the terms, equals

$$K[\log(m+1)\cdot(m+1)^{-\sigma} + \sum_{k=m+1}^{n-1} (\log(k+1) - \log(k)) \cdot (k+1)^{-\sigma} - \log(n)\cdot(n+1)^{-\sigma})]$$

$$= K[\log(m+1)\cdot(m+1)^{-\sigma} - \log(n)\cdot(n+1)^{-\sigma} + \sum_{k=m+1}^{n-1} \log(1+\frac{1}{k})\cdot(k+1)^{-\sigma}] \ .$$

Next note that for $0 \leq u \leq 1$, $\log(1+u) = u - u^2/2 + u^3/3 + \ldots = u + \beta_u$, where $|\beta_u| \leq u^2/2$, as in the Leibniz convergence test for series with alternating signs (in fact this remains true for $0 \leq u$). The previous sum thus equals

$$K \cdot \left[\frac{\log(m+1)}{(m+1)^{\sigma}} - \frac{\log(n)}{(n+1)^{\sigma}} + \sum_{k=m+1}^{n-1} (\frac{1}{k} + \beta_k) \cdot (k+1)^{-\sigma} \right]$$

$$\leq K \cdot \left[\frac{\log(m+1)}{(m+1)^{\sigma}} + \sum_{k=m+1}^{\infty} k^{-1-\sigma} + \frac{1}{2} \sum_{k=m+1}^{\infty} k^{-2-\sigma} \right] \ .$$

For $\sigma > 0$ the two summations are absolutely convergent, so clearly taking m sufficiently large will guarantee that each of the three terms in the above formula will be smaller than $\varepsilon/(3K)$, completing the proof. □

Since the derivative of n^s is $\log(n) \cdot n^s$, we can use Theorem 1.9 to obtain a corollary similar to 1.8.

1.10 Corollary : For $f(s) = \eta(s)$ or for $f(s) = L(s, \chi)$ with χ any non-trivial Dirichlet character modulo q, the Dirichlet series for $f'(s)$ converges for $\sigma > 0$.

3 Convergence of Euler Products for $\sigma > .5$

In this section we simply present some numerical evidence for convergence of many Euler products in the half plane $\sigma > .5$. The Euler products for any Dirichlet L-function and the Riemann zeta function are well known to converge absolutely for $\sigma > 1$. We now give some numerical evidence here that for an L-function coming from a primitive character χ mod q, $q \geq 3$, the Euler product converges for $\sigma > .5$ and diverges for smaller σ. Three primitive characters are considered, χ_2^3 which takes values $0, 1, -1$ as n is respectively congruent to $0, 1, 2$ modulo 3, χ_2^4 which takes values $0, 1, 0, -1$ as n is respectively congruent to $0, 1, 2, 3$ modulo 4, and χ_2^5 which takes values $0, 1, i, -i, -1$ as n is respectively congruent to $0, 1, 2, 3, 4$ modulo 5. We consider $s = \sigma + 30i$, for $\sigma = .4$ (showing divergence) and for $\sigma = .6, .7, .9, 1.1$, which show stronger and stronger convergence as σ increases. Of course for

$\sigma = 1.1$ convergence is known and is absolute. We choose $t = 30$ as a fairly typical t value, similar results can be seen for other t values.

The tables show the s values, and $\Delta(10^k)$ is the absolute value of the error in computing the Euler product to 10^k terms.

Table 1. χ_2^3

s	$\Delta(10^2)$	$\Delta(10^3)$	$\Delta(10^4)$	$\Delta(10^5)$	$\Delta(10^6)$	$\Delta(10^7)$
$.4+30i$	2.63	.948	1.038	1.527	1.279	.92895
$.6+30i$	.3834	.1310	.1071	.0628	.03348	.02217
$.7+30i$	.1667	.04784	.03132	.01389	.005819	.003036
$.9+30i$	.03522	.00646	2.64×10^{-3}	7.015×10^{-4}	1.798×10^{-4}	5.655×10^{-5}
$1.1+30i$	.008014	.000889	2.28×10^{-4}	3.63×10^{-5}	$5.653\times10^{=6}$	1.056×10^{-6}

Table 2. χ_2^4

s	$\Delta(10^2)$	$\Delta(10^3)$	$\Delta(10^4)$	$\Delta(10^5)$	$\Delta(10^6)$	$\Delta(10^7)$
$,4+30i$	.5990	.9698	1.4292	1.8426	.57275	1.2107
$.6+30i$	.09977	.11541	.05695	.048676	.012220	.02360
$.7+30i$	.042703	.041387	.013952	.010361	.0019669	.0032676
$.9+30i$	.008350	.005571	.009151	.0005152	6.1523×10^{-5}	6.6160×10^{-5}
$1.1+30i$	.0017451	.0007744	6.6285×10^{-5}	2.6603×10^{-5}	2.1386×10^{-6}	1.377×10^{-6}

Table 3. χ_2^5

s	$\Delta(10^2)$	$\Delta(10^3)$	$\Delta(10^4)$	$\Delta(10^5)$	$\Delta(10^6)$	$\Delta(10^7)$
$.4+30i$	1.396	1.753	1.491	1.801	.7668	1.8641
$.6+30i$	.3098	.1627	.07835	.08715	.02307	.03472
$.7+30i$	.1455	.05550	.01905	.01846	.003667	.004391
$.9+30i$	.03254	.006930	1.197×10^{-3}	8.735×10^{-4}	9.447×10^{-5}	7.376×10^{-5}
$1.1+30i$	.007414	.0009013	8.134×10^{-5}	4.339×10^{-5}	2.549×10^{-6}	1.328×10^{-6}

4 Theory of Euler Product Convergence

We start this section with a brief discussion of infinite products and the related convergence issues, and conclude with a theorem that seems to give an approach to proving that Euler products of the type considered in Section 3 converge, for $\sigma > 1/2$. Intuitively one would say that $\Pi_{n=1}^{\infty} u_n$, $u_n \in \mathbb{C}$, converges when $\lim_{N\to\infty} \Pi_{n=1}^{N} u_n = L$ exists, and then define $\Pi_{n=1}^{\infty} u_n = L$.

But this has complications, especially if any $u_n = 0$. For the most general definition see Apostol's text [1], p. 207. This definition has quite a few cases and even a few surprises, e.g. if $u_n = 1/n$ we say $\Pi_{n=1}^{\infty} u_n$ *diverges* to 0. For our purposes it suffices to avoid these complications by using a subset of the Apostol definition and requiring :
(a) $u_n \neq 0$ for all n, and
(b) $\lim_{n\to\infty} u_n = 1$.

Then convergence of $\Pi_{n=1}^{\infty} u_n$ is now equivalent to convergence of the infinite series $\Sigma_{n=1}^{\infty} \log(u_n)$ provided we are a little careful with the multi-valued logarithmic function, as follows. From (a) $\log(u_n)$ is defined for all n, and from (b), discarding a finite number of terms if so required (which has no effect on convergence issues), we can suppose $|u_n - 1| < 1/2$ for all sufficiently large n. We then choose the branch of the logarithm for which $\log(1) = 0$. This also implies $\lim(\log(u_n)) = 0$. In case $\Pi_{n=1}^{\infty} u_n = L$ converges it is now clear that $\Sigma_{n=1}^{\infty} \log(u_n) = \log(L)$, furthermore that $L \neq 0$ and that $\Pi_{n=1}^{\infty} u_n^{-1} = L^{-1}$. It is also standard to call $\Pi_{n=1}^{\infty} u_n$ absolutely convergent if and only if $\Sigma_{n=1}^{\infty} \log(u_n)$ is absolutely convergent, i.e. $\Sigma_{n=1}^{\infty} |\log(u_n)|$ is convergent.

We shall henceforth write $u_n = 1 - \alpha_n$, and next give two results that connect the convergence of $\Pi_{n=1}^{\infty} u_n$ with the convergence of $\Sigma_{n=1}^{\infty} \alpha_n$. The first concerns absolute convergence and is found in many texts, cf. [3]. The second concerns convergence and can be found in [2], p.405, at least for the case $u_n \in \mathbb{R}$.

4.1 Theorem : Let $\alpha_n \in \mathbb{C} \setminus \{1\}$ and suppose $\Sigma_{n=1}^{\infty} \alpha_n$ is absolutely convergent. Then $\Pi_{n=1}^{\infty} u_n$ is absolutely convergent.

Sketch of proof (following [3]): The hypotheses on α_n imply conditions (a), (b) for u_n hold, so we consider $\Sigma_{n=1}^{\infty} \log(u_n) = \Sigma_{n=1}^{\infty} \log(1 - \alpha_n)$. Discarding a finite number of α_n if necessary we have $|\alpha_n| < 1/2$, whence

$$\log(1 - \alpha_n) = -\alpha_n - \frac{\alpha_n^2}{2} - \frac{\alpha_n^3}{3} - \ldots = -\alpha_n(1 + \frac{\alpha_n}{2} + \frac{\alpha_n^2}{3} + \ldots) \ .$$

It is then easily seen that $|\log(1 - \alpha_n)| \leq (3/2) \cdot |\alpha_n|$ and the convergence of $\Sigma_{n=1}^{\infty} \alpha_n$ thus implies convergence of $\Sigma_{n=1}^{\infty} \log(u_n)$. □

The proofs of next lemma and theorem follow Bartle's proofs for the case $\alpha_n \in \mathbb{R}$ (cf. [2], given as a "Project"), with a couple of changes that are discussed in Remark 4.5 below.

4.2 Lemma : Let $z \in \mathbb{C}$, $|z| < 1/2$. Then $(1/6)|z|^2 < |z + \log(1 - z)| < (5/6)|z|^2$.

Proof: We have

$$z + \log(1 - z) = z - z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots = -\frac{z^2}{2}(1 + R), \text{ where } R = \sum_{n=1}^{\infty} \frac{2z^n}{n+2} \ .$$

Now $|R| \leq (2/3)|z| + (2/4)|z|^2 + (2/5)|z|^3 \leq (2/3)|z|(1 + |z| + |z|^2 + \ldots)$ $= \dfrac{2|z|}{3}\dfrac{1}{1 - |z|} < \dfrac{2}{3} \cdot \dfrac{1}{2} \cdot \dfrac{1}{1 - 1/2} = \dfrac{2}{3}$, whence $1/3 < |1 + R| < 5/3$. Multiplying by $|z|^2/2$ completes the proof. □

4.3 Theorem : Let $\alpha_n \in \mathbb{C} \setminus \{1\}$ and suppose $\Sigma_{n=1}^{\infty} \alpha_n$ is convergent. Then $\Pi_{n=1}^{\infty} u_n$ is convergent if $\Sigma_{n=1}^{\infty} |\alpha_n|^2$ is convergent.

Proof: We start as in the proof of 4.1 and have (again n is assumed sufficiently large so $|\alpha_n| < 1/2$) $\log(1-\alpha_n) = -\alpha_n + \beta_n$, where $\beta_n = \alpha_n + \log(1-\alpha_n)$, so by Lemma 4.2 $(1/6)|\alpha_n|^2 < |\beta_n| < (5/6)|\alpha_n|^2$. By hypothesis $\Sigma_{n=1}^{\infty}\alpha_n$ converges, thus $\Sigma_{n=1}^{\infty}\log(1-\alpha_n)$ converges if and only if $\Sigma_{n=1}^{\infty}\beta_n$ converges. But by the above (right-hand) inequality this will follow from the convergence of $\Sigma_{n=1}^{\infty}|\alpha_n|^2$, indeed $\Sigma_{n=1}^{\infty}\beta_n$ is absolutely convergent here. □

4.4 Corollary : Let $\alpha_n \in \mathbb{C}\setminus\{1\}$ and suppose $\Sigma_{n=1}^{\infty}\alpha_n$ is convergent. Then $\Pi_{n=1}^{\infty}u_n$ is convergent if $\alpha_n = O(n^{-r})$, $r > 1/2$.

4.5 Remark : In the real case $\alpha_n \in \mathbb{R}$, as in [2], one actually obtains the following stronger result : Let $\alpha_n \in \mathbb{R}$, $\alpha_n < 1$, and suppose $\Sigma_{n=1}^{\infty}\alpha_n$ is convergent. Then $\Pi_{n=1}^{\infty}u_n$ is convergent if and only if $\Sigma_{n=1}^{\infty}|\alpha_n|^2$ is convergent. To see this one simply observes, for n sufficiently large, that

$$\beta_n = -\frac{\alpha_n^2}{2} - \frac{\alpha_n^3}{3} - \ldots = -\frac{\alpha_n^2}{2}\left(1 + \frac{2\alpha_n}{3} + \frac{2\alpha_n^2}{4} + \ldots\right) < 0\ .$$

Hence $\sum \beta_n$ converges if and only if $\sum |\beta_n|$ converges, and then the left hand side of the inequality mentioned in the above proof can be used, showing that $\sum \beta_n$ converges implies $\sum |\alpha_n|^2$ converges.

5 Examples and Questions

Our first example is standard. Here as usual Π_p (respectively $\sum_p$) denotes a product (respectively sum) over the prime numbers, and p_n is the n'th prime.

5.1 Example : The Euler product for $\zeta(s)$, or for any Dirichlet L-function $L(s,\chi)$, is absolutely convergent for $\sigma > 1$.

To see this, e.g. for $\zeta(s)$, consider the Euler product $\zeta^{-1}(s) = \Pi_p(1-p^{-s})$. Then, using the notation of Section 4, $\alpha_n = p_n^{-s}$ so $\sum_n \alpha_n$ is clearly absolutely convergent for $\sigma > 1$, and Theorem 4.1 then implies that $\zeta^{-1}(s) = \Pi_n(1-\alpha_n)$ is absolutely convergent. As seen in Section 4 this is equivalent to absolute convergence of the Euler product for ζ. The proof for the L-functions is similar.

The following examples will involve Theorem 4.3 (or its Corollary 4.4) and be less straightforward. As in Example 5.1 we will generally look at the inverse of the function in question for convergence, without specific mention.

5.2 Example : Let $f(s) = \Pi_{n\geq 2}\dfrac{1}{1-(-1)^n n^{-s}} = \dfrac{1}{1-2^{-s}}\cdot\dfrac{1}{1+3^{-s}}\cdots$. Then the infinite product $f(s)$ converges for $\sigma > 1/2$.

Proof: Here $\alpha_n = (-1)^n n^{-s}$, $n \geq 2$. Using Theorem 1.7 shows $\Sigma_{n\geq 2}\alpha_n$ is convergent, $\sigma > 0$. The convergence of $f(s)$ follows by Corollary 4.4. □

The next example is very similar to Example 5.2 but n is replaced by p_n so that it is an Euler type product.

5.3 Example : Let

$$g(s) = \Pi_{n\geq 1}\frac{1}{1+(-1)^n p_n^{-s}} = \frac{1}{1-2^{-s}}\cdot\frac{1}{1+3^{-s}}\cdot\frac{1}{1-5^{-s}}\cdot\frac{1}{1+7^{-s}}\cdot\frac{1}{1-11^{-s}}\cdots\ .$$

Then the infinite product $g(s)$ converges for $\sigma > 1/2$.

Proof: Similar to that of Example 5.2.

We now turn to Euler products that arise from Dirichlet L-functions. For the next example we consider $L(s, \chi_2^3)$, defined in Section 4.

Example 5.4 : Let $\chi = \chi_2^3$ and consider

$$L(s,\chi) = \Pi_{n\geq 1} \frac{1}{1-\chi(p_n)\cdot p_n^{-s}} = \frac{1}{1+2^{-s}}\cdot\frac{1}{1+5^{-s}}\cdot\frac{1}{1-7^{-s}}\cdot\frac{1}{1+11^{-s}}\cdots .$$

Here $\alpha_n = \chi(p_n)\cdot p_n^{-s}$. The second condition for convergence in Corollary 4.4 is satisfied, but unfortunately the first condition, that $\sum \alpha_n$ is convergent (at least for $\sigma > 1/2$), appears to be very difficult to prove.

The same situation holds for any primitive Dirichlet character χ modulo q , where $q \geq 3$.

We close this section with some potentially interesting questions.

Question 5.5 : Do the functions f, g in Examples 5.2, 5.3 (particularly 5.3), also satisfy a functional equation relating the function values at $z, 1-z$. Are they in the Selberg class? If so they may give an example of a function in the Selberg class that satisfies RH. Is there another example of this type?

Question 5.6 : As mentioned in Example 5.4, can one prove $\sum \alpha_n$ convergent here, or in similar examples for primitive Dirichlet characters mod $q,\ q \geq 3$? The difficulty seems to be, e.g. for χ_2^3, that although density theorems imply the density of primes congruent to 1 mod 3 is 1/2, and similarly for primes congruent to 2 mod 3, there could be arbitrarily long sequences of each type. Of course any such proof would imply GRH for the corresponding L-function(s).

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