PRELIMINARIES, NOTATION

$$s = \sigma + it \qquad |s| = \sqrt{\sigma^2 + t^2}$$

$$\mathrm{Re}(s) = \sigma, \quad \mathrm{Im}(s) = t$$

$$\log(s) = \ln(s)$$

$$\sum_p, \quad \prod_p$$

$$\pi(x) = \sum_{p \le x} 1 = |\{p : p \le x\}|$$

$$\psi(x) = \sum_{p^n \le x} \log(p)$$

Bernoulli numbers $\dfrac{x}{e^x - 1} = \sum_{n \ge 0} \dfrac{B_n}{n!} x^n$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$,
$B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $\ldots$, $B_3 = B_5 = B_7 = \ldots = 0$

$$\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x}\, dx, \quad \sigma > 0, \quad \Gamma(n+1) = n!$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx, \quad \sigma > 1$$

PNT = Prime Number Theorem
RH = Riemann Hypothesis

HISTORY THROUGH 1859

1350 Oresme
1650 Mengoli
1734 Euler
1792-93 Gauss (15-16 years old)
1798 Legendre
1837 Dirichlet
1845 Bertrand
1849 Gauss
1848, 1850 Chebychev ЧЕБЫШЕВ
1852 "
1859 Riemann (1826 - 1866)

HISTORY AFTER 1859

1876 Weierstrass
1896 de la Vallée-Poussin, Hadamard
1903 Gram
1914 Backlund
1925 Hutchinson
1914 Hardy - Littlewood, Ramanujan
1932 Siegel
1942 Selberg (Fields Medal 1960), 1968 Spir
1974 Levinson Bombieri (Fields Medal
1989 Conrey 1978 Deligne (Fields Medal
1985 Voronin

LEONHARD EULER 1707-1783

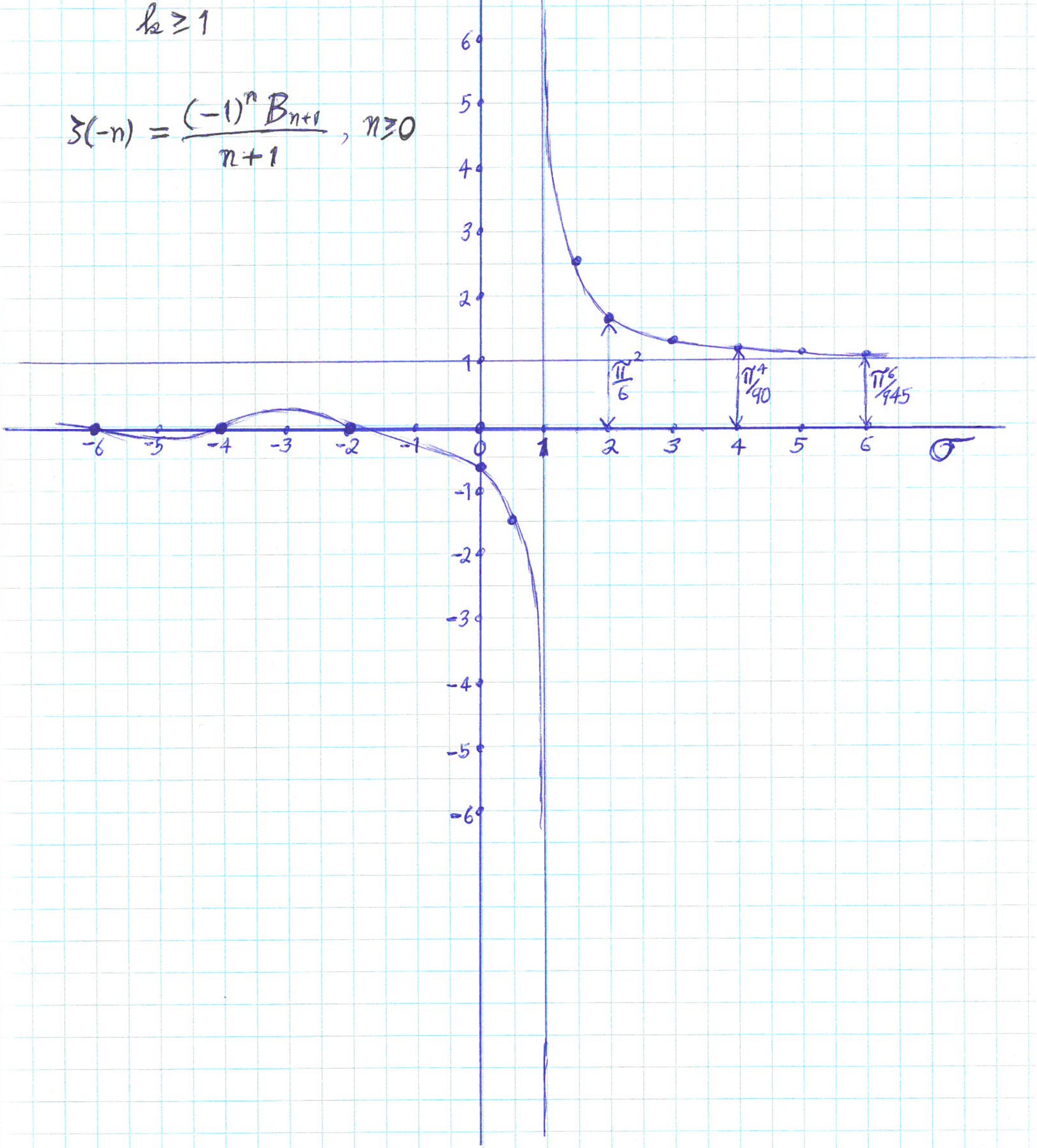
$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!}$$

$k \geq 1$

$$\zeta(-n) = \frac{(-1)^n B_{n+1}}{n+1}, \quad n \geq 0$$

$\zeta(\sigma)$

Euler's contribution
$\zeta(\sigma)$, $\sigma \in \mathbb{R}$



Gauß 3, Encke ?
Briefe

Hochzuverehrender Freund. 75

Vorallem statte ich Ihnen für die gewogentliche Übersendung des Jahrbuchs von 1852 meinen verbindlichsten Dank ab.

Die gütige Mittheilung Ihrer Bemerkungen über die Frequenz der Primzahlen ist mir in mehr als einer Beziehung interessant gewesen. Sie haben mir meine eignen Beschäftigungen mit demselben Gegenstande in Erinnerung gebracht, deren erste Anfänge in eine sehr entfernte Zeit fallen, ins Jahr 1792 oder 1793, wo ich mir die Lambertschen Supplemente zu den Logarithmentafeln angeschafft hatte. Es war noch ehe ich mit feinern Untersuchungen aus der höhern Arithmetik mich befasst hatte eines meiner ersten Geschäfte, meine Aufmerksamkeit auf die abnehmende Frequenz der Primzahlen zu richten, zu welchem Zweck ich dieselben in den einzelnen Chiliaden abzählte, und die Resultate auf einem der angehefteten weissen Blätter verzeichnete. Ich erkannte bald, dass unter allen Schwankungen diese Frequenz durchschnittlich nahe dem Logarithmen verkehrt proportional sei, so dass die Anzahl aller Primzahlen unter einer gegebenen Grenze n nahe durch das Integral

$$\boxed{\int \frac{dn}{\log n}}$$

ausgedrückt werde, wenn der hyperbolische Logarithm verstanden werde. In späterer Zeit, als mir die in Vegas Tafeln (von 1796) ~~bekannt~~ abgedruckte Liste bis 400031 bekannt wurde, dehnte ich meine Abzählung weiter aus, was ~~die~~ jenes Verhältniss bestätigte. Eine grosse Freude machte mir 1811 die Erscheinung von Chernacs cribrum, und ich habe (da ich

[illegible]

TABLE I

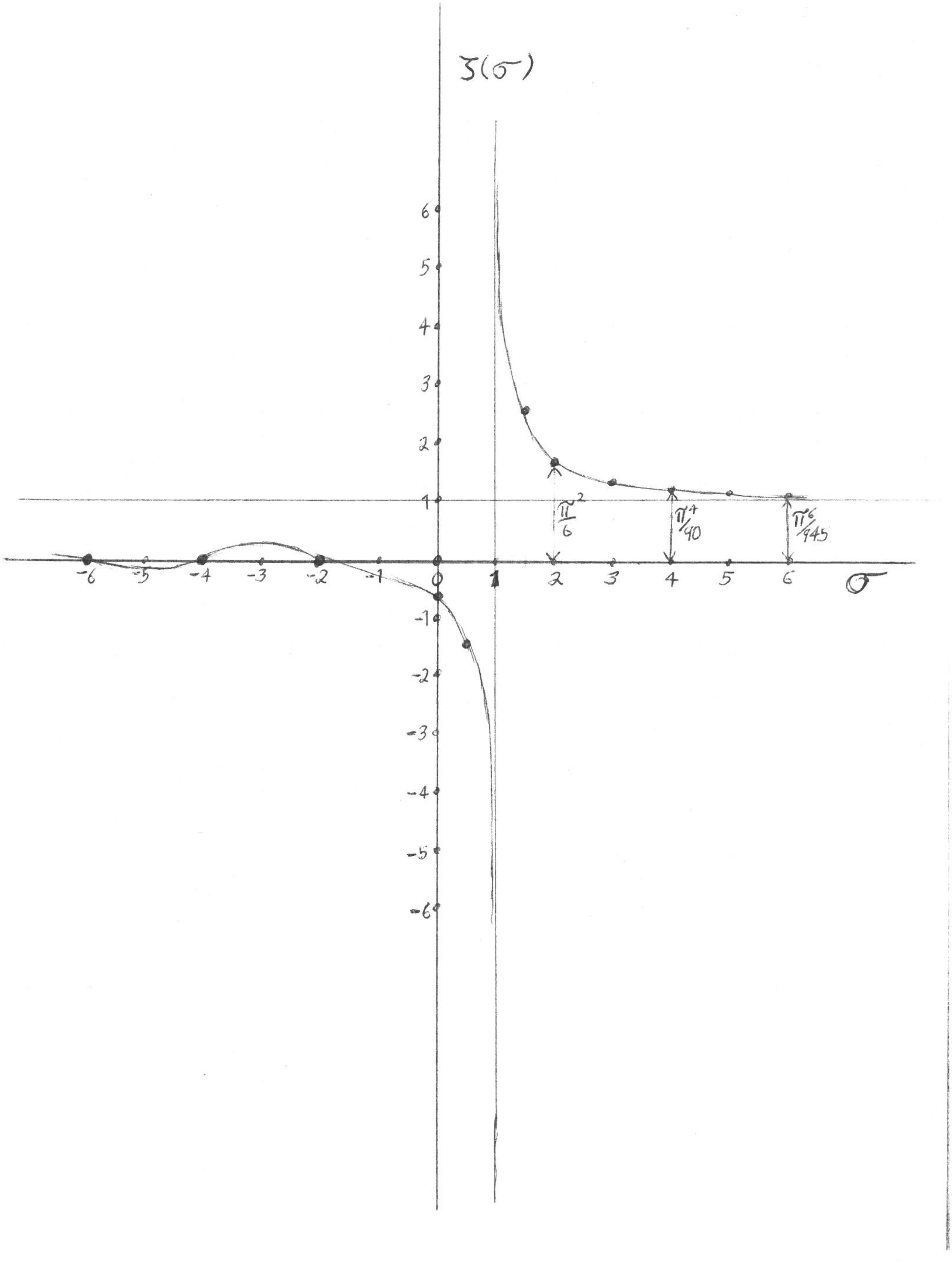
x	$\pi(x)$	$Li(x)$	$Li(x) - \pi(x)$
500,000	41,538	41,606	68
1,000,000	78,498	78,628	130
1,500,000	114,155	114,263	108
2,000,000	148,933	149,055	122
2,500,000	183,072	183,245	173
3,000,000	216,816	216,971	155

TABLE II

GRAM (1903)

$\alpha_1 = 14.134\,725$	$\alpha_6 = 37.586\,176$	$\alpha_{11} = 52.8$
$\alpha_2 = 21.022\,040$	$\alpha_7 = 40.918\,720$	$\alpha_{12} = 56.4$
$\alpha_3 = 25.010\,856$	$\alpha_8 = 43.327073$	$\alpha_{13} = 59.4$
$\alpha_4 = 30.424\,878$	$\alpha_9 = 48.005\,150$	$\alpha_{14} = 61.0$
$\alpha_5 = 32.935\,057$	$\alpha_{10} = 49.773\,832$	$\alpha_{15} = 65.0$

$$\zeta\left(\frac{1}{2} \pm i\alpha_j\right) = 0$$



ζ(σ)
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
σ
π²/6
π⁴/90
π⁶/945

RIEMANN

1826 - 1866

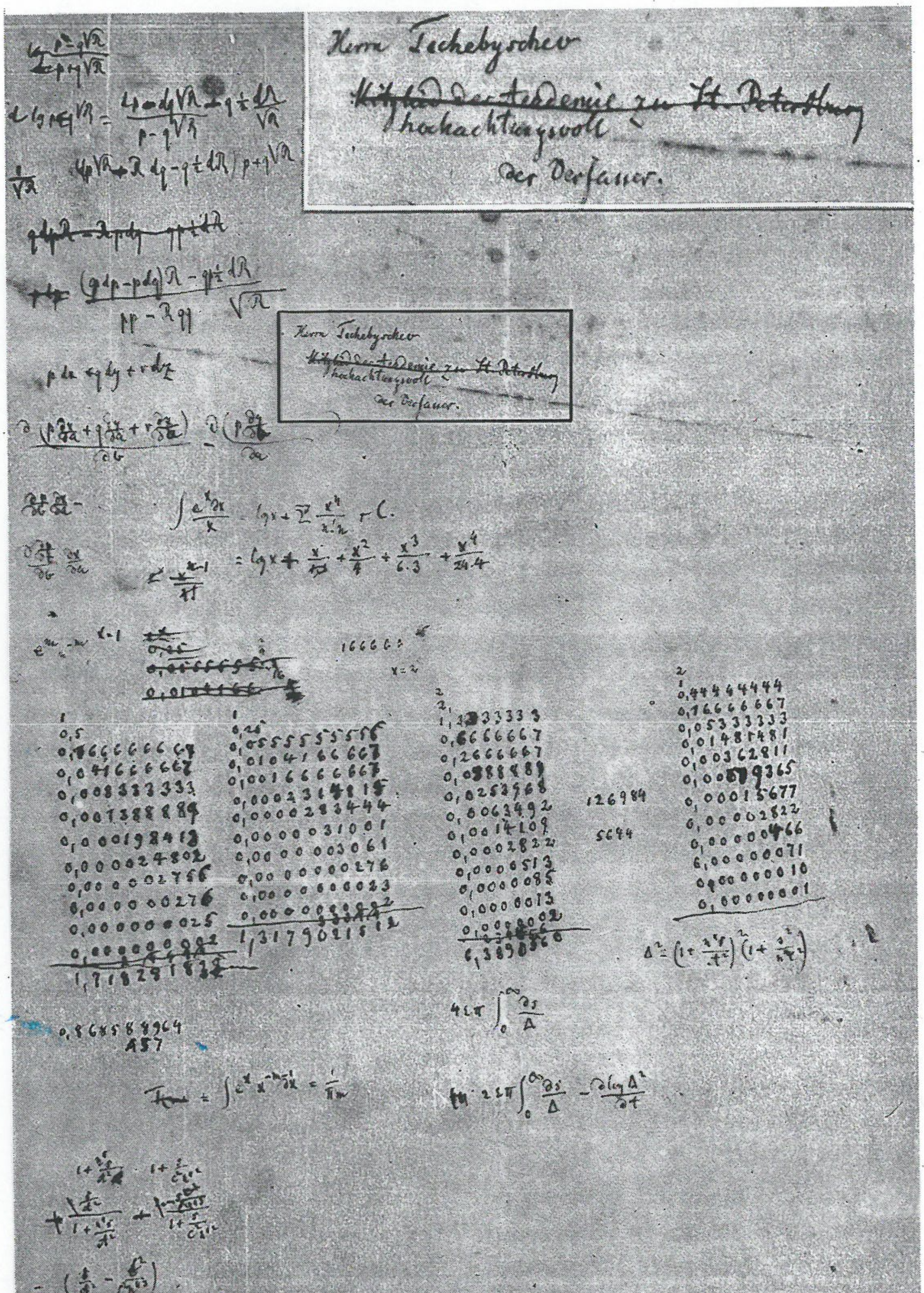
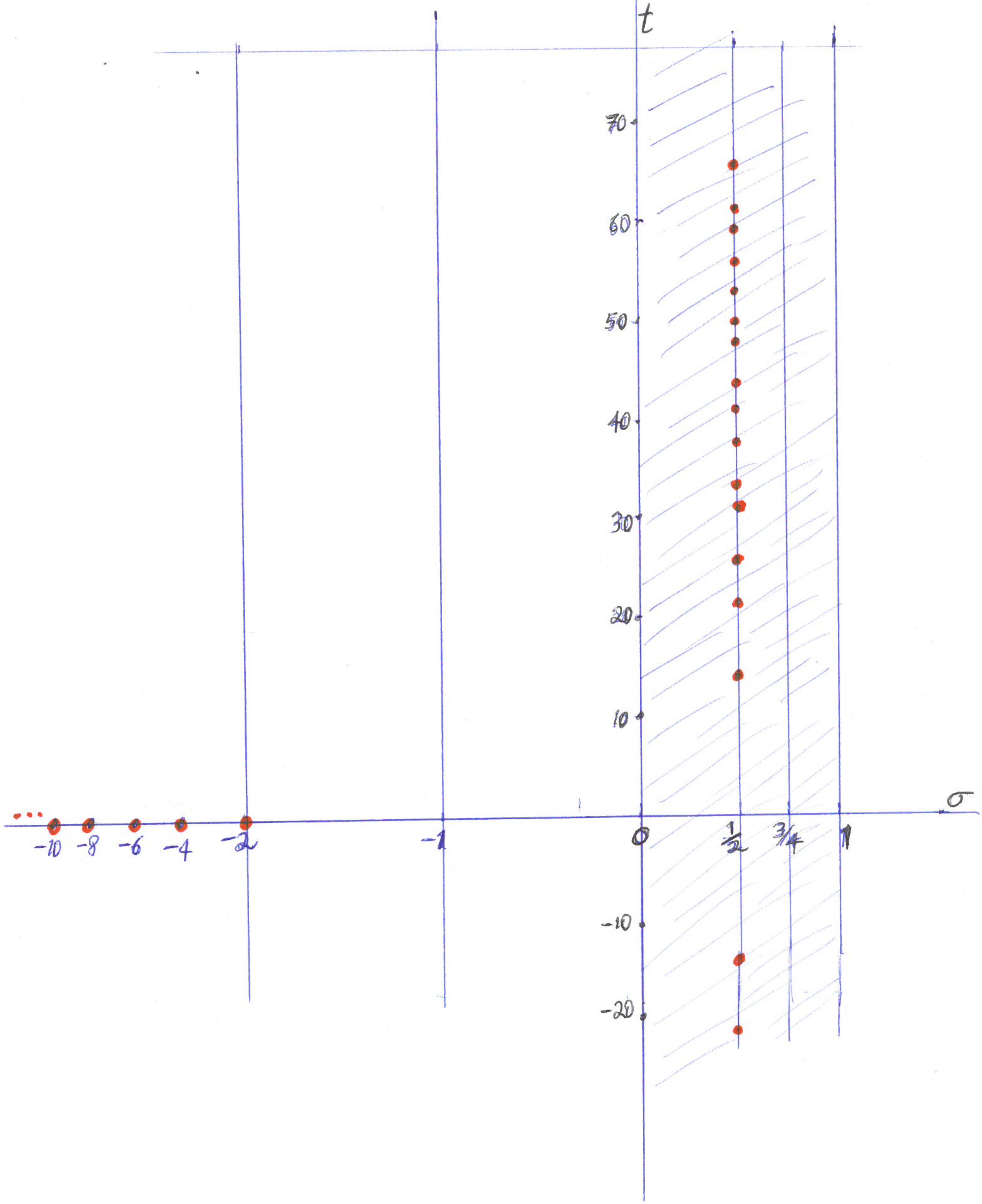


Fig. 1. A scrap sheet used to hold some other loose sheets in Riemann's papers. The note seems to prove that Riemann was aware of Chebyshev's work and intended to send him an offprint of his own paper. In all likelihood Riemann was practicing his penmanship in forming Roman, rather than German, letters to write a dedication to Chebyshev. (Reproduced with the permission of the Niedersächsische Staats- und Universitätsbibliothek, Handschriftenabteilung, Göttingen.)

Zeros of ζ   Gram 1903



Functional Equation

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) \cdot \zeta(1-s)$$

$$\xi(s) := \pi^{-s/2}(s-1)\,\Gamma\left(\frac{s}{2}+1\right) \cdot \zeta(s)$$

$$\xi(s) = \xi(1-s)$$

Explicit formula (von Mangoldt 1885)

$$\psi(x) = x - \sum_{m \geq 1} \frac{x^{-2m}}{-2m} - \sum_{\substack{\zeta(\rho)=0 \\ \mathrm{Im}(\rho) \neq 0}} \frac{x^{\rho}}{\rho} - \log(2\pi)$$

And

$$\Gamma(s)\,\eta(s) = \int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx\,, \quad \sigma > 0$$

$$\log \zeta(s) = s \int_2^\infty \frac{\pi(x)}{x(x^s-1)}\,dx$$

Expository references

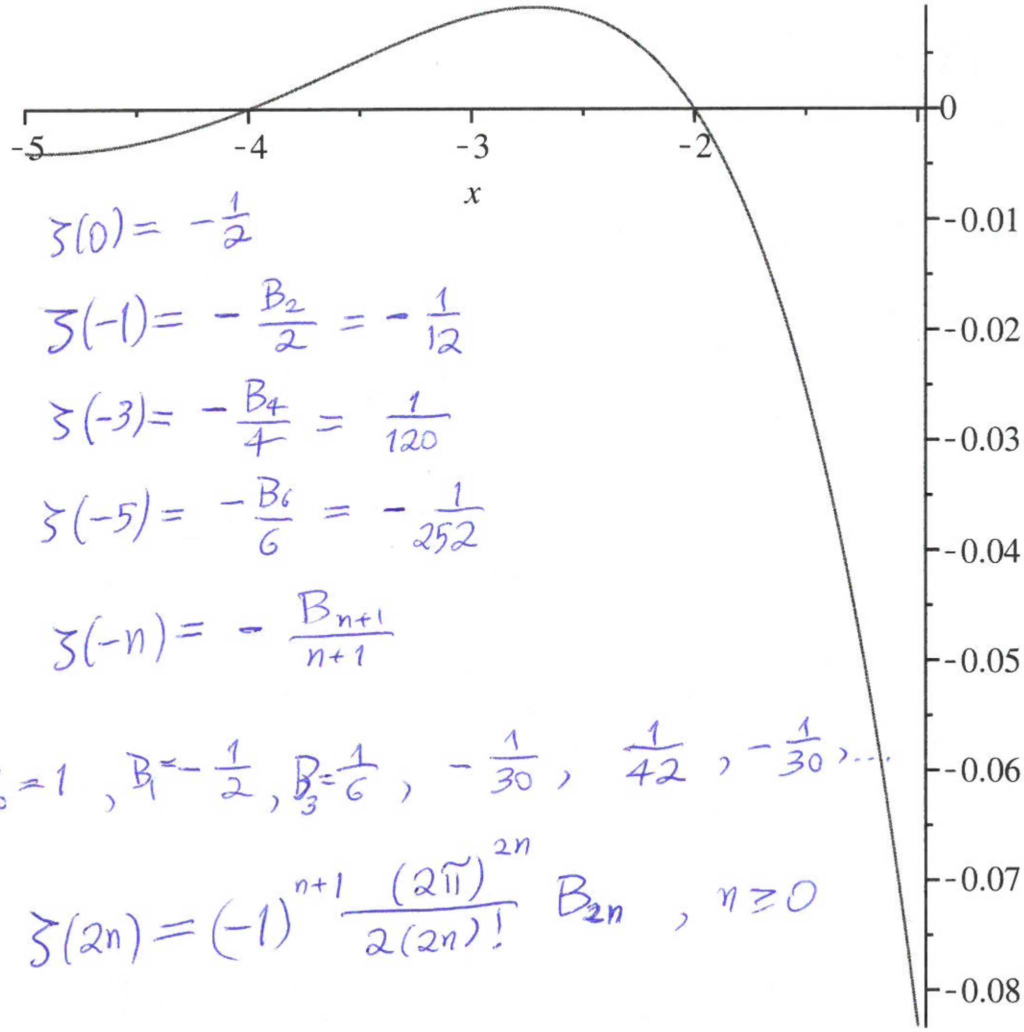
Edwards, H.M., Riemann's Zeta Function, Academic Press NY (1974), reprinted by Dover Pub. (2001)

Kudryavtseva, E., Saidak, F., Zv, P.,

Morfismos Vol. 9, No. 2 (2005), 1–48
http://chucha.math.cinvestav.mx/morfismos

Conrey, J.B., Notices AMS (50), 3 (2003), 341–353

> plot(Zeta(x), x=-5..-1)



$\zeta(0) = -\frac{1}{2}$

$\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}$

$\zeta(-3) = -\frac{B_4}{4} = \frac{1}{120}$

$\zeta(-5) = -\frac{B_6}{6} = -\frac{1}{252}$

$\zeta(-n) = -\frac{B_{n+1}}{n+1}$

$B_0 = 1, \; B_1 = -\frac{1}{2}, \; B_3 = \frac{1}{6}, \; -\frac{1}{30}, \; \frac{1}{42}, \; -\frac{1}{30}, \ldots$

$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad n \geq 0$

>

Littlewood's thesis

Hilbert ∩ Millenium Problems

500 years sleep

Riemann, B. Ueber die Anzahl der Primzahlen unter einer gegeben Grösse
(On the number of primes less than a given magnitude)

"Man findet nun in der That etwa so viele reele Nullstellen innerhalb dieser Grenzen, und es ist sehr warscheinlich, dass alle Wurzeln reel sind."
(One finds about this many real roots within these bounds, and it is very likely that all roots [of $\xi(\frac{1}{2}+it)$] are real.)

"Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indes die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien."

(One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof, after some fleeting vain attempts, because it is not necessary for the next objective of my investigation.)

Homework Problem in Lang, Complex Analysis, 3rd Ed. p.421

1. (a) Show $\zeta(s)$ has zeros of order 1 at the negative integers
   (b) Show that the only other zeros are such that $0 \leq Re(s) \leq 1$
   (c) Prove that the zeros of (b) actually have $Re(s) = 1/2$. [You can ask the professor teaching the course for a hint on that one.]

With Gopala Srinivasan

irregular behaviour vanishes as soon as one moves a small distance off the real axis, a property of the function that does not appear to have been observed in the literature. The purpose of this short note is to prove the following:

Theorem 1.1. Let $s = \sigma + it$, with $|t| \geq 5/4$. Then $|\Gamma(s)|$ is strictly monotone increasing with respect to σ.

We remark that the lower bound 5/4 given in Theorem 1.1 is close to but not the best possible. Using MAPLE, it seems that the best lower bound is approximately given by $|t| > 1.04794998$, and in Section 3 it is shown that monotonicity fails for $|t| = 1$. We also remark that the same property does not seem to hold for the digamma function $\Psi(s)$, in particular for σ sufficiently large negative $|\Psi(\sigma + it)|$ oscillates with respect to σ, no matter how large $|t|$ is taken. A recent paper of Alzer [2] deals with a monotonicity property for the Hurwitz zeta function, and an earlier paper [1] with monotonicity of the gamma function, in both cases along the real axis.

In the following section we prove an elementary lemma that enables one to detect monotonicity of the modulus of any holomorphic function f. We also state a few results from the theory of the gamma function that we need for the proof of Theorem 1.1 given in Section 3.

Figure 1 below gives a beautiful illustration of Theorem 1.1, showing the poles at $s = 0, -1, -2, -3, -4$ along the real axis, and how the modulus becomes monotone increasing as $|t|$ becomes somewhat larger than 1 (e.g. the bottom curve in the figure represents $|\Gamma(\sigma - 2.5 \cdot t)|$). It is taken from the book of Jahnke and Emde [9], written well before the age of computer graphics.

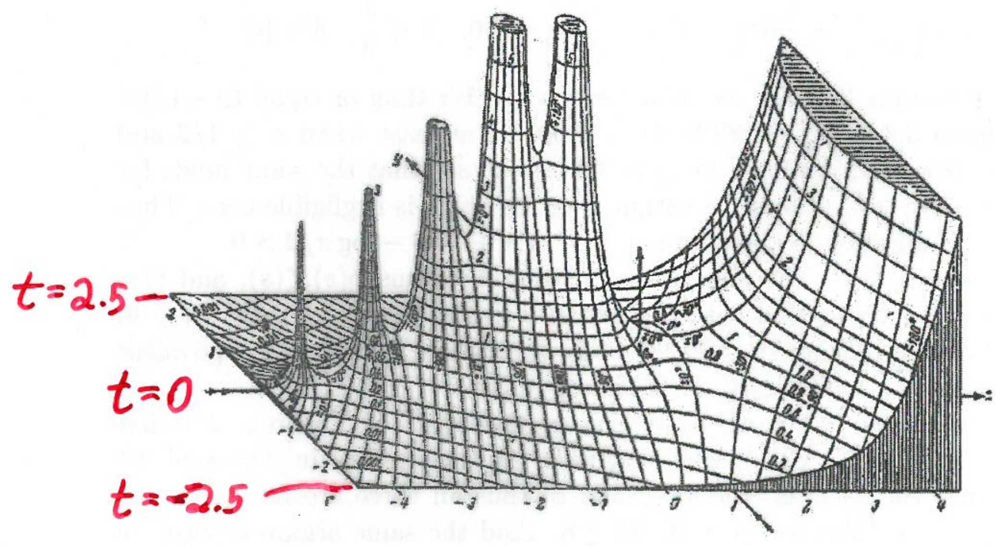
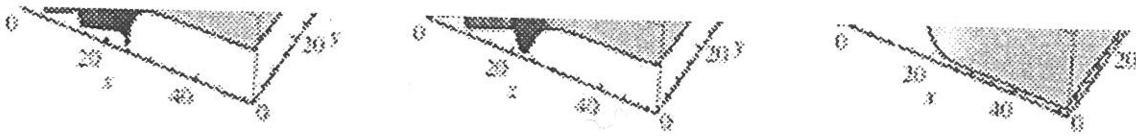
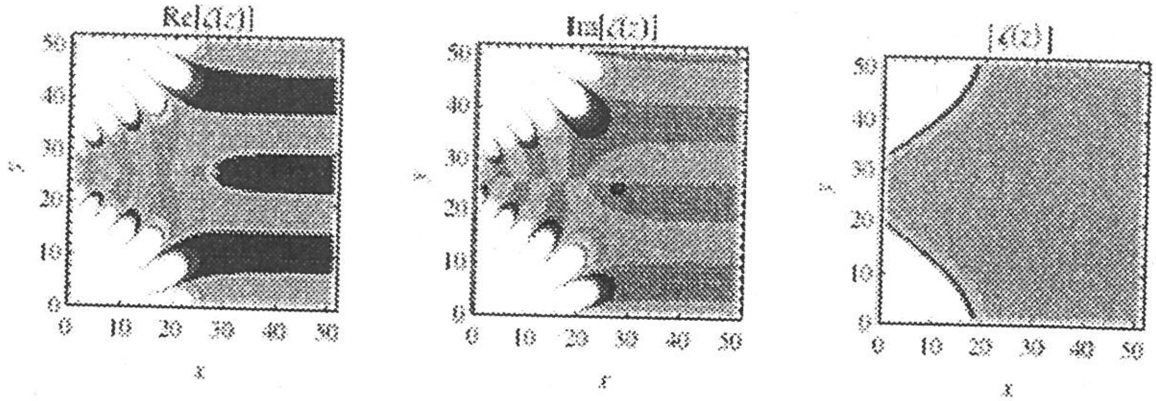


Figure 1: The Modulus of the Gamma Function



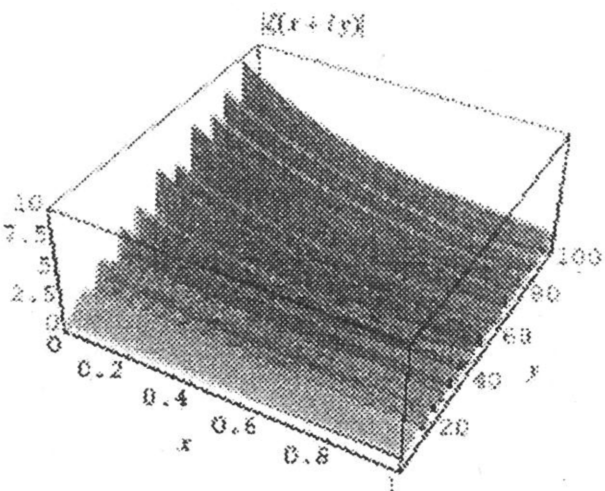


	Min	Max
Re	-15	15
Im	-15	15

Replot

POWERED BY webMathematica

The Riemann zeta function is an extremely important special function of mathematics and physics that arises in definite integration and is intimately related with very deep results surrounding the prime number theorem. While many of the properties of this function have been investigated, there remain important fundamental conjectures (most notably the Riemann hypothesis) that remain unproved to this day. The Riemann zeta function $\zeta(s)$ is defined over the complex plane for one complex variable, which is conventionally denoted s (instead of the usual z) in deference to the notation used by Riemann in his 1859 paper that founded the study of this function (Riemann 1859). It is implemented in *Mathematica* as Zeta[s].



The plot above shows the "ridges" of $|\zeta(x + i y)|$ for $0 < x < 1$ and $1 < y < 100$. The fact that the ridges appear to decrease monotonically for $0 \leq x \leq 1/2$ is not a coincidence since it turns out that monotonic decrease implies the Riemann hypothesis (Zvengrowski and Saidak 2003; Borwein and Bailey 2003, pp. 95-96).

On the real line with $x > 1$, the Riemann zeta function can be defined by the integral

$$\zeta(x) \equiv \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1}\, d u,$$

where $\Gamma(x)$ is the gamma function. If x is an integer n, then we have the identity

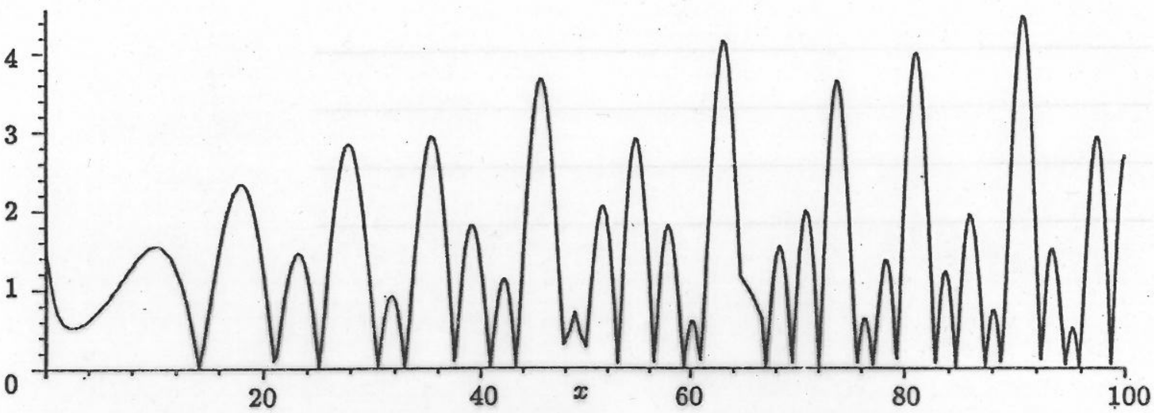


FIGURE 2. $|\zeta(\sigma + it)|$ for $\sigma = \frac{1}{2}$, and $0 \leq t \leq 100$.

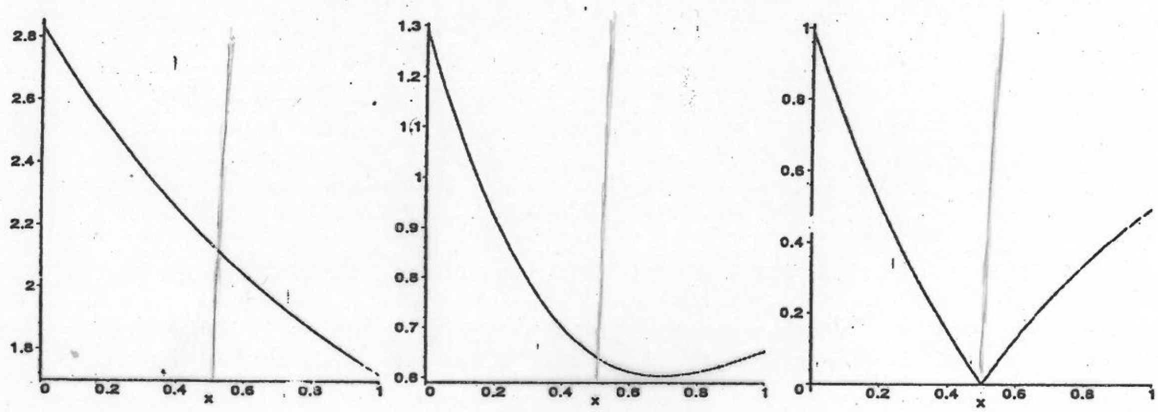
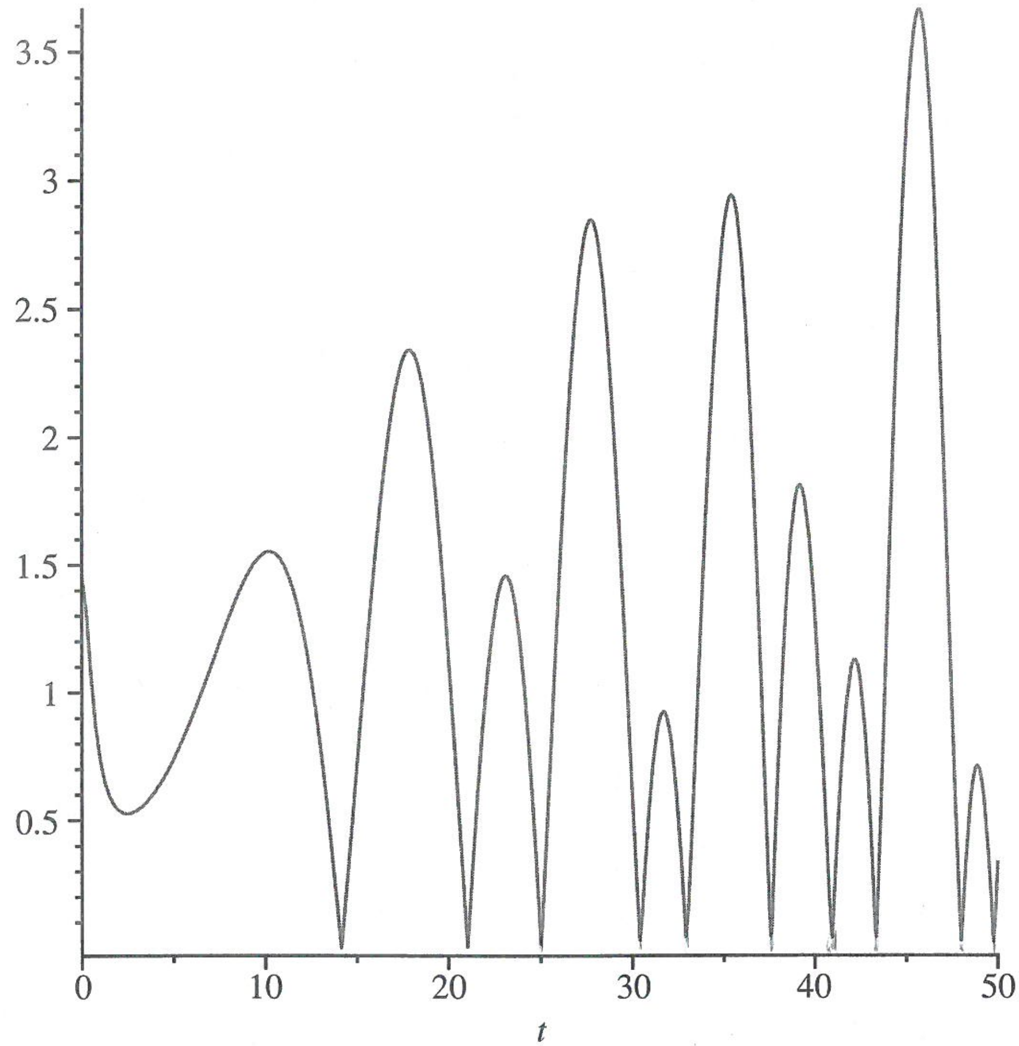


FIGURE 1. $|\zeta(\sigma + it_0)|$ for $t_0 = 17$, 24.5 and 25.011, with $0 \leq \sigma \leq 1$.

Note. The vertical scale varies for the graphs in Figure 1, and we also remark that the graphs in Figure 1, if plotted for a larger range of σ, are essentially no more complicated than in the range $0 \leq \sigma \leq 1$ depicted (of course, we always have $\left|\zeta(\sigma + it_0)\right| \to 1$, as $\sigma \to \infty$).

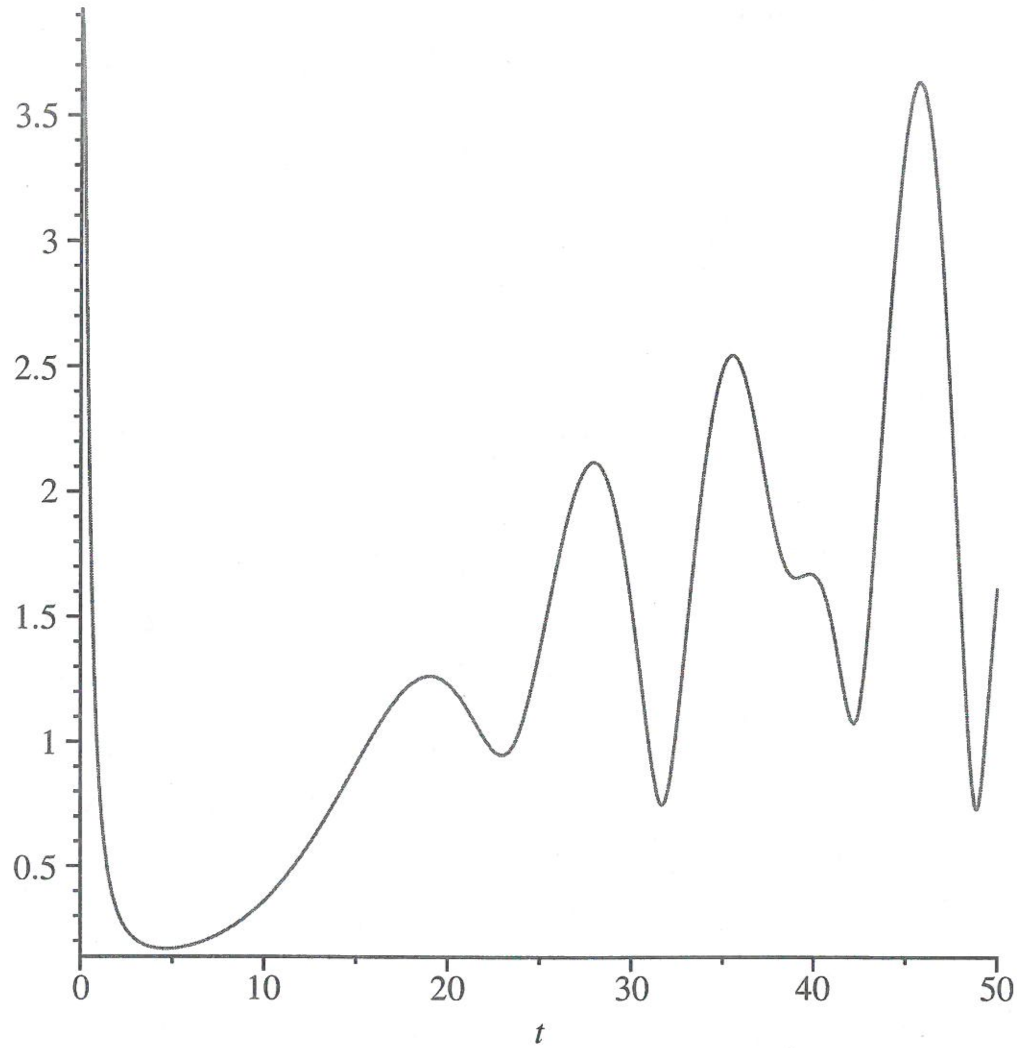
> $plot\left(\text{abs}\left(\text{Zeta}\left(\frac{1}{2}+I\cdot t\right)\right), t=0..50\right)$



>

$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ , \quad 0 \leq t \leq 50$

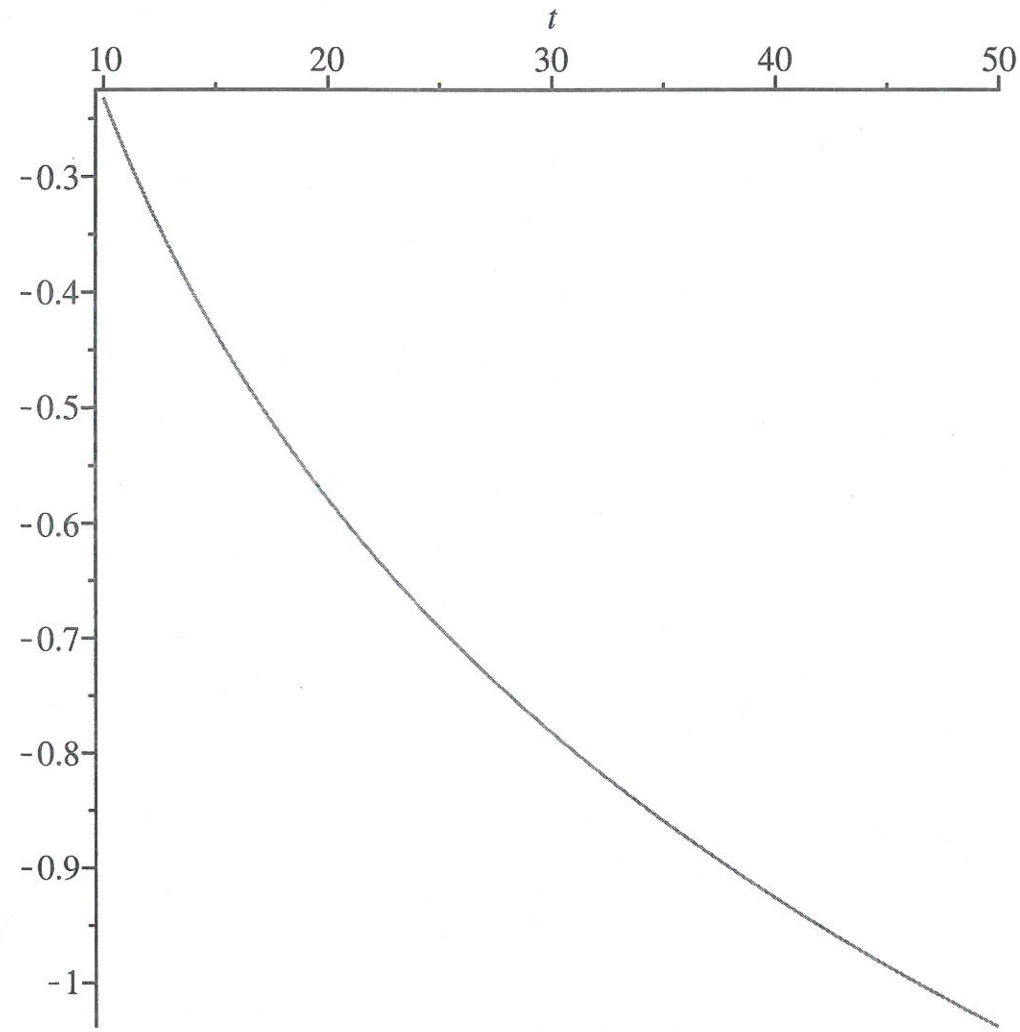
> $plot\left(\text{abs}\left(\text{Zeta}\left(1, \frac{1}{2} + I \cdot t\right)\right), t = 0..50\right)$



>

$\left|\zeta'\left(\frac{1}{2} + it\right)\right|, \quad 0 \le t \le 50$

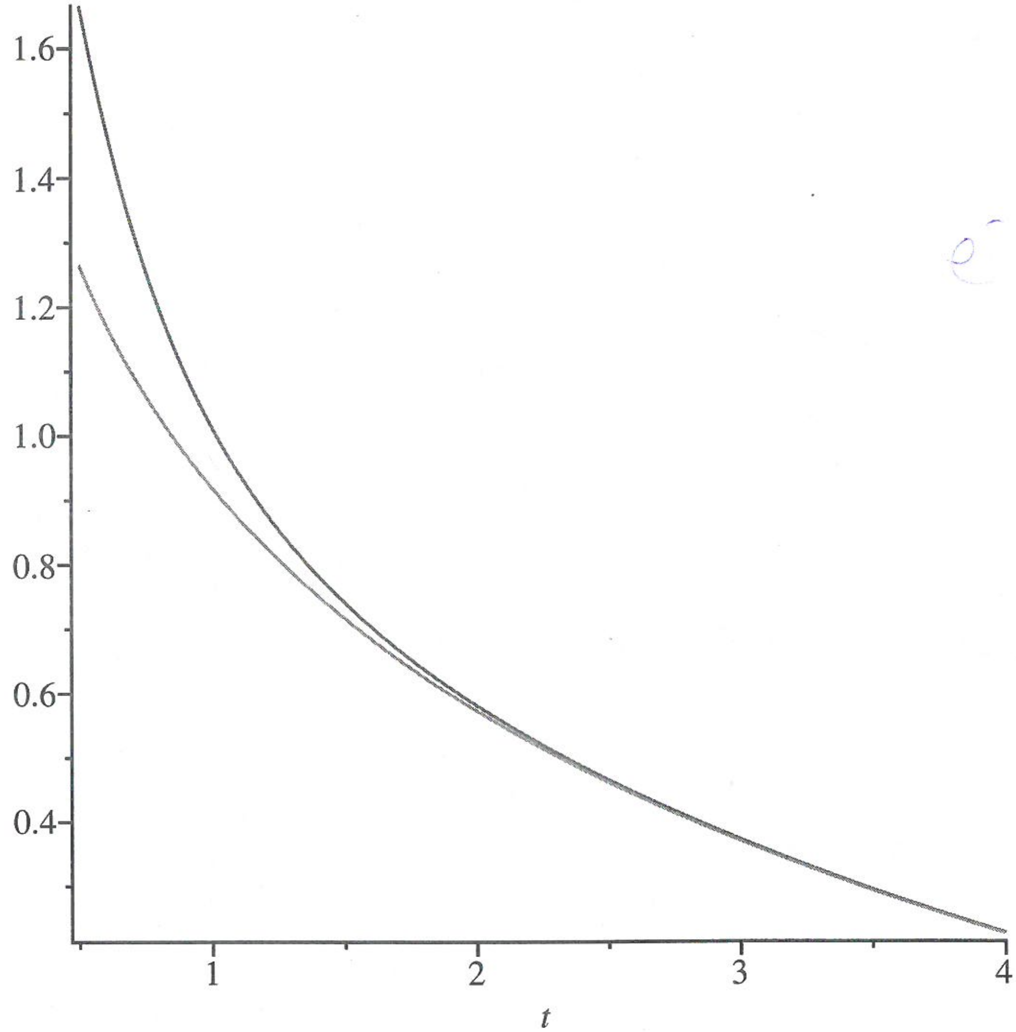
$$> plot\left(\left[\mathrm{Re}\left(\frac{\mathrm{Zeta}\left(1, \frac{1}{2} + I \cdot t\right)}{\mathrm{Zeta}\left(\frac{1}{2} + I \cdot t\right)}\right), -.5 \cdot \log\left(\frac{t}{2 \cdot \mathrm{Pi}}\right)\right], t = 10 ..50\right)$$



$$\mathrm{Re}\left(\frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)}\right) \text{ and } -\frac{1}{2}\log\left(\frac{t}{2\pi}\right)$$

$$10 \le t \le 50$$

$$> plot\left(\left[\mathrm{Re}\left(\frac{\mathrm{Zeta}\left(1, \frac{1}{2} + I \cdot t\right)}{\mathrm{Zeta}\left(\frac{1}{2} + I \cdot t\right)}\right), -.5 \cdot \log\left(\frac{t}{2 \cdot \mathrm{Pi}}\right)\right], t = .5 \,..4\right)$$



e^{-t}

Same $\frac{1}{2} \le t \le 4$

Where are the Zeros of Zeta of s?

Tom Apostol (a little editing by P. Zvengrowski)

Where are the zeros of $\zeta(s)$?
George Bernhard Riemann has made a good guess.
"They're all on the critical line," saith he,
"with density $\frac{1}{2\pi \log t}$."

Now this statement by Riemann has set off a trigger,
and many a good man with vim and with vigour,
Has tried to find with mathematical rigour,
What happens to ζ as $|t|$ gets bigger.

The names Hardy and Landau, Titschmarsh and Cramér,
Littlewood and Ramanujan are there,
But in spite of their skill and with all their finesse,
For finding the zeros there's been little success.

In 1914 G. H. Hardy did find,
An infinite number that lie on the line,
Too bad that his theorem won't rule out the case,
There might be some zeros in some other place!

So - where are the zeros of $\zeta(s)$?
We must know exactly, we cannot just guess.
if we wish to refine the prime number theorem,
the path of integration must not get too near'em.