Nachdiplomvorlesung an der ETH Zürich im FS 2016

# Geometric and topological aspects of Coxeter groups and buildings

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# LECTURE 1

# MOTIVATION AND EXAMPLES

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## 1.1 Coxeter Groups

Before we give a proper definition of Coxeter groups let us give some examples.

### 1.1.1 One-dimensional examples

**Example 1.1** (One-dimensional unit sphere)**.** Let us first consider the one-dimensional unit sphere centered at the origin and two lines through the origin with dihedral angle $\frac{\pi}{m}$, $(m \in \{2, 3, 4, \dots\})$; see Figure 1.1. Further let $s_1$ and $s_2$ denote the reflections across the lines respectively. Note that $s_1 s_2$ is rotation by $\frac{2\pi}{m}$. Hence the group $\langle s_1 s_2 \rangle$ generated by $s_1 s_2$ is cyclic of order $m$ (i.e. isomorphic to $C_m$).

The full group $W = \langle s_1, s_2 \rangle$ is the dihedral group of order $2m$, denoted by $D_{2m}$, which has the presentation

$$W = \langle s_1, s_2 \,|\, s_i^2 = 1 \quad \forall i = 1, 2, \quad (s_1 s_2)^m = 1 \rangle$$

**Example 1.2** (One-dimensional Euclidean space)**.** Let us now consider the real line and the reflections at 0 and 1, denoted by $s_1$ and $s_2$ respectively; see Figure 1.2. Hence $s_2 s_1$ is the translation by 2, such that $\langle s_2 s_1 \rangle \cong \mathbb{Z}$. The full group $W = \langle s_1, s_2 \rangle$ has in this case the presentation

$$W = \langle s_1, s_2 \,|\, s_i^2 \, \forall i = 1, 2, \quad (s_1 s_2)^\infty = 1 \rangle$$

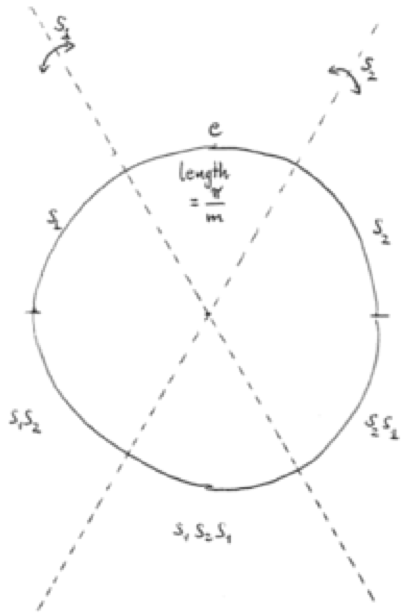



Figure 1.1: One-dimensional unit sphere

### 1.1.2 Examples in dimension $n \geq 2$

*Notation.* $\mathbb{X}^n$ denotes either ...

- ... the $n$-dimensional sphere $\mathbb{S}^n$,
- ... the $n$-dimensional Euclidean space $\mathbb{E}^n$, or
- ... the $n$-dimensional hyperbolic space $\mathbb{H}^n$

**Definition 1.3.** A *convex polytope* $P^n \subseteq \mathbb{X}^n$ is a convex, compact intersection of a finite number of half-spaces; e.g. see Figure 1.3.

The *link* $lk(v)$ of a vertex $v$ of $P^n$ is the $(n-1)$-dimensional spherical polytope obtained by intersecting $P$ with a small sphere around $v$.

$P$ is *simple* if for every vertex $v$ of $P$ its link $\mathrm{lk}(v)$ is a simplex.

**Definition 1.4.** Suppose $G \curvearrowright X$. A *fundamental domain* is a closed connected subset $C$ of $X$, such that $Gx \cap C \neq \emptyset$ for every $x \in X$, and $|Gx \cap C| = 1$ for every $x \in \mathrm{int}(C)$. A fundamental domain $C$ is called *strict*, if $Gx \cap C = \{x\}$ for every $x \in C$, i.e. $C$ has exactly one point from each $G$-orbit. For example $[0,1]$ is a strict fundamental domain for $D_\infty \curvearrowright \mathbb{E}^1$, whereas $\langle s_1 s_2 \rangle \cong \mathbb{Z} \curvearrowright \mathbb{E}^1$ does not have a strict fundamental domain.

**Theorem 1.5.** *Let $P^n$ be a simple convex polytope in $\mathbb{X}^n$ with codimension-one faces $F_i$. Suppose that $\forall i \neq j$ the dihedral angle between $F_i$ and $F_j$, if $F_i \cap F_j \neq \emptyset$, is $\frac{\pi}{m_{ij}}$ for*

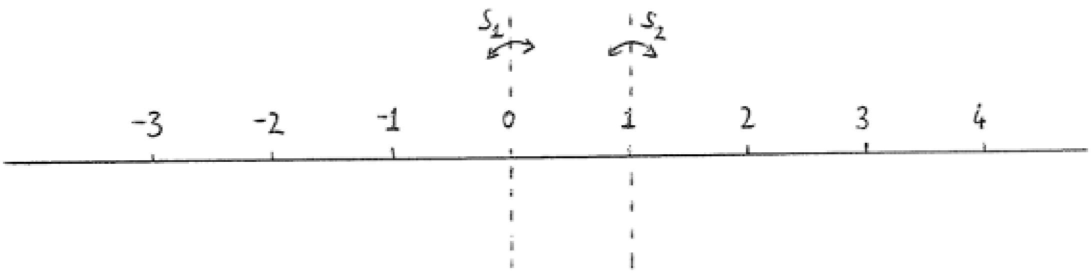



Figure 1.2: One-dimensional Euclidean space

*some* $m_{ij} \in \{2, 3, 4, \ldots\}$*. Set* $m_{ii} = 1$ *for every* $i$*, and* $m_{ij} = \infty$ *if* $F_i \cap F_j = \emptyset$*. Let* $s_i$ *be the isometric reflection of* $\mathbb{X}^n$ *across the hyperplane supported by* $F_i$ *and* $W = \langle s_i \rangle$ *the group generated by these reflections. Then:*

1. $W = \langle s_i \,|\, s_i^2 = 1\,\forall i, \quad (s_i s_j)^{m_{ij}} = 1 \rangle$,
2. *$W$ is a discrete subgroup of* $Isom(\mathbb{X}^n)$*, and*
3. *$P^n$ is a strict fundamental domain for the $W$ action on $\mathbb{X}^n$ and $P^n$ tiles $\mathbb{X}^n$.*

*Proof.* Later . . . □

**Definition 1.6.** A group $W$ as in the one-dimensional examples or as in Theorem 1.5 is called a *geometric reflection group*.

### Examples

**Example 1.7** (Spherical)**.** For $n = 2$ we may project classical polytopes to the sphere and consider their symmetry groups; see Figure 1.4. For arbitrary $n$ one may also consider the symmetry groups $S_n$ = symmetries of $(n-1)$-simplex; see Figure 1.5.

**Example 1.8** (Euclidean)**.** 1. Taking an equilateral triangle in $\mathbb{E}^2$ we get the situation depicted in Figure 1.6.

This amounts to:

$$m_{ij} = \begin{cases} 1 & \text{, if } i \neq j \\ 3 & \text{, else} \end{cases}$$

$W = \langle s_0, s_1, s_2 \rangle = \langle s_i \,|\, s_i^2 = 1\,\forall i, \quad (s_i s_j)^3 = 1\,\forall i \neq j \rangle$

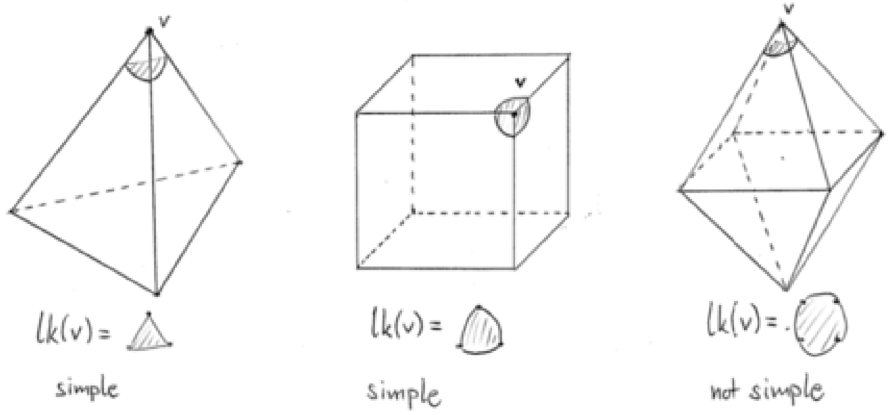



Figure 1.3: Three convex polytopes in $\mathbb{E}^n$. Two of them are simple and one is not.

2. Similarly one may take another tesselation of $\mathbb{E}^2$ by triangles, i.e. $P$ is one of the triangles depicted in Figure 1.7.

3. If we take $P$ to be a square (see Figure 1.8) we get

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } |i - j| = 1 \\ \infty, & \text{else} \end{cases} .$$

Note that $m_{ij} = 2$ if and only if $s_i$ and $s_j$ commute. ($W$ is an example of a right-angled Coxeter group).

Further note, that $P$ is a product of simplices. This generalizes to the following theorem by Coxeter:

**Theorem 1.9.** *All Euclidean $P^n$ are products of simplices.*

It is worth noting, that Coxeter actually classified all spherical and Euclidean $P^n$.

**Example 1.10** (Hyperbolic)**.** 1. There are infinitely many triples $(p, q, r)$ such that

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi.$$

Hence there are infinitely many hyperbolic triangle groups.

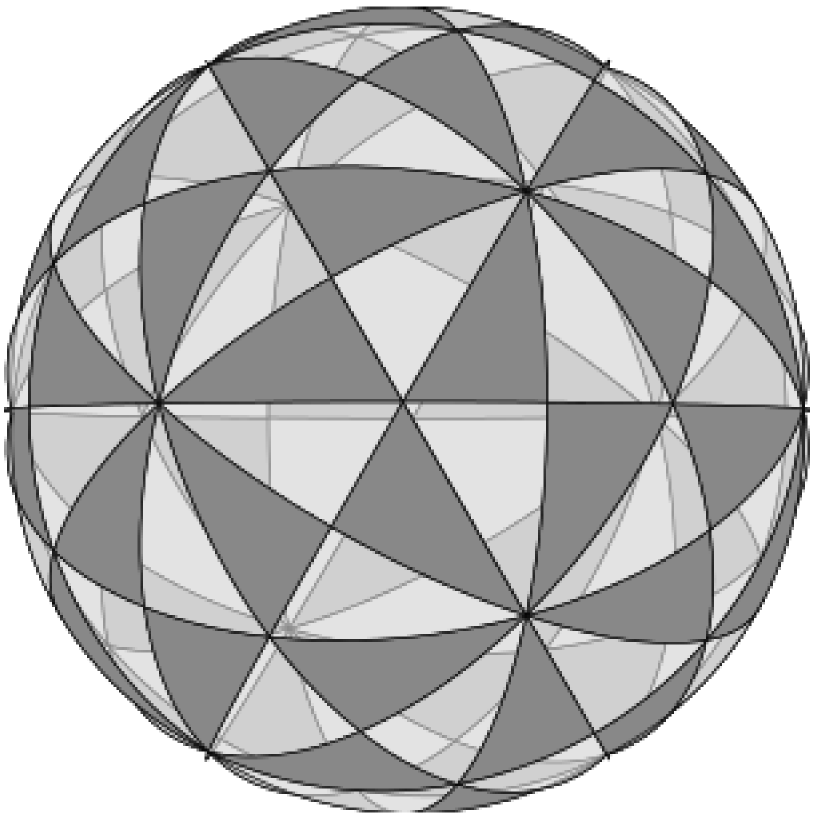


Figure 1.4: Spherical tiling induced by symmetry group of the icosahedron.

2. In $\mathbb{H}^2$ there are right-angled $p$-gons for $p \geq 5$. Here

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, & \text{if } |i-j| = 1 \\ \infty, & \text{else} \end{cases}$$

Now $W$ induces a tessellation of $\mathbb{H}^2$; see Figure 1.9.

3. In $\mathbb{H}^3$, $P$ could be a dodecahedron with all dihedral angles $\frac{\pi}{2}$; see Figure 1.10.

**Definition 1.11** (Tits, 1950s)**.** Let $S = \{s_i\}_{i \in I}$ be a finite set. Let $M = (m_{ij})_{i,j \in I}$ be a matrix such that

- $m_{ii} = 1 \; \forall i \in I$,
- $m_{ij} = m_{ji} \; \forall i \neq j$, and
- $m_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\} \; \forall i \neq j$.

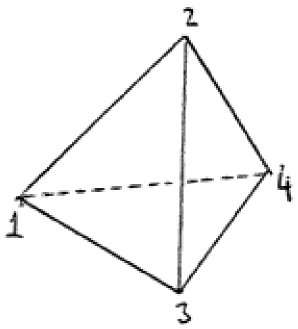



Figure 1.5: Simplex

Then $M$ is called a *Coxeter matrix*. The associated *Coxeter group* $W = W_M$ is defined by the presentation

$$W = \langle S \,|\, (s_i s_j)^{m_{ij}} \,\forall i, j \rangle.$$

The pair $(W, S)$ is called a *Coxeter system*.

*Remark* 1.

1. Theorem 1.5 says that geometric reflection groups are Coxeter groups. So all examples above are Coxeter groups.
2. A finite Coxeter groups is sometimes called a spherical Coxeter group. The reason is, that all finite Coxeter groups can be realised as geometric reflection groups with $\mathbb{X}^n = \mathbb{S}^n$.
3. In the next lecture we'll show:
   - all $s_i$'s are pairwise distinct,
   - each $s_i$ has order 2, and
   - each $s_i s_j$ has order $m_{ij}$.

   Also we'll construct an embedding $W \hookrightarrow \mathrm{GL}(N, \mathbb{R})$, where $N = |S|$. This gives us our first geometric realisation for a general Coxeter group.
4. Coxeter groups arise in Lie theory as Weyl groups of root systems, e.g.
   a) type $A_2$ root system has Weyl group

   $$W = \langle s_\alpha \,|\, \alpha \text{ in the root system} \rangle,$$

   where $s_\alpha$ is the reflection in the hyperplane orthogonal to $\alpha$, i.e.

   $$W = \langle s_1, s_2 \,|\, s_i^2 = 1, (s_1 s_2)^3 = 1 \rangle = D_6 \cong S_3.$$

   See for example Figure 1.11.

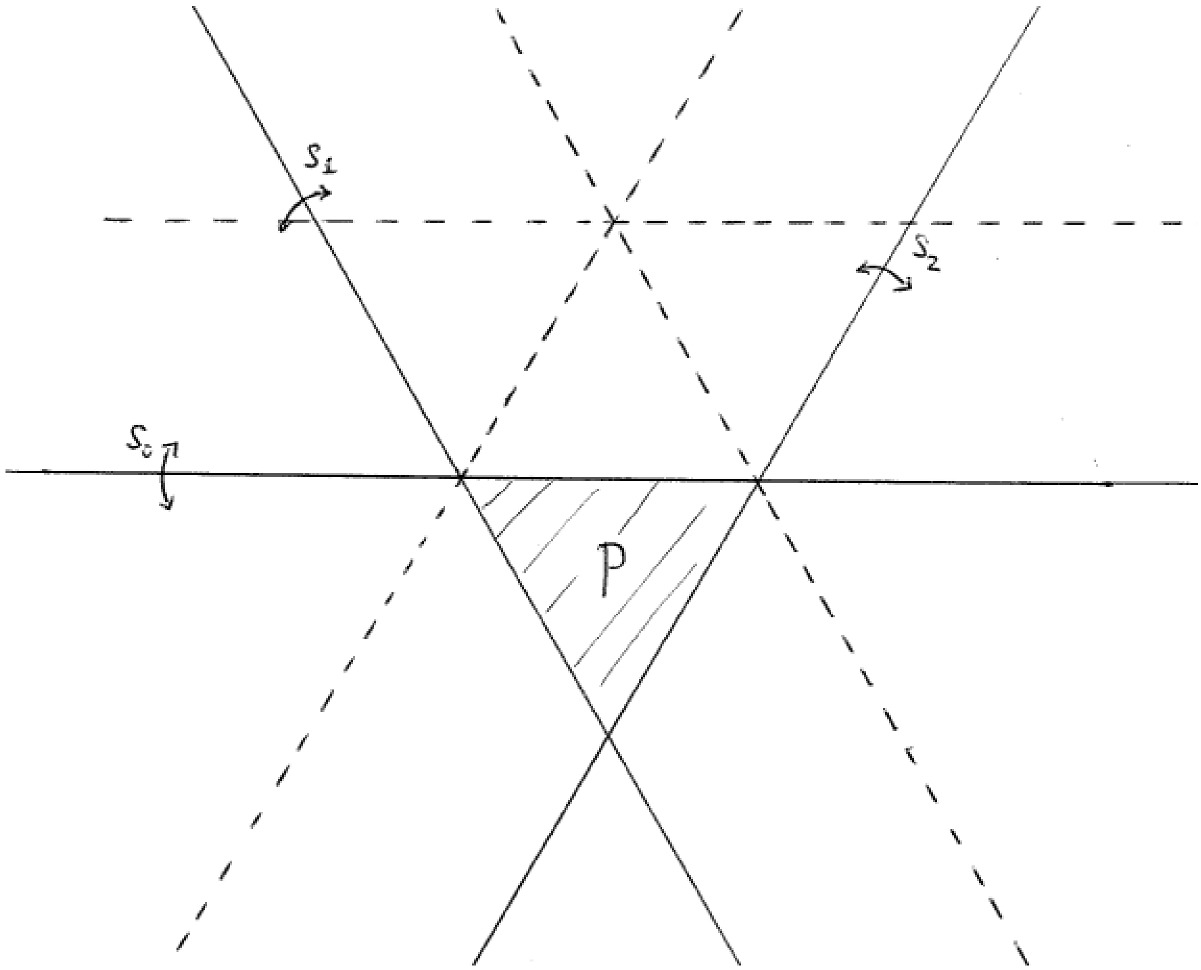



Figure 1.6: $P$ is an equilateral triangle.

b) Euclidean geometric groups can arise as "affine Weyl groups" for algebraic groups over local fields with a discrete valuation, e.g. $\mathrm{SL}_3(\mathbb{Q}_p)$.

The affine Weyl group of type $\tilde{A}_2$ is

$$W = \langle s_0, s_1, s_2 \rangle = \langle s_1, s_2 \rangle \ltimes \mathbb{Z}^2.$$

See for example Figure 1.6. Hence, $\langle s_1, s_2 \rangle$ is the subgroup of $W$ which fixes the origin and $\mathbb{Z}^2$ is the subgroup of $W$ consisting of translations.

c) infinite non-euclidean Coxeter groups can arise as "Kac-Moody Weyl groups".

A Coxeter matrix $M$ satisfies the *crystallographic restriction* if $m_{ij} \in \{2, 3, 4, 6, \infty\}$ for $i \neq j$.

Provided this restriction is satisfied, $W = W_M$ is the Weyl group for some Kac-Moody algebra.

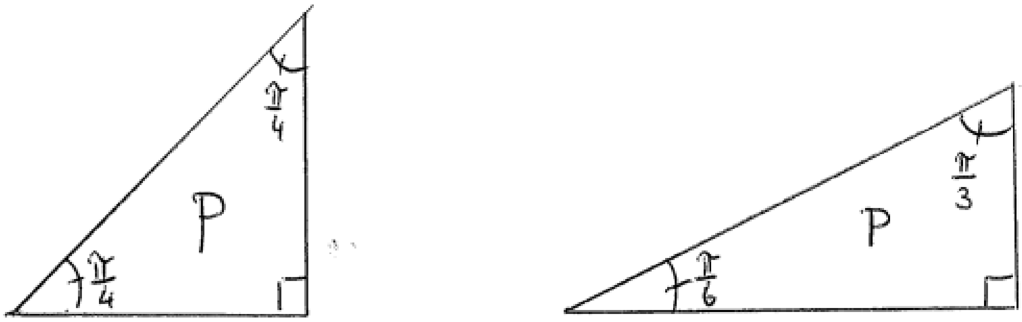



Figure 1.7: Each $P$ tiles $\mathbb{E}^2$.

5. Tits formulated the general definition of a Coxeter group in order to formulate the definition of a building.

## 1.2 Buildings

**Definition 1.12.** A *polyhedral complex* is a finite-dimensional CW-complex in which each $n$-cell is metrised as a convex polytope in $\mathbb{X}^n$ ($\mathbb{X}^n$ should be the same for each cell) and the restriction of each attaching map to a codimension-one face is an isometry. We will discuss later conditions under which a polyhedral complex is a metric space.

**Example 1.13.**

- the tessellation of $\mathbb{X}^n$ by copies of $P$; see Figure 1.6 or Figure 1.8.
- a simplicial tree; see Figure 1.12.

**Definition 1.14.** Let $P = P^n$ be as in Theorem 1.5 above, $S = \{s_i\}, W = \langle S \rangle$. A *building of type* $(W, S)$ is a polyhedral complex $\Delta$, which is a union of subcomplexes called *apartments*. Each apartment is isometric to the tiling of $\mathbb{X}^n$ by copies of $P$, and each such copy of $P$ is called a *chamber*. The apartments and chambers satisfy:

1. Any two chambers are contained in a common apartment.
2. Given any two apartments $A$ and $A'$, there is an isometry $A \to A'$ fixing $A \cap A'$ pointwise.

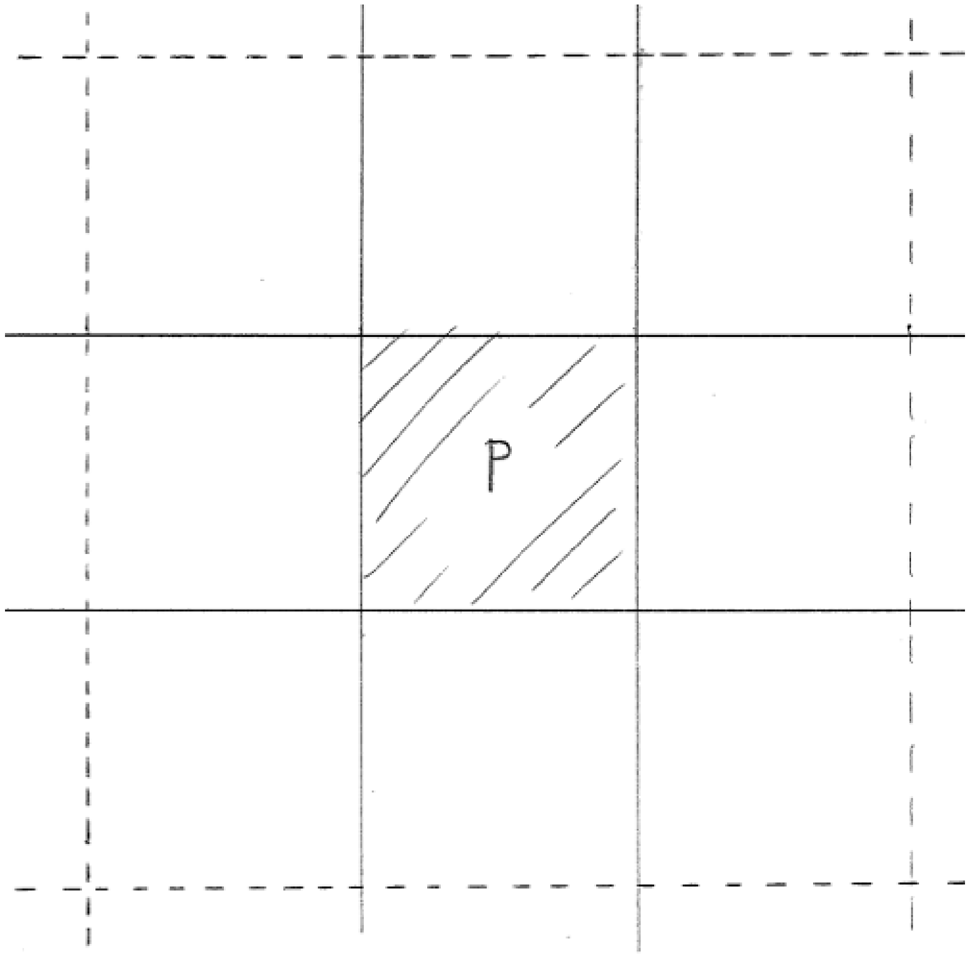



Figure 1.8: $P$ is a square.

**Example 1.15.** 1. A single copy of $\mathbb{X}^n$ tiled by copies of $P$ is a *thin building*, i.e. there is a single apartment.

2. Spherical:

   Let us consider

   $$W = \langle s_1, s_2 \,|\, s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle \cong D_4.$$

   Then there is a (thin) spherical building of type $(W, S)$ as depicted in Figure 1.13. Hereby each edge is a chamber and the only apartment is actually the complete bipartite graph $K_{2,2}$.

   However there is also a thick building of type $(W, S)$ given by $K_{3,3}$; see Figure 1.14.

3. Euclidean:

If we consider $W = D_\infty$ as in Example 1.2, we get an apartment as depicted in Figure 1.15. We can now put these together to the regular three-valent tree (see Figure 1.12) and get a Euclidean building.

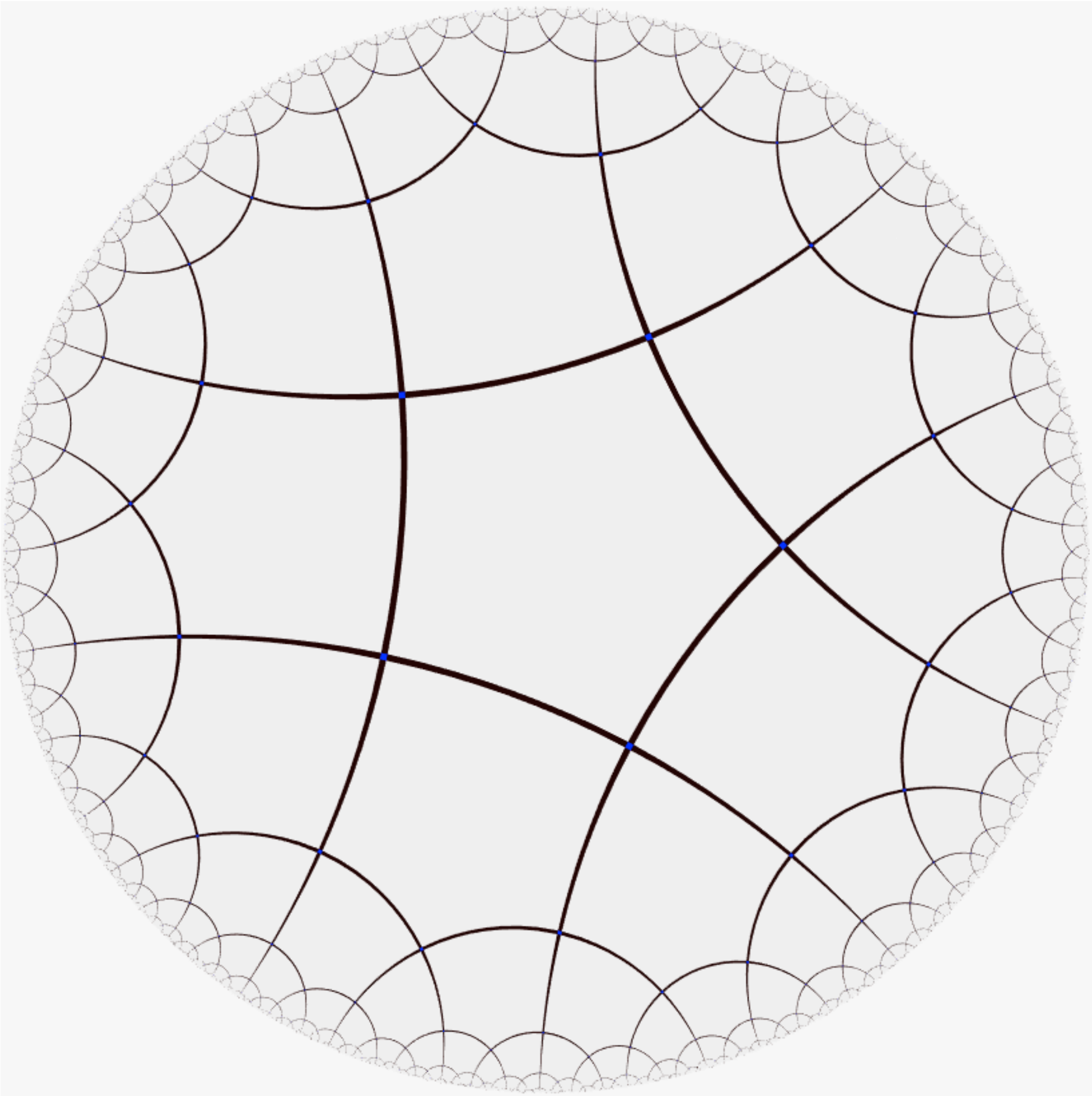


Figure 1.9: $P$ is a right-angled pentagon in $\mathbb{H}^2$. This image was created by Jeff Weeks' free software KaleidoTile.

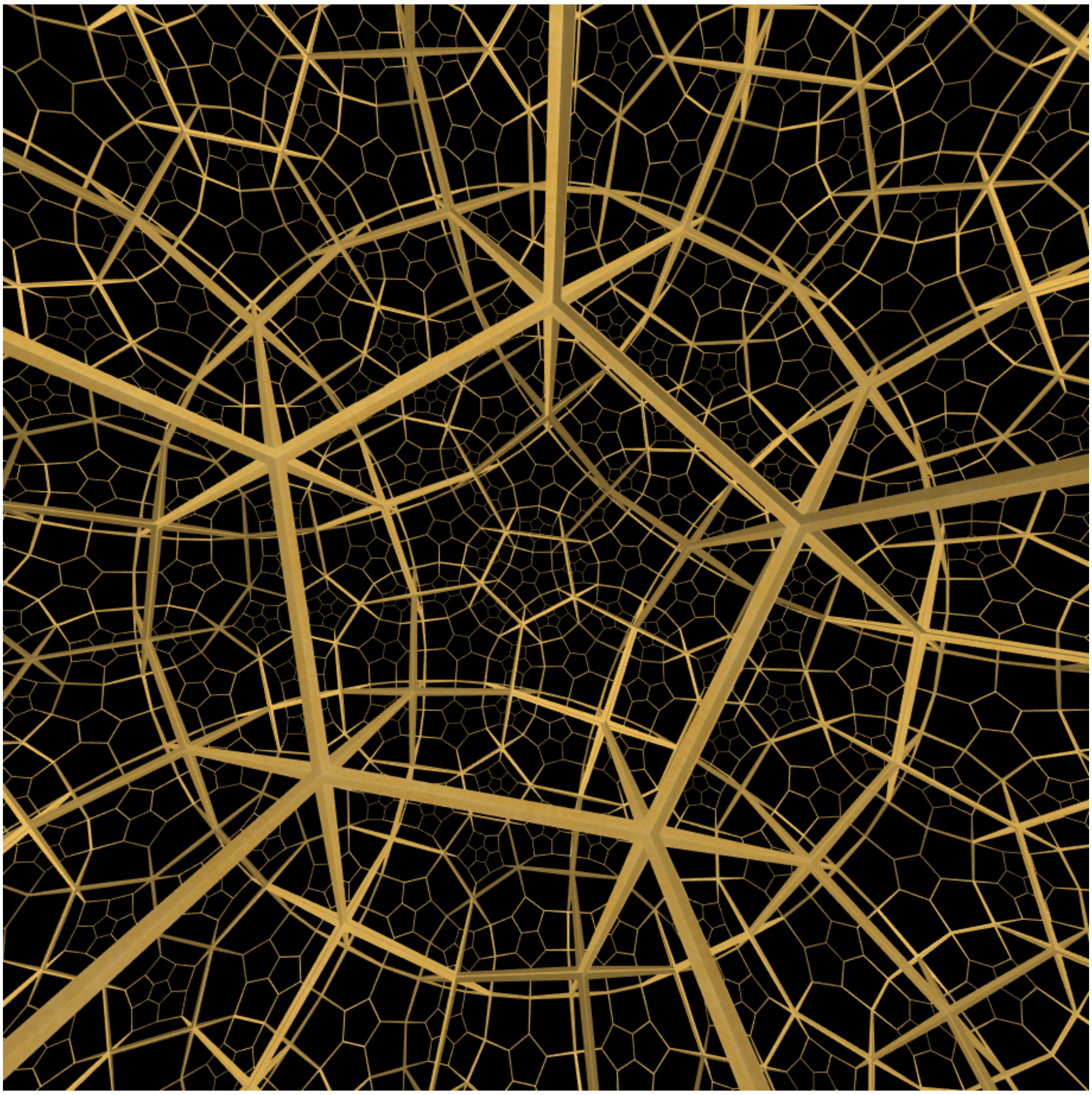


Figure 1.10: $P$ is a dodecahedron in $\mathbb{H}^3$ with all dihedral angles $\pi/2$.. This image was created by Jeff Weeks' free software CurvedSpaces.

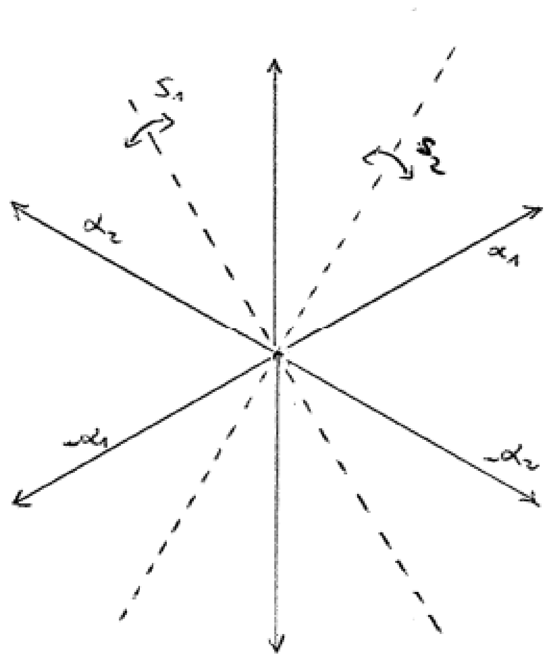



Figure 1.11: Coxeter groups as Weyl group of the root system $A_2$.

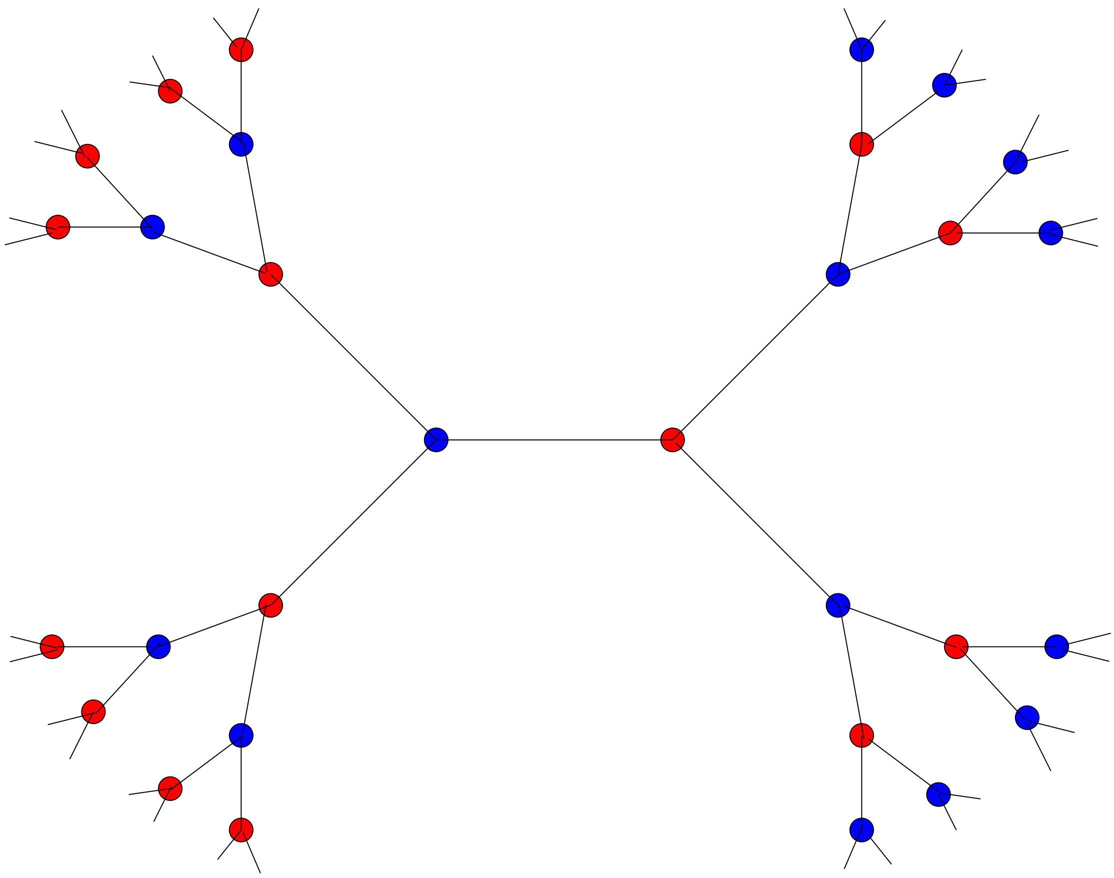


Figure 1.12: The three-valent regular tree.

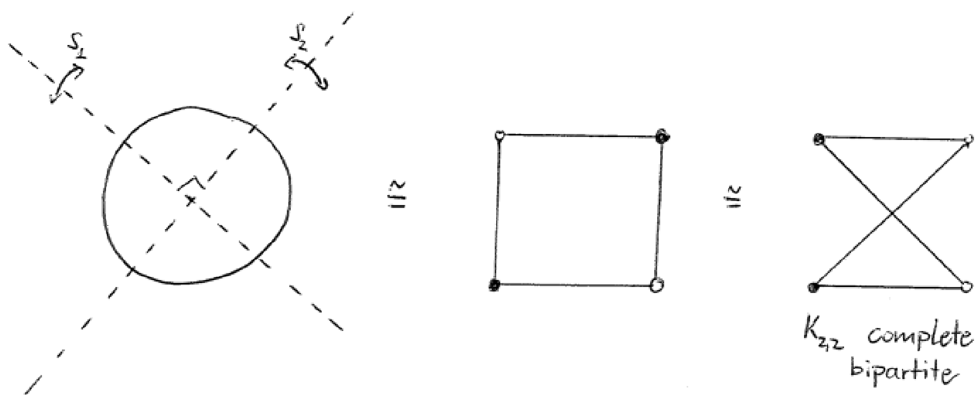



Figure 1.13: The graph $K_{2,2}$ as a spherical building.

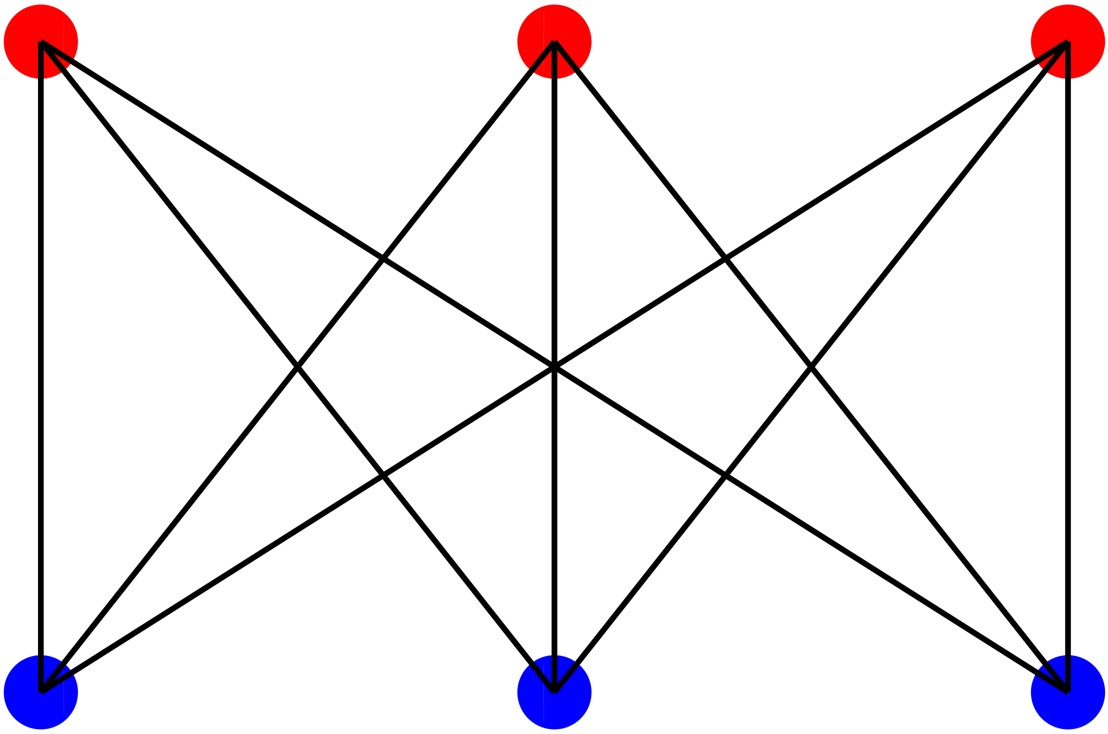


Figure 1.14: The graph $K_{3,3}$.

Figure 1.15: One apartment of the three-valent tree regarded as a euclidean building.

# LECTURE 2

# SOME COMBINATORIAL THEORY OF COXETER GROUPS

09.03.2016

Let $G$ be a group generated by a set $S$ with $1 \notin S$.

## 2.1 Word metrics and Cayley graphs

**Definition 2.1.** The *word length* with respect to $S$ is

$$\ell_S(g) = \min\{n \in \mathbb{N}_0 \mid \exists s_1, \ldots, s_n \in S \cup S^{-1} \text{ such that } g = s_1 \ldots s_n\}.$$

If $\ell_S(g) = n$ and $g = s_1 \ldots s_n$ then the word $(s_1, \ldots, s_n)$ is a *reduced expression* for $g$.

The *word metric* on $G$ with respect to $S$ is $d_S(g, h) = \ell_s(g^{-1}h)$.

**Definition 2.2.** The *Cayley graph Cay*$(G, S)$ of $G$ with respect to $S$ is the graph with vertices $V = G$ and (directed) edges

$$E = \{(g, gs) \mid g \in G, s \in S\}.$$

However, if $s$ is an involution (i.e. has order 2), we will put a single undirected edge labelled by $s$.

**Example 2.3.**

1. The Cayley graph of $D_6 = \langle s_1, s_2 \mid s_i^2 = 1, (s_1 s_2)^3 = 1\rangle$ is depicted in Figure 2.1.
2. The Cayley graph of $D_\infty = \langle s_1, s_2 \mid s_i^2 = 1\rangle$ is depicted in Figure 2.2.
3. If $W$ is the $(3, 3, 3)$ triangle group, Cay$(W, S)$ is the dual graph to the tesselation of $\mathbb{R}^2$ by equilateral triangles; see Figure 2.3.
4. If $W$ is generated by the reflections in the sides of a square, Cay$(W, S)$ is depicted in Figure 2.4.

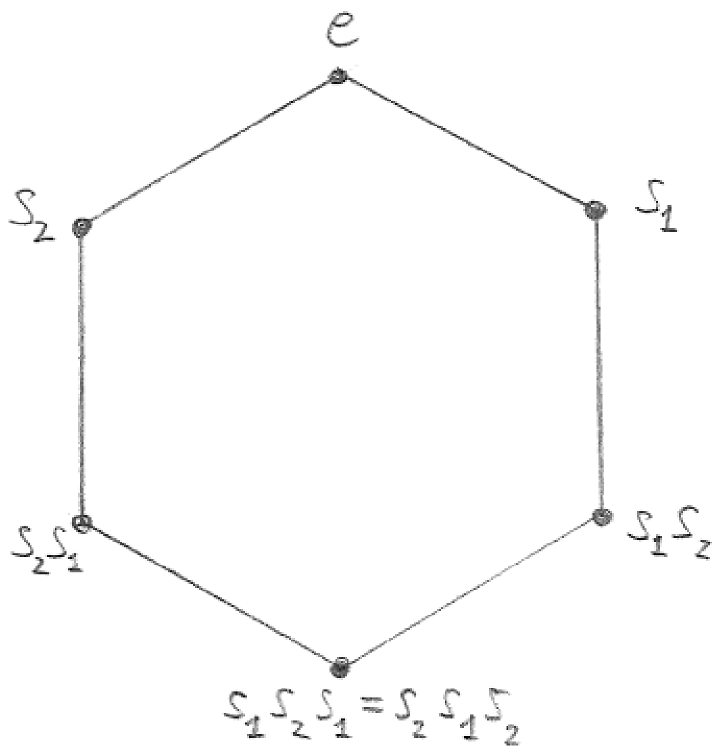



Figure 2.1: Cayley graph of $D_6$.

Since $S$ generates $G$, $\mathrm{Cay}(G, S)$ is connected. The word metric $d_S(\cdot, \cdot)$ on $G$ extends to the path metric on $\mathrm{Cay}(G, S)$. Note that $G$ acts on $\mathrm{Cay}(G, S)$ on the left by graph automorphisms.

This action is also isometric with respect to $d_S(\cdot, \cdot)$:

$$d_S(hg, hg') = \ell_S((hg)^{-1}hg') = \ell_S(g^{-1}g') = d_S(g, g')$$

If $s \in S$ is an involution, the group element $gsg^{-1}$ flips the edge $(g, gs) \leftrightarrow (gs, g)$ onto itself. In fact, $gsg^{-1}$ is the unique group element which does this, since $hg = gs$ if and only if $h = gsg^{-1}$.

## 2.2 Coxeter systems

Recall from the first lecture the following definition (cf. Definition 1.11): A Coxeter matrix $M = (m_{ij})_{i,j \in I}$ has $m_{ii} = 1$, $m_{ij} = m_{ji} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ if $i \neq j$.

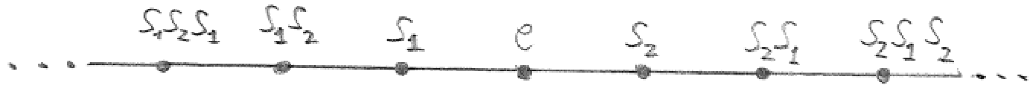



Figure 2.2: Cayley graph of $D_\infty$.

The corresponding Coxeter group is

$$W = \langle S = \{s_i\}_{i \in I} \,|\, (s_i s_j)^{m_{ij}} = 1 \rangle$$

and $(W, S)$ is called a Coxeter system.

**Lemma 2.4.** Let $(W, S)$ be a Coxeter system. Then there is an epimorphism

$$\varepsilon : W \to \{-1, 1\}$$

induced by $\varepsilon(s) = -1$ for all $s \in S$.

**Corollary 2.5.** Each $s \in S$ is an involution.

**Corollary 2.6.** Write $\ell = \ell_S$. Then $\forall w \in W, s \in S : \ell(ws) = \ell(w) \pm 1$ and $\ell(sw) = \ell(w) \pm 1$.

**Theorem 2.7** (Tits)**.** *Let $(W, S)$ be a Coxeter system. Then there is a faithful representation*

$$\rho : W \to GL(N),$$

*where $N = |S|$, such that:*

- *$\rho(s_i) = \sigma_i$ is a linear involution with fixed set a hyperplane. (This is NOT necessarily an orthogonal reflection!)*
- *If $s_i, s_j$ are distinct then $\sigma_i \sigma_j$ has order $m_{ij}$.*

**Corollary 2.8.** In a Coxeter system $(W, S)$ the elements of $S$ are distinct involutions in $W$.

*Proof of Theorem 2.7.* Let $V$ be a vector space over $\mathbb{R}$ with basis $\{e_1, \ldots, e_N\}$. Now define a symmetric bilinear form $B$ by

$$B(e_i, e_j) = \begin{cases} -\cos(\frac{\pi}{m_{ij}}), & \text{if } m_{ij} \text{ is finite} \\ -1, & \text{if } m_{ij} = \infty \end{cases}.$$

Note that $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ if $i \neq j$.

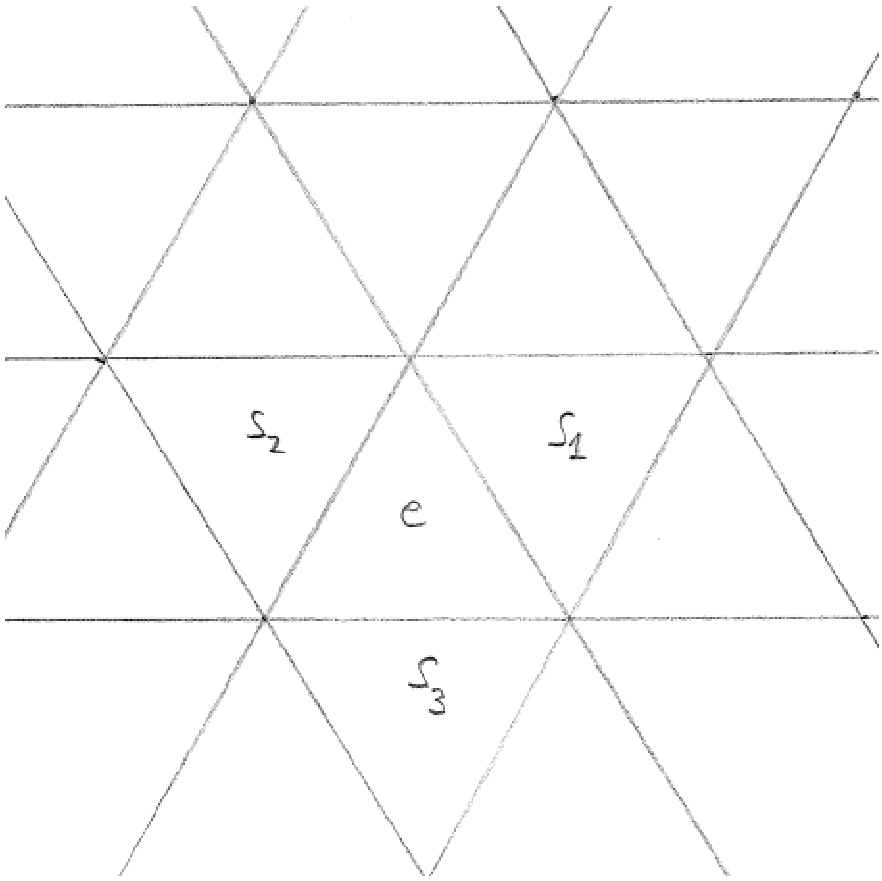



Figure 2.3: Tesselation of $\mathbb{R}^2$ by the $(3, 3, 3)$ triangle group.

Let us consider the hyperplane $H_i = \{v \in V \mid B(e_i, v) = 0\}$, and $\sigma_i : V \to V$ given by

$$\sigma_i(v) = v - 2B(e_i, v)e_i.$$

It is easy to check, that $\sigma_i(e_i) = -e_i$, $\sigma_i$ fixes $H_i$ pointwise, $\sigma_i^2 = \mathrm{id}$, and that $\sigma_i$ preserves $B(\cdot, \cdot)$. The theorem will then follow from the following two claims, whose proofs we postpone for now.

**Claim 1:** The map $s_i \mapsto \sigma_i$ extends to a homomorphism $\rho : W \to \mathrm{GL}(V)$.

**Claim 2:** $\rho$ is faithful.

□

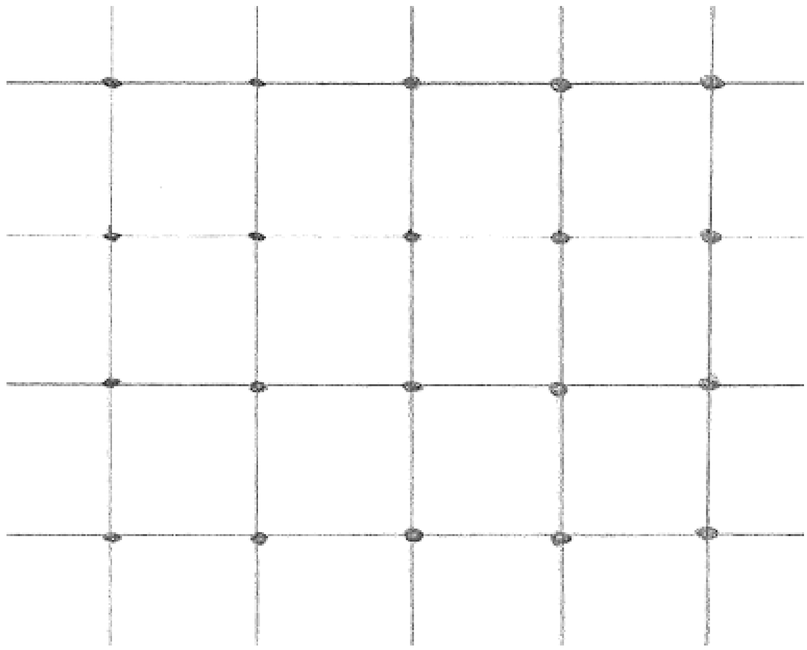


Figure 2.4: Cayley graph of the group generated by the reflection in the sides of a square.

## 2.3 Reflection Systems

**Definition 2.9.** A *pre-reflection system* for a group $G$ is a pair $(X, R)$, where $X$ is a connected simplicial graph, $G$ acts on $X$, and $R$ is a subset of $G$, such that:

1. each $r \in R$ is an involution;
2. $R$ is closed under conjugation, i.e. $\forall g \in G \, \forall r \in R : grg^{-1} \in R$;
3. $R$ generates $G$;
4. for every edge $e$ in $X$ there is a unique $r = r_e \in R$, which flips e; and
5. for every $r \in R$ there is at least one edge $e$ in $X$, which is flipped by $r$.

**Example 2.10.** If we consider again $W = D_6$, $X = \mathrm{Cay}(G, S)$, and set $R = \{s_1, s_2, s_1s_2s_1\}$, then we get the situation depicted in Figure 2.5.

**Example 2.11.** If $(W, S)$ is any Coxeter system, let $X = \mathrm{Cay}(W, S)$ and set $R = \{wsw^{-1} \mid w \in W, s \in S\}$. Then $(X, R)$ is a pre-reflection system; indeed $wsw^{-1}$ flips the edge $(w, ws)$.

**Definition 2.12.** Let $(X, R)$ be a pre-reflection system. For each $r \in R$, the *wall* $H_r$ is the set of midpoints of edges which are flipped by $r$. A pre-reflection system $(X, R)$ is a *reflection system*, if in addition

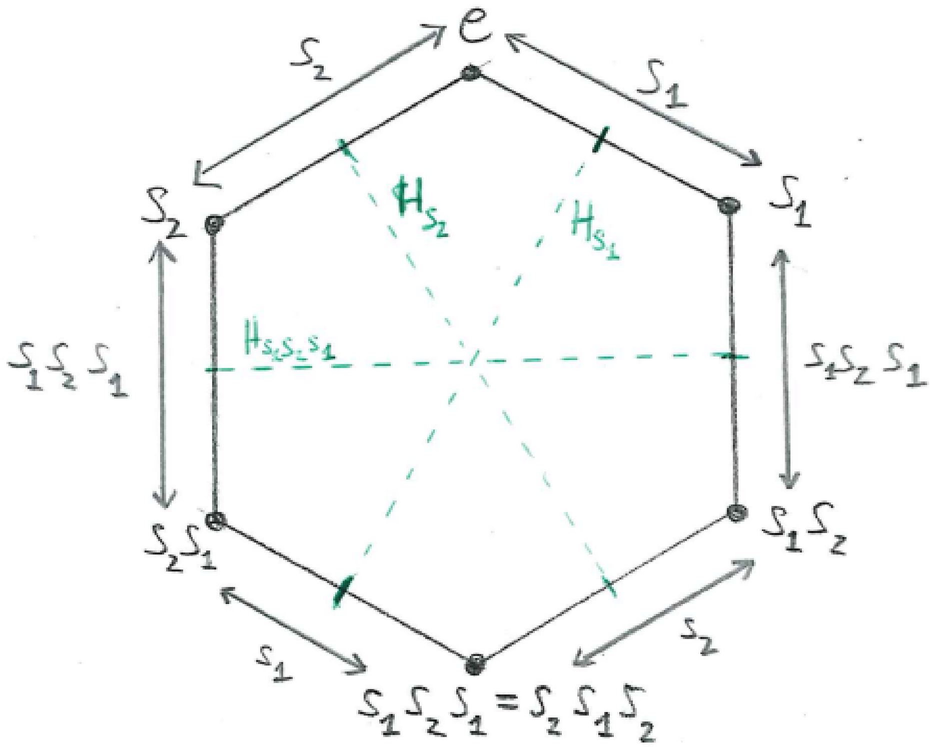



Figure 2.5: Cayley graph of $D_6$ with the reflections $R = \{s_1, s_2, s_1s_2s_1\}$ and corresponding walls $H_r$ for $r \in R$.

6. for each $r \in R$, $X \backslash H_r$ has exactly two components. (These will be interchanged by $r$).

We call $R$ the set of *reflections*.

**Theorem 2.13.** *Suppose a group $W$ is generated by a set of distinct involutions $S$. Then the following are equivalent:*

1. *$(W, S)$ is a Coxeter system;*
2. *if $X = Cay(W, S)$ and $R = \{wsw^{-1} \,|\, w \in W, s \in S\}$, then $(X, R)$ is a reflection system;*
3. *$(W, S)$ satisfies the* Deletion Condition*:*

*if* $(s_1, \dots, s_k)$ *is a word in* $S$ *with* $\ell(s_1, \dots, s_k) < k$*, then there are* $i < j$*, such that*

$$s_1 \dots s_k = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k,$$

*where* $\hat{s}_i$ *means, we delete this letter;*

*4.* $(W, S)$ *satisfies the* Exchange Condition*:*
*if* $(s_1, \dots, s_k)$ *is a reduced expression for* $w \in W$ *and* $s \in S$*, either* $\ell(sw) = k + 1$ *or there is an index* $i$*, such that*

$$s_1 \dots s_k = ss_1 \dots \hat{s}_i \dots s_k.$$

*Proof.* We will sketch 1. $\implies$ 2. $\implies$ 3. $\implies$ 4. $\implies$ 1.

**1.** $\implies$ **2.:** There is a bijection

$$\{\text{words in } S\} \longleftrightarrow \{\text{paths in } X = \text{Cay}(W, S) \text{ starting at } e\}$$

mapping a word $(s_1, \dots, s_k)$ to the path with vertices $e$, $s_1$, $s_1s_2$, $s_1s_2s_3$, $\dots$, $s_1s_2 \dots s_k$; see Figure 2.6.

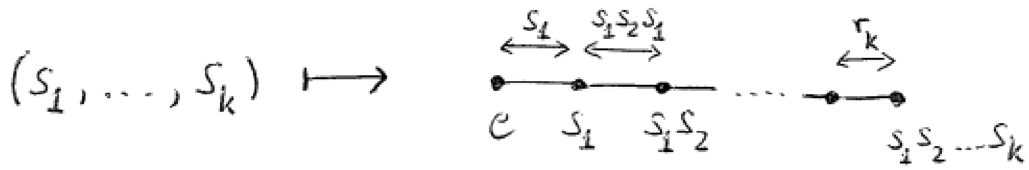


Figure 2.6: The map sending a word $(s_1, \dots, s_k)$ to the path $e$, $s_1$, $s_1s_2$, $s_1s_2s_3$, $\dots$, $s_1s_2 \dots s_k$.

The word $(s_1, \dots, s_k)$ has a canonical associated sequence of reflections

$$\begin{aligned} r_1 &= s_1 \\ r_2 &= s_1s_2s_1 \\ r_3 &= s_1s_2s_3s_2s_1 \\ &\vdots \end{aligned}$$

Further we have the following key lemma.

**Lemma 2.14.** If $w \in W$ and $r \in R$, any word for $w$ crosses $H_r$ the same number of times mod 2, i.e. if $\underline{s}, \underline{s}'$ are words for $w$, and $n(r, \underline{s}), n(r, \underline{s}')$ are the number of times, these paths cross $H_r$, then $(-1)^{n(r,\underline{s})} = (-1)^{n(r,\underline{s}')}$.

*Proof.* Define a homomorphism $\varphi : W \to \mathrm{Sym}(R \times \{-1, 1\})$ by extending $\varphi(s)(r, \varepsilon) = (srs, \varepsilon(-1)^{\delta_{rs}})$, $\varepsilon \in \{\pm 1\}$. □

We can use this lemma to show that each $H_r$ separates $\mathrm{Cay}(W, S)$; namely, $w$ and $w'$ are on the same side of $H_r$ if and only if any path from $w$ to $w'$ crosses $H_r$ an even number of times.

**2. $\Longrightarrow$ 3.:**

**Lemma 2.15.** Let $(s_1, \ldots, s_k)$ be a word in $S$ with associated reflections $(r_1, \ldots, r_k)$ as above. If $r_i = r_j$ for $i < j$, then $s_1 \ldots s_k = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_k$.

*Proof.* Let $r = r_i = r_j$ and let $w_k := s_1 \ldots s_k$ for each $k$. If we now apply $r$ to the subpath from $w_i$ to $w_{j-1}$, we get a new path of the type

$$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j-1}, s_{j+1}, \ldots s_k),$$

as depicted in Figure 2.7.

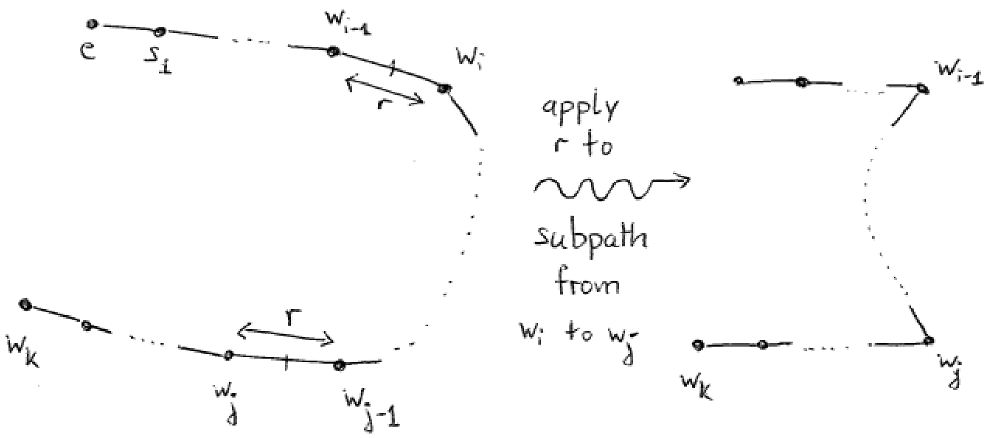



Figure 2.7: Applying the reflection $r$ to the subpath from $w_i$ to $w_{j-1}$.

□

**Lemma 2.16.** If $(s_1, \ldots, s_k)$ is a word in $S$ with associated reflections $(r_1, \ldots, r_k)$ as above, then this word is a reduced expression if and only if the $r_i$ are pairwise distinct.

*Proof.* If some $r_i = r_j$, the word is non-reduced by the previous lemma. Now let $w = s_1 \ldots s_k$ and $R(e, w) = \{r \mid H_r \text{ separates } e \text{ from } w\}$. Then

$$r \in R(e, w) \implies r = r_i \text{ for some } i \implies \ell(w) \geq |R(e, w)|.$$

Hence if all $r_i$ are pairwise distinct, then $|R(e, w)| \geq k$. On the other hand $\ell(w) \leq k$, and so $\ell(w) = k$, i.e. the word is reduced. The Deletion Condition follows. □

**3. $\implies$ 4.:** Suppose $(s_1, \ldots, s_k)$ is a reduced word and $s \in S$. We set $w = s_1 \ldots s_k$. If $\ell(sw) = k + 1$, there is nothing to show. Hence let us assume that $\ell(sw) \leq k$. In this case $(s, s_1, \ldots, s_k)$ is non-reduced and by the Deletion Condition we can delete two letters. However $(s_1, \ldots, s_k)$ is reduced, such that one of the two letters has to be $s$. Thus

$$\begin{aligned} ss_1 \ldots s_k = s_1 \ldots \hat{s}_i \ldots s_k &\implies sw = s_1 \ldots \hat{s}_i \ldots s_k \\ &\implies sw = ss_1 \ldots \hat{s}_i \ldots s_k. \end{aligned}$$

**4. $\implies$ 1.:** We will use Tits' solution to the word problem in Coxeter groups.

**Definition 2.17.** Suppose $W$ is generated by a set of distinct involutions $S$. If $s, t \in S$, $s \neq t$, let $m_{st}$ be the order of $st$ in $W$. If $m_{st}$ is finite, a *braid move* on a word in $S$ replaces a subword $(s, t, s, \ldots)$ by a subword $(t, s, t, \ldots)$, where each subword has $m_{st}$ letters. For example in $D_6$: $(s_1, s_2, s_1) \leftrightarrow (s_2, s_1, s_2)$.

**Theorem 2.18** (Tits)**.** *Suppose a group $W$ is generated by a set of distinct involutions and the Exchange Condition holds. Then:*

1. *A word $(s_1, \ldots, s_k)$ is reduced if and only if it cannot be shortened by a sequence of*
   - *deleting a subword $(s, s)$, $s \in S$; or*
   - *carrying out a braid move.*
2. *Two reduced expressions represent the same group element if and only if they are related by a sequence of braid moves.*

*Proof.* Use induction on $\ell(w)$. Prove 2., then 1. □

To show $(W, S)$ is a Coxeter system: Let $m'_{ij}$ be the order of $s_i s_j$ in $W$. Further let $W'$ be the Coxeter group with Coxeter matrix $M' = (m'_{ij})$. Finally use Theorem 2.18 to show that $W' \to W$ is injective. □

**Definition 2.19.** For each $T \subseteq S$, the *special subgroup* $W_T$ of $W$ is $W_T = \langle T \rangle$. Sometimes these are also called *parabolic subgroups* or *visual subgroups*. We shall also use the alternative notation: if $J \subseteq I$, $W_J = \langle s_j \mid j \in J \rangle$. If $T = \emptyset$, we define $W_\emptyset$ to be the trivial group.

Using Theorem 2.13 we can show, that for each $T \subseteq S$:

1. $(W_T, T)$ is a Coxeter system.

2. For every $w \in W_T$, $\ell_T(w) = \ell_S(w)$, and any reduced expression for $w$ only uses letters in $T$, i.e. $\mathrm{Cay}(W_T, T)$ embeds isometrically as a convex subgraph of $\mathrm{Cay}(W, S)$.

3. If $T, T' \subseteq S$, then $W_T \cap W_{T'} = W_{T \cap T'}$. There is a bijection

$$\{\text{subsets of } S\} \longleftrightarrow \{\text{special subgroups}\},$$

which preserves inclusion.

# LECTURE 3

# THE TITS REPRESENTATION

16.03.2016

## 3.1 Proof of Tits' representation theorem

We will now return the proof of Theorem 2.7. Let us briefly recall the statement and what we have said so far.

**Theorem** (Theorem 2.7 (Tits))**.** *Let $(W, S)$ be a Coxeter system. Then there is a faithful representation*

$$\rho : W \to GL(N, \mathbb{R}),$$

*where $N = |S|$, such that:*

- *$\rho(s_i) = \sigma_i$ is a linear involution with fixed set a hyperplane. (This is NOT necessarily an orthogonal reflection!)*
- *If $s_i, s_j$ are distinct then $\sigma_i \sigma_j$ has order $m_{ij}$.*

**Definition 3.1.** This representation is called the *Tits representation*, or the standard (geometric) representation.

*Continuation of the proof.* So far we had the following. Let $V$ be a vector space over $\mathbb{R}$ with basis $\{e_1, \ldots, e_N\}$. Now define a symmetric bilinear form $B$ by

$$B(e_i, e_j) = \begin{cases} -\cos(\frac{\pi}{m_{ij}}), & \text{if } m_{ij} \text{ is finite} \\ -1, & \text{if } m_{ij} = \infty. \end{cases}$$

Note that $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ if $i \neq j$.

Let us consider the hyperplane $H_i = \{v \in V \mid B(e_i, v) = 0\}$, and $\sigma_i : V \to V$ given by

$$\sigma_i(v) = v - 2B(e_i, v)e_i;$$

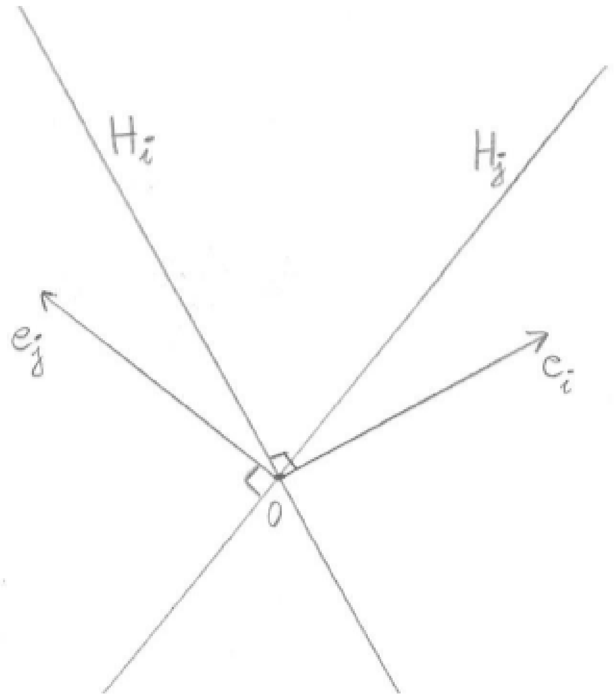



Figure 3.1: The basis vectors $e_i$, $e_j$ and their corresponding hyperplanes $H_i$, $H_j$.

see Figure 3.1.

It is easy to check that $\sigma_i(e_i) = -e_i$, $\sigma_i$ fixes $H_i$ pointwise, $\sigma_i^2 = \text{id}$, and that $\sigma_i$ preserves $B(\cdot,\cdot)$. The theorem will then follow from the following two claims.

**Claim 1:** The map $s_i \mapsto \sigma_i$ extends to a homomorphism $\rho : W \to \text{GL}(V)$.

*Proof of Claim 1.* It is enough to show that $\sigma_i\sigma_j$ has order $m_{ij}$: Let $V_{ij}$ be the subspace $\text{span}(e_i, e_j)$. Then $\sigma_i$ and $\sigma_j$ preserve $V_{ij}$, so we will consider the restriction of $\sigma_i\sigma_j$ to $V_{ij}$.

**Case I ($m_{ij}$ is finite):** Let $v = \lambda_i e_i + \lambda_j e_j \in V_{ij}$. If $v \neq 0$ then

$$\begin{aligned} B(v,v) &= \lambda_i^2 - 2\lambda_i\lambda_j \cos\left(\frac{\pi}{m_{ij}}\right) + \lambda_j^2 \\ &= \left(\lambda_i - \lambda_j \cos\left(\frac{\pi}{m_{ij}}\right)\right)^2 + \lambda_j^2 \sin\left(\frac{\pi}{m_{ij}}\right)^2 > 0 \end{aligned}$$

So $B$ is positive definite on $V_{ij}$, however not necessarily so on the whole of $V$. Thus we can identify $V_{ij}$ with Euclidean two-space and $B|_{V_{ij}}$ with the standard inner product.

The maps $\sigma_i$ and $\sigma_j$ are now orthogonal reflections. Since

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right) = \cos\left(\pi - \frac{\pi}{m_{ij}}\right),$$

the angle between $e_i$ and $e_j$ (in $V_{ij}$) is $\pi - \frac{\pi}{m_{ij}}$. Hence the dihedral angle between $H_i$ and $H_j$ is $\frac{\pi}{m_{ij}}$ and so $\sigma_i\sigma_j$ is a rotation by the angle $2\frac{\pi}{m_{ij}}$. This shows that $\sigma_i\sigma_j$ has order $m_{ij}$ when restricted to the subspace $V_{ij}$.

Let us now consider $V_{ij}^{\perp} = \{v' \in V \mid B(v', v) = 0 \quad \forall v \in V_{ij}\}$. Since $B$ is positive definite on $V_{ij}$,

$$V = V_{ij} \oplus V_{ij}^{\perp}.$$

Now $\sigma_i\sigma_j$ fixes $V_{ij}^{\perp}$ pointwise. Hence $\sigma_i\sigma_j$ has order $m_{ij}$ on $V$.

**Case II ($m_{ij} = \infty$):** Again let $v = \lambda_i v_i + \lambda_j v_j \in V_{ij}$. Then

$$\begin{aligned} B(v, v) &= \lambda_i^2 - 2\lambda_i\lambda_j + \lambda_j^2 \\ &= (\lambda_i - \lambda_j)^2 \geq 0, \end{aligned}$$

with equality if and only if $\lambda_i = \lambda_j$. So $B$ is positive semi-definite, but not positive definite on $V_{ij}$. Consider

$$\begin{aligned} \sigma_i\sigma_j(e_i) &= \sigma_i(e_i + 2e_j) \\ &= -e_i + 2(e_j + 2e_i) = e_i + 2(e_i + e_j). \end{aligned}$$

By induction we get for all $k \geq 1$:

$$(\sigma_i\sigma_j)^k(e_i) = e_i + 2k(e_i + e_j).$$

Thus $\sigma_i\sigma_j$ has infinite order on $V_{ij}$ and hence also on the whole of $V$. This finishes the proof of our first claim and we have a representation $\rho : W \to GL(N, \mathbb{R})$. □

Before we move on to the proof of the faithfulness of $\rho$ let us discuss the geometry of the second case above. Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

This is the matrix for $B|_{V_{ij}}$ in the basis $\{e_i, e_j\}$ of $V_{ij}$ when $m_{ij} = \infty$. Since $B$ is positive semi-definite, but not positive definite on $V_{ij}$, the matrix $A$ has an one-dimensional nullspace of vectors $v$ such that $B(v, v) = 0$:

$$\text{null}(A) = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \text{span}(e_i + e_j) = \{v \in V_{ij} \mid B(v, v) = 0\}.$$

Thus $B$ induces a positive definite form on $V_{ij}/\text{null}(A)$ and the latter can be identified with one-dimensional Euclidean space. Let $W_{ij} = \langle s_i, s_j \rangle \cong D_\infty$. Note:

1. $W_{ij}$ acts faithfully on $V_{ij}$.
2. We have
$$\sigma_i(e_i + e_j) = \sigma_j(e_i + e_j) = e_i + e_j,$$
so $W_{ij}$ fixes null$(A)$ pointwise.

Now consider the dual vector space
$$V_{ij}^* = \{\text{linear functionals } \varphi : V_{ij} \to \mathbb{R}\}.$$
The group $W_{ij}$ acts on $V_{ij}^*$ via $(w \cdot \varphi)(v) = \varphi(w^{-1} \cdot v)$ $(w \in W_{ij},\ \varphi \in V_{ij}^*,\ v \in V_{ij})$ and this action is faithful because the original one was. So we have a faithful action of $D_\infty$.

Consider the codimension-one subspace of $V_{ij}^*$
$$Z = \{\varphi \in V_{ij}^* \mid \varphi(e_i + e_j) = 0\}.$$
Since $W_{ij}$ fixes $e_i + e_j$, it preserves $Z$.

We may now identify
$$Z \longleftrightarrow (V_{ij}/\text{null}(A))^* .$$

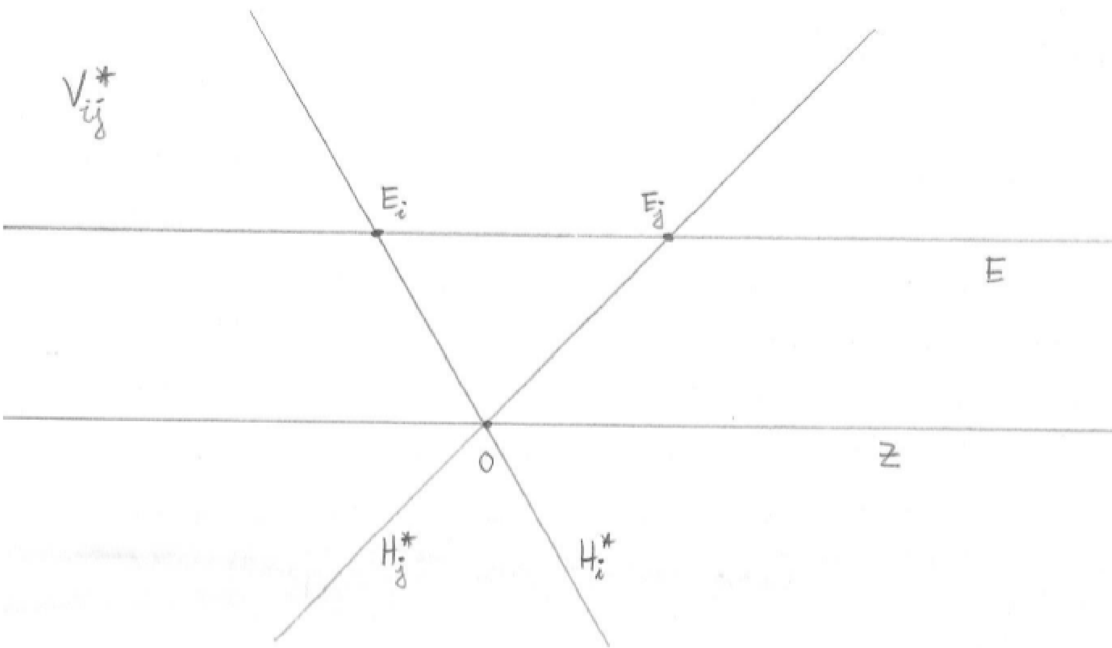



Figure 3.2: The dual space $V_{ij}^*$ with the (affine) subspaces $E$ and $Z$.

So $Z$ has an one-dimensional Euclidean structure as well. Let $E$ be the codimension-one affine subspace
$$E = \{\varphi \in V_{ij}^* \mid \varphi(e_i + e_j) = 1\} = Z + \mathbb{1}.$$

Therefore also $E$ has an one-dimensional Euclidean structure. Since $W_{ij}$ fixes $e_i + e_j$, it stabilizes $E$. Now $E$ spans $V_{ij}^*$ and $W_{ij}$ acts faithfully on $V_{ij}^*$, so the $W_{ij}$-action on $E$ is faithful. Let

$$H_i^* = \{\varphi \in V_{ij}^* \,|\, \varphi(e_i) = 0\}.$$

Then $H_i^* \neq Z$, so $H_i^* \cap E =: E_i \neq \emptyset$ is a codimension-one hyperplane of $E$. The same holds for $j$. Observe that $s_i \cdot e_i = -e_i$, so $s_i$ acts on $E$ as an isometric reflection with fixed hyperplane $E_i$. We get an isometric action of $W_{ij} \cong D_\infty$ on $E$ generated by reflections.

**Claim 2:** $\rho$ is faithful.

*Sketch of proof of Claim 2.* Consider the dual representation $\rho^* : W \to GL(V^*)$ given by

$$(\rho^*(w)(\varphi))(v) = \varphi(\rho(w^{-1})(v))$$

for all $\varphi \in V^*, w \in W, v \in V$.

Define elements $\varphi_i \in V^*$ by $\varphi_i(v) = B(e_i, v)$. Now define

$$H_i^* = \{\varphi \in V^* \,|\, \varphi(e_i) = 0\}.$$

Then $\sigma_i^* := \rho^*(s_i)$ is $\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\varphi_i$. Using this it is easy to check that $\sigma_i^*(\varphi_i) = -\varphi_i$, $(\sigma_i^*)^2 = \mathrm{id}$ and that $\sigma_i^*$ fixes $H_i^*$ pointwise.

Define the *chamber* $C$ by

$$C = \{\varphi \in V^* \,|\, \varphi(e_i) \geq 0 \quad \forall i\}.$$

**Example 3.2.** If $W = D_{2m}$ (Case I), $V^*$ is $\mathbb{E}^2$; see Figure 3.3.

If $W = W_{ij} = D_\infty$ (Case II), we have the situation as in Figure 3.4.

This is the "simplicial cone" cut out by the hyperplanes $H_i^*$. Let

$$\mathring{C} = \mathrm{int}(C) = \{\varphi \in V^* \,|\, \varphi(e_i) > 0\}.$$

**Theorem 3.3** (Tits)**.** *Let $w \in W$. If $w\mathring{C} \cap \mathring{C} \neq \emptyset$, then $w = 1$.*

*Sketch.* Holds for each $W_{ij} = \langle s_i, s_j \rangle$ by Cases I and II above. Use a combinatorial lemma of Tits to promote to $W$ ... □

**Corollary 3.4.** $\rho^*$ is faithful $\implies$ $\rho$ is faithful.

□

This finished the proof of our second claim and hence Theorem 2.7 has been proven.
□

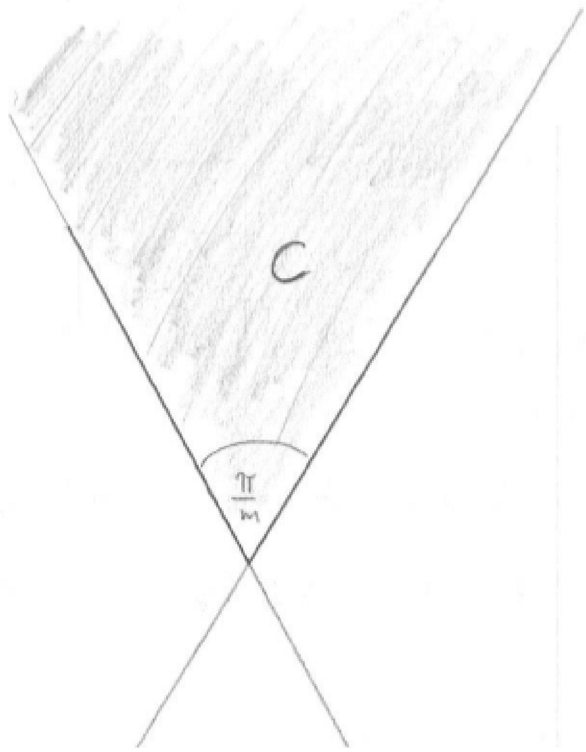



Figure 3.3: The chamber $C$ for $W = D_{2m}$.

## 3.2 Some corollaries of Tits' representation theorem

**Definition 3.5.** The *Tits cone* of $W$ is the subset of $V^*$ given by $\bigcup_{w \in W} wC$ where $C$ is the chamber defined above.

**Example 3.6.**

1. If $W = D_{2m}$, the Tits cone is all of $\mathbb{E}^2$.
2. If $W = D_\infty$, the Tits cone is $\{\varphi \in V_{ij}^* \,|\, \varphi(e_i + e_j) > 0\} \cup \{0\}$.

**Corollary 3.7.** $\rho(W)$ is a discrete subgroup of $GL(N, \mathbb{R})$.

*Proof.* Consider the $W$-action on the interior of the Tits cone. This action has finite point stabilisers. □

**Definition 3.8.** A group $G$ is *linear* (over $\mathbb{R}$) if there is a faithful representation $\varphi : G \to GL(n, \mathbb{R})$ for some $n \in \mathbb{N}$.

**Corollary 3.9.** Coxeter groups and their subgroups are linear.

This is particularly nice because of the following two theorems on linear groups.

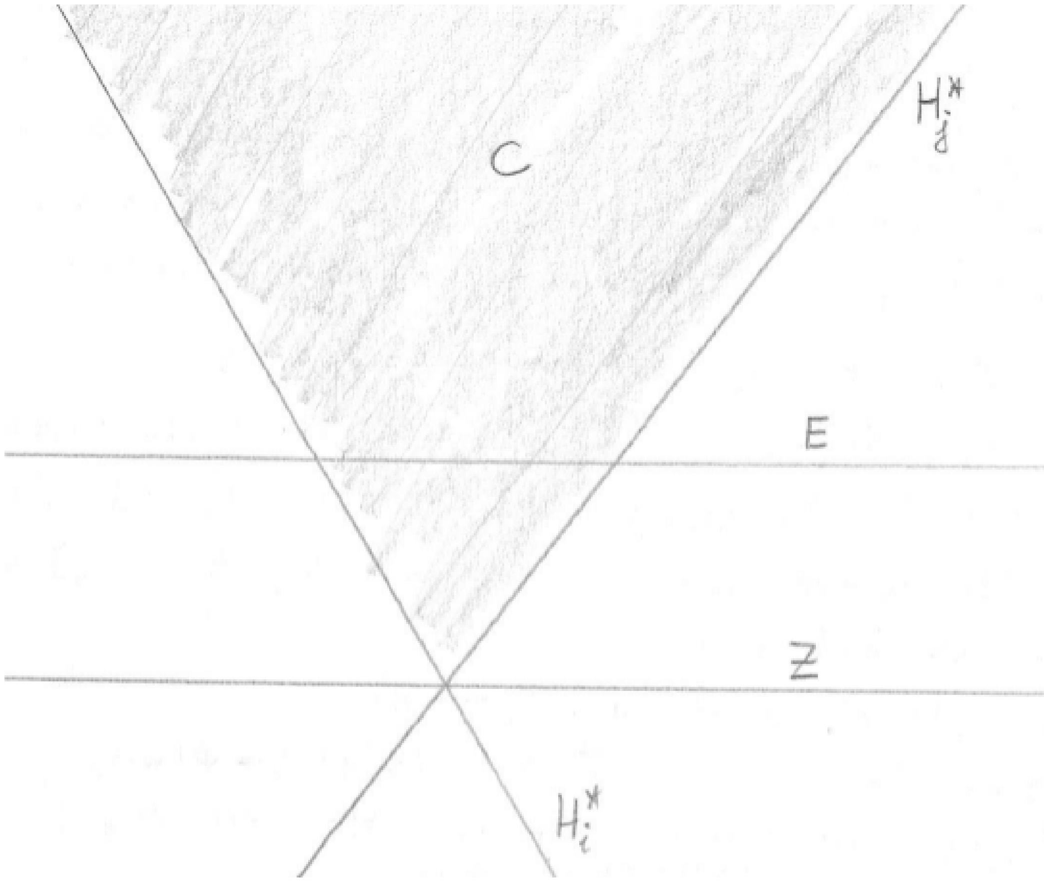



Figure 3.4: The chamber $C$ for $W = D_\infty$.

**Theorem 3.10** (Selberg)**.** *Finitely generated linear groups are virtually torsion-free, i.e. they have a torsion-free subgroup with finite index.*

**Theorem 3.11** (Malcev)**.** *Finitely generated linear groups are residually finite: For every $g \in G, g \neq 1$, there is a finite group $H_g$ and a homomorphism $\varphi : G \to H_g$ such that $\varphi(g) \neq 1$.*

**Definition 3.12.** A Coxeter system $(W, S)$ is *reducible* if $S = S' \sqcup S''$, $S' \neq \emptyset, S'' \neq \emptyset$, such that everything in $S'$ commutes with everything in $S''$, i.e. $m_{ij} = 2 \,\forall s_i \in S', s_j \in S''$. Then $W = \langle S' \rangle \times \langle S'' \rangle = W_{S'} \times W_{S''}$.

$(W, S)$ is *irreducible* if it is not reducible.

**Theorem 3.13.** *Suppose $(W, S)$ is irreducible and $n = |S|$. Then:*

1. *$B$ is positive definite if and only if $W$ is finite. In this case, $W$ is a geometric reflection group (cf. Definition 1.6) generated by reflections in codimension-one faces of a simplex in $\mathbb{S}^n$ with dihedral angles $\frac{\pi}{m_{ij}}$.*

2. *If $B$ is positive semi-definite, then $W$ is a geometric reflection group on $\mathbb{E}^{n-1}$ generated by reflections in codimension-one faces of either an interval if $n = 2$ ($D_\infty$), or a simplex if $n \geq 3$, with dihedral angles $\frac{\pi}{m_{ij}}$.*

*Proof.* To 2.: We can find a codimension-one affine Euclidean subspace $E$ in $V^*$ on which $W$ acts by isometric reflections. If $n \geq 3$, $H_i^*$ and $H_j^*$ meet at an angle of $\frac{\pi}{m_{ij}}$ in $E$. The subspace $E$ is a "slice" across the Tits core. □

*Remark* 2. The positive definite $B$, and the positive semi-definite $B$ but not definite $B$, can be classified using graphs. This gives a classification of irreducible finite Coxeter groups $W$, and of irreducible affine Coxeter groups. This may be found in any book on Coxeter groups and was first done by Coxeter himself.

## 3.3 Geometry for $W$ finite

Let $W$ be finite, $C = \{\varphi \in V^* \mid \varphi(e_i) \geq 0\} \subseteq V^* \cong \mathbb{E}^n$. Now take $x \in \mathring{C}$ and act on $x$ by $W$. The orbit then has $|W|$ points. By regarding its convex hull we get a convex Euclidean polytope (in general not regular), which is stabilised by $W$. In fact, its one-skeleton is isomorphic as a non-metric graph to $\mathrm{Cay}(W, S)$. This polytope is another geometric realisation for $W$. See for example Figure 3.5.

Later on, we will past together these polytopes to get a piecewise Euclidean geometric realisation for arbitrary $W$. This polytope is (depending on who you are talking to) called *Coxeter polytope*, *$W$-permutahedron*, *$W$-associahedron*, *weight polytope*.

## 3.4 Motivation for other geometric realisation

Let $W_n = \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_{i+1})^2 = 1 \text{ for } i \in \mathbb{Z}/n\mathbb{Z}\rangle$. The Tits representation gives $W_n \hookrightarrow GL(n, \mathbb{R})$. But for $n \geq 5$, $W_n$ is a two-dimensional hyperbolic reflection group, i.e. $W_n$ is generated by reflections in the sides of a right-angled hyperbolic $n$-gon (see Figure 1.9). The finite special subgroups of $W_n$ are all $C_2 \times C_2$ and the Coxeter polytope for this is a square.

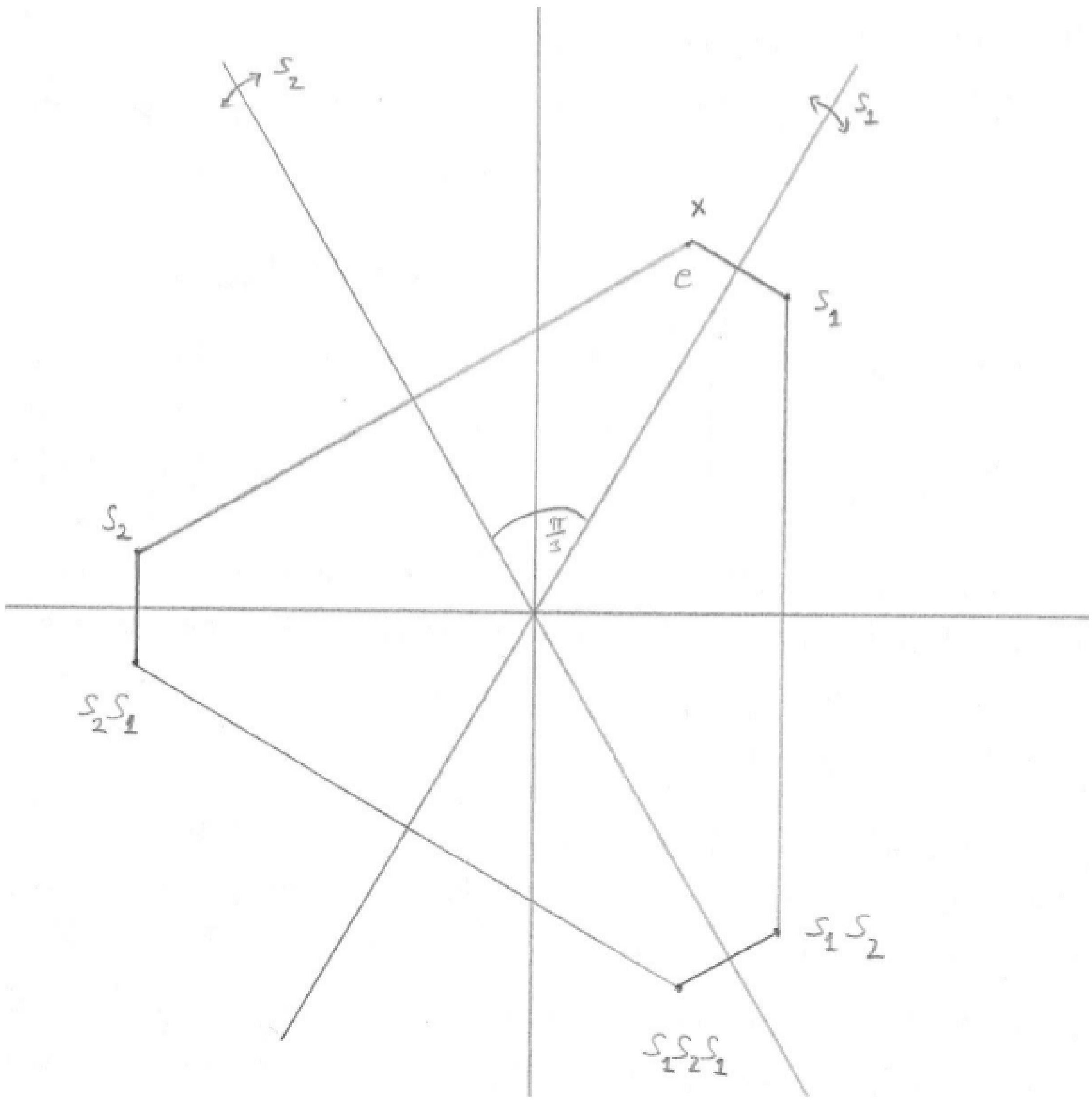



Figure 3.5: A Coxeter polytope for the group $W = D_6$.

# LECTURE 4

# THE BASIC CONSTRUCTION OF A GEOMETRIC REALISATION

23.03.2016

The term “geometric realisation” is not a formally defined mathematical term. It gets used in various situations where $W$ acts on some space $X$ such that the elements of $S$ are in some sense reflections. The action might not be by isometries.

Today we want to give a “universal” construction of geometric realisations for a Coxeter group.

## 4.1 Simplicial complexes

**Definition 4.1.** An *abstract simplicial complex* consists of a set $V$, possibly infinite, called the *vertex set* and a collection $X$ of finite subsets of $V$ such that

1. $\{v\} \in X$ for all $v \in V$,
2. If $\Delta \in X$ and $\Delta' \subseteq \Delta$ then $\Delta' \in X$.

An element of $X$ is called a *simplex*. If $\Delta$ is a simplex and $\Delta' \subsetneq \Delta$ then $\Delta'$ is a *face* (sometimes “facet” for codimension one). The *dimension* of a simplex $\Delta$ is $\dim \Delta = |\Delta| - 1$. A *k-simplex* is a simplex of dimension $k$, a 0-simplex is a *vertex*, a 1-simplex is an *edge*. The *k-skeleton* $X^{(k)}$ consists of all simplices of dimension $k$. This is also a simplicial complex.

The dimension of $X$ is $\dim X = \max\{\dim(\Delta) \mid \Delta \in X\}$ if this exists. A simplicial complex is called *pure* if all its maximal simplices have the same definition. We do not assume that $X$ is pure. However we do assume that $\dim(X)$ is finite.

We will frequently identify an abstract simplicial complex $X$ with the following simplicial cell complex $X$ and refer to both as simplicial complexes. The *standard n-simplex*

$\Delta^n$ is the convex hull of the $(n+1)$ points $(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)$ in $\mathbb{R}^{n+1}$; see Figure 4.1.

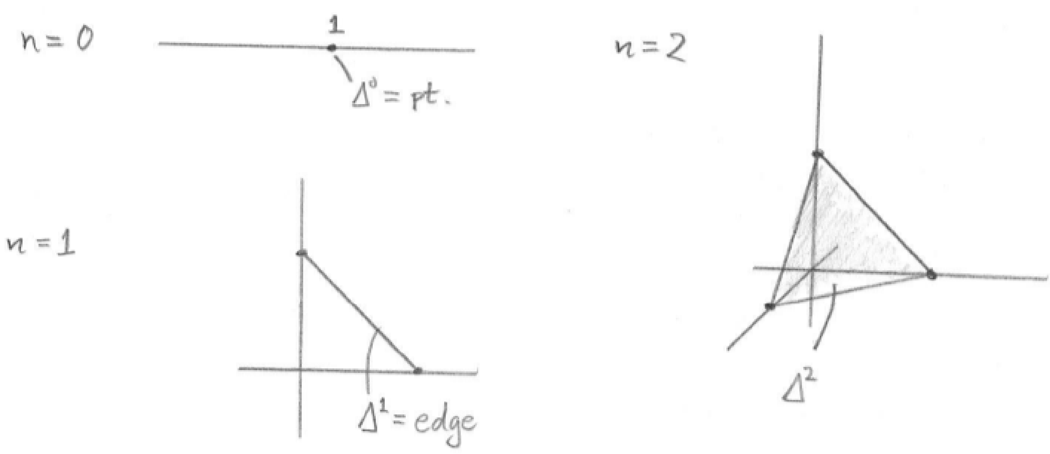



Figure 4.1: Simplices of different dimensions.

For each $n$-simplex $\Delta$ in $X$, we identify $\Delta$ with $\Delta^n$. This gives the $n$-cells in $X$. The attaching maps are obtained by gluing faces accordingly.

Conversely, define $V = V(X) = X^{(0)}$. Then $\Delta \subseteq V$ is in $X \iff \Delta$ spans a copy of $\Delta^n$.

## 4.2 The "Basic construction"

**Definition 4.2.** Let $(W,S)$ be any Coxeter system and let $X$ be a connected, Hausdorff topological space. A *mirror structure* on $X$ over $S$ is a collection $(X_s)_{s\in X}$ where each $X_s$ is a non-empty closed subspace of $X$. We call $X_s$ the *s-mirror*.

**Idea:** The basic construction $\mathcal{U}(W,X)$ is a geometric realisation for $W$ obtained by gluing together $W$-many copies of $X$ along mirrors.

**Example 4.3** (Examples of mirror structures)**.**

1. Let $X$ be the cone on $|S|$ vertices $\{\sigma_s \mid s \in S\}$. Put $X_s = \sigma_s$; see Figure 4.2.
2. Let $X$ be the $n$-simplex where $|S| = n+1$, with codimension-one faces $\{\Delta_s \mid s \in S\}$. Put $X_s = \Delta_s$. E.g. $S = \{s,t,u\}$; see Figure 4.3.

   Note that we can view $X$ as a cone on $X^{(n-1)}$, which is complete.
3. Let $P^n$ be a simple convex polytope in $\mathbb{X}^n \in \{\mathbb{S}^n, \mathbb{E}^n, \mathbb{H}^n\}$, $n \geq 2$ with codimension-one faces $\{F_i\}_{i\in I}$ such that if $i \neq j$ and $F_i \cap F_j \neq \emptyset$ then the dihedral angle between

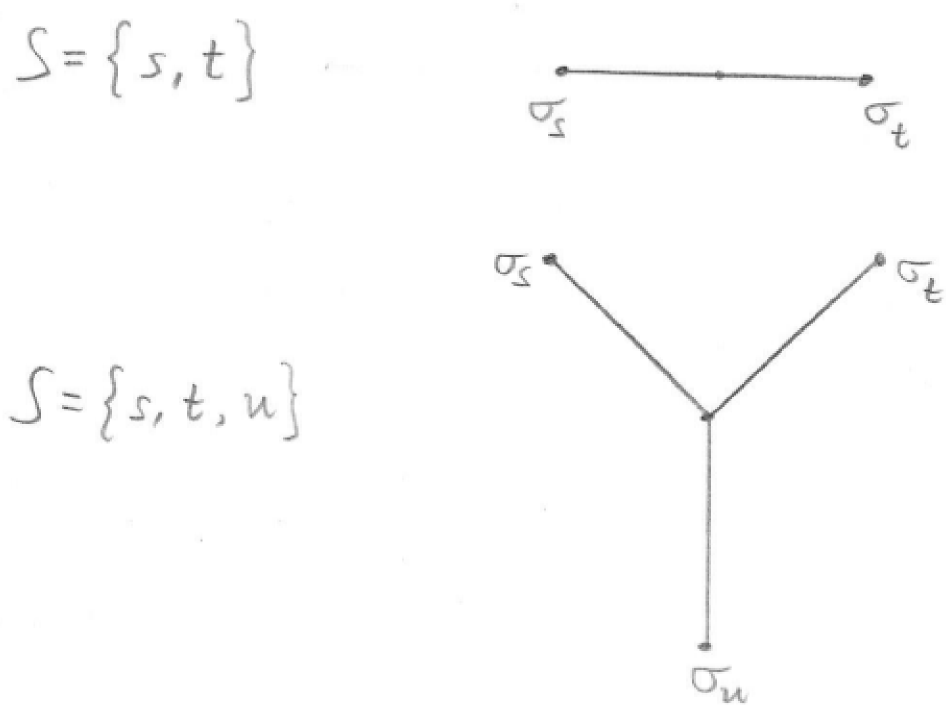



Figure 4.2: $X$ is the cone on $|S|$ vertices $\{\sigma_s \mid s \in S\}$ and $X_s = \sigma_s$.

them is $\frac{\pi}{m_{ij}}$ where $m_{ij} \geq 2$ is an integer. Put $m_{ii} = 1$ and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$. Let $(W, S)$ be the Coxeter system with Coxeter matrix $(m_{ij})$. Put $X = P^n$ and $X_{s_i} = F_i$. In the next lecture we will prove that $\mathcal{U}(W, P^n)$ is isometric to $\mathbb{X}^n$. This will then imply Theorem 1.5.

4. Let $C \subseteq V^*$ be the chamber associated to the Tits representation. Put $X = C$, $X_{s_i} = C \cap H_i^*$.

5. If $W$ is finite, the Tits representation gives $\rho : W \to O(n, \mathbb{R})$ with $n = |S|$. Let $C = \{v \in \mathbb{R}^n \mid \langle v, e_i \rangle \geq 0 \quad \forall i\}$. Let $x \in \mathring{C}$ and take the convex hull of $W.x$, i.e. consider the associated Coxeter polytope. Put $X = C \cap$ Coxeter polytope, $X_{s_i} = X \cap H_i$; see Figure 4.4.

6. Recall: If the bilinear form $B$ for the Tits representation is positive semi-definite and not definite, we get a tiling of $\mathbb{E}^{n-1}$ by intersecting the Tits cone with an affine subspace $E$. Put $X = C \cap E$, $X_{s_i} = X \cap H_i^*$.

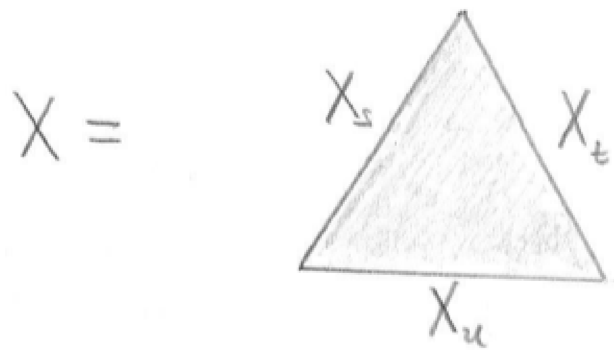



Figure 4.3: $X$ is a 2-simplex, with codimension-one faces $\{\Delta_s \,|\, s \in S\}$ where $S = \{s, t, u\}$.

**Construction of $\mathcal{U}(W, X)$:** For each $x \in X$, define $S(x) \subseteq S$ by

$$S(x) := \{s \in S \,|\, x \in X_s\}$$

.

**Example 4.4.** 1. In the first example of Example 4.3 above

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s \,|\, s \in S\} \\ \{s\}, & \text{if } x = \sigma_s \end{cases}$$

2. In the second example of Example 4.3 above

$$\{S(x) \,|\, x \in X\} = \{T \subsetneq S\}.$$

Recall: If $T \subseteq S$, the special subgroup $W_T$ is $\langle T \rangle$ with $W_\emptyset = 1$.

Now let us define a relation on $W \times X$ by $(w, x) \sim (w', x') \iff x = x'$ and $w^{-1}w' \in W_{S(x)}$. Check: this is an equivalence relation. Equip $W$ with the discrete topology and $W \times X$ with the product topology. Define

$$\mathcal{U}(W, X) = W \times X/\sim .$$

We write $[w, x]$ for the equivalence class of $(w, x)$ and $wX$ for the image of $\{w\} \times X$ in $\mathcal{U}(W, X)$. This is well-defined since $x \mapsto [w, x]$ is an embedding. Each $wX$ is called a *chamber*.

**Example 4.5.** 1. Let $W = \langle s, t, u \,|\, s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1\rangle$, i.e. $W$ is the $(3, 3, 3)$-triangle group. Let $X = \text{Cone}\{\sigma_s, \sigma_t, \sigma_u\}$. Now

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s, \sigma_t, \sigma_u\} \\ \{s\}, \{t\}, \{u\}, & \text{as } x = \sigma_s, \sigma_t, \sigma_u \text{ resp.} \end{cases}$$

So $W_{S(x)}$ are either 1, $\{1, s\}$, $\{1, t\}$, or $\{1, u\}$. Thus if $x \notin \{\sigma_s, \sigma_t, \sigma_u\}$ then the equivalence class $[w, x] = \{(w, x)\}$. If $x = \sigma_s$ then $(w, \sigma_s) \sim (w', \sigma_s) \iff w^{-1}w' \in \{1, s\} \iff w = w'$ or $w' = ws$. So $[w, \sigma_s] = \{(w, \sigma_s), (ws, \sigma_s)\}$. Hence we glue $wX$ and $wsX$ along $\sigma_s$; see Figure 4.5.

The space $\mathcal{U}(W, X)$ is the Cayley graph $\mathrm{Cay}(W, S)$ up to subdivision. In general, for any Coxeter system $(W, S)$: If $X = \mathrm{Cone}\{\sigma_s \mid s \in S\}$ and $X_s = \sigma_s$ then $\mathcal{U}(W, X) = \mathrm{Cay}(W, S)$ (up to subdivision).

2. Let $W$ be the same as in 1. Let $X$ be a two-simplex and $X_s = \Delta_s$ its codimension-one faces. Then $\mathcal{U}(W, X)$ is a tesselation of $\mathbb{E}^2$. If $x \in \Delta_s \cap \Delta_t$, then $W_{S(x)} = \langle s, t\rangle \cong D_6$; see Figure 4.6.

   For any Coxeter system $(W, S)$: If $X$ = simplex with codimension-one faces $\{\Delta_s \mid s \in S\}$, $X_s = \Delta_s$, then the simplicial complex $\mathcal{U}(W, X)$ is called the *Coxeter complex*. If $(W, S)$ is irreducible affine, the Coxeter complex is the tessellation $E\cap$ Tits cone.

## 4.3 Properties of $\mathcal{U}(W, X)$

**Lemma 4.6.** $\mathcal{U}(W, X)$ is connected as a topological space.

*Proof.* Since $\mathcal{U}(W, X) = W \times X/\sim$ has the quotient topology, $A \subseteq \mathcal{U}(W, X)$ is open (resp. closed) if and only if $A \cap wX$ is open (resp. closed) for all chambers $wX$. Suppose $A \subseteq \mathcal{U}(W, X)$ is both open and closed. Assume $A \neq \emptyset$. Since $X$ is connected, for any $w \in W$, $A \cap wX$ is either $\emptyset$ or $wX$. So $A$ is a non-empty union of chambers $A = \bigcup_{v \in V} vX$ where $\emptyset \neq V \subseteq W$. Let $v \in V$ and $s \in S$. Since $X_s \neq \emptyset$, if $x \in X_s$ then any open neighbourhood of $[v, x] \in vX$ must contain $[vs, x] \in vsX$. So $V_S \subseteq V$. But $S$ generates $W$, so $V = W$ and $A = \mathcal{U}(W, X)$. □

**Definition 4.7.** We say $\mathcal{U}(W, X)$ is *locally finite* if for every $[w, x] \in \mathcal{U}(W, X)$ there is an open neighbourhood of $[w, x]$ which meets only finitely many chambers.

**Lemma 4.8.** The following are equivalent:

- $\mathcal{U}(W, X)$ is locally finite;
- $\forall x \in X : W_{S(x)}$ is finite;
- $\forall T \subseteq S$ such that $W_T$ is infinite we have $\bigcap_{x \in T} X_t = \emptyset$.

**Example 4.9.** Let $W = \langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^3 = 1\rangle$. Then the Coxeter complex is not locally finite; see Figure 4.7.

Next time we will construct for a general Coxeter system $(W, S)$ a chamber $X = K$ with mirror structure $(K_s)_{s \in S}$ such that $\mathcal{U}(W, K)$ is locally finite and contractible.

## 4.4 Action of $W$ on $\mathcal{U}(W, X)$

The group $W$ acts on $W \times X$ by $w' \cdot (w, x) = (w'w, x)$. Check: This action preserves the equivalence relation $\sim$, such that $W$ acts on $\mathcal{U}(W, X) = W \times X/\sim$. This also induces an action on the set of chambers: $w \cdot w'X = (ww')X$. This action is transitive on the set of chambers, and is free on the set of chambers provided there is some point $x \in X$ which is not contained in any mirror. In this situation, the map $w \mapsto wX$ is a bijection from $W$ to the set of chambers.

Stabilisers: The point $[w, x] \in \mathcal{U}(W, X)$ has stabiliser

$$\begin{aligned}\{w' \in W \mid w' \cdot (w, x) \sim (w, x)\} &= \{w' \in W \mid (w'w, x) \sim (w, x)\} \\ &= \{w' \in W \mid (w'w)^{-1}w \in W_{S(x)}\} \\ &= \{w' \in W \mid w^{-1}w'w \in W_{S(x)}\} = wW_{S(x)}w^{-1},\end{aligned}$$

i.e. the stabiliser of $[w, x]$ is a conjugate of $W_{S(x)}$.

**Definition 4.10.** The action by homeomorphisms of a discrete group $G$ on a Hausdorff space $Y$ (not necessarily locally compact) is called *properly discontinuous* if

1. $Y/G$ is Hausdorff;
2. $\forall y \in Y : G_y = \mathrm{stab}_G(Y)$ is finite;
3. $\forall y \in Y$ there is an open neighbourhood $U_y$ of $y$ which is stabilised by $G_y$ and $gU_y \cap U_y = \emptyset$ for all $g \in G \backslash G_y$.

**Lemma 4.11.** The $W$-action on $\mathcal{U}(W, X)$ is properly discontinuous if and only if $W_{S(x)}$ is finite for every $x \in X$.

*Proof.* Let us first assume that the $W$-action on $\mathcal{U}(W, X)$ is properly discontinuous. As we have seen before the stabiliser of a point $[w, x] \in \mathcal{U}(W, X)$ is $wW_{S(x)}w^{-1}$. Thus $W_{S(x)}$ is finite by 2.

Let us now assume that $W_{S(x)}$ is finite for every $x \in X$. All that needs to be seen is 3. Without loss of generality we consider $y = [1, x] \in \mathcal{U}(W, X)$. Let $V_x = X - \bigcup\{\text{mirrors which do not contain } x\}$. The sought for neighbourhood of $y$ is then given by $U_y = W_{S(x)}V_x$. □

## 4.5 Universal property of $\mathcal{U}(W, X)$

$\mathcal{U}(W, X)$ satisfies the following universal property.

**Theorem 4.12** (Vinberg)**.** *Let $(W, S)$ be any Coxeter system. Suppose $W$ acts by homeomorphisms on a connected Hausdorff space $Y$ such that, for every $s \in S$, the fixed point set $Y^s$ of $s$ is non-empty. Suppose further that $X$ is a connected Hausdorff space* *with a mirror structure* $(X_s)_{s\in S}$*. Then if* $f : X \to Y$ *is a continuous map such that* $f(X_s) \subseteq Y^s$ *for all* $s \in S$*, there is a unique extension of* $f$ *to a* $W$*-equivariant map* $\tilde{f} : \mathcal{U}(W,X) \to Y$ *given by* $\tilde{f}([w,x]) = w \cdot f(x)$*.*

Next time we will apply the above theorem to Theorem 1.5.

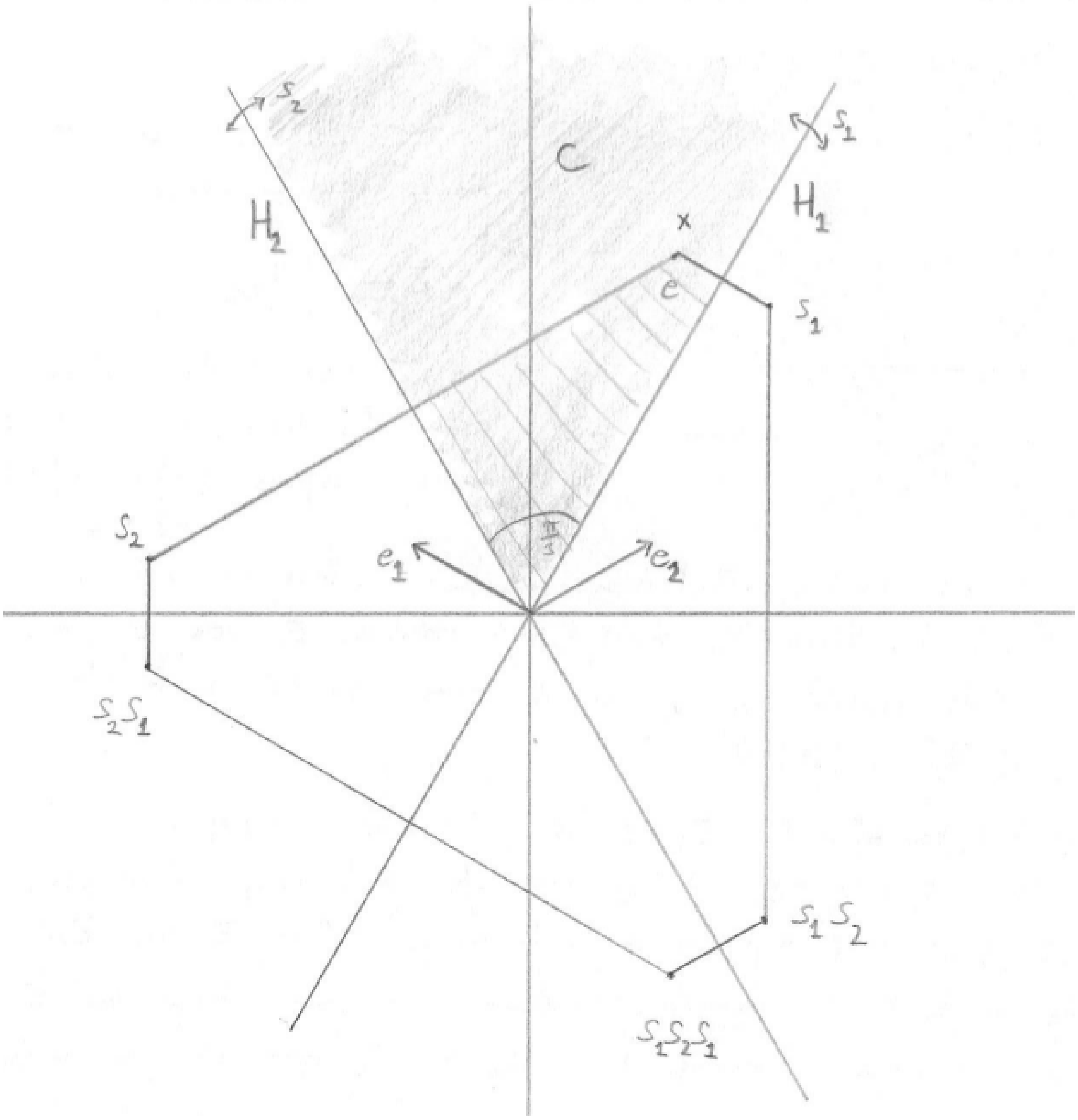



Figure 4.4: $W = D_6$, $X = C \cap$ Coxeter polytope, $X_{s_i} = X \cap H_i$.

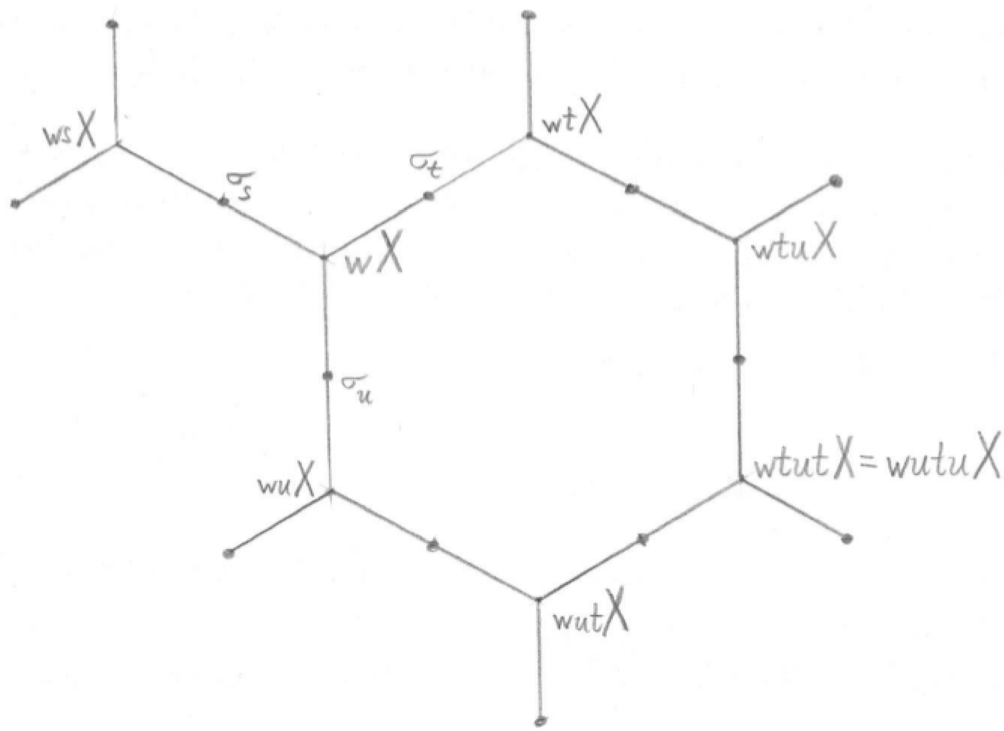



Figure 4.5: $\mathcal{U}(W, X)$ depicted for $W = D_6$ and $X = \text{Cone}\{\sigma_s, \sigma_t, \sigma_u\}$.

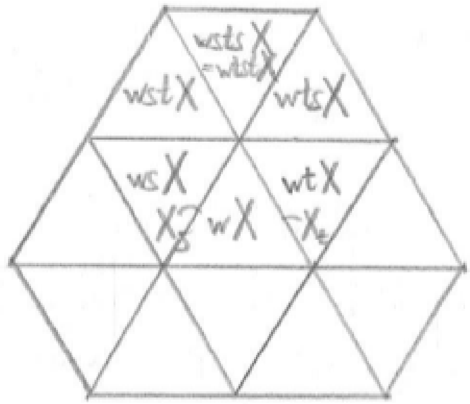



Figure 4.6: $\mathcal{U}(W, X)$ depicted for $W = D_6$ and $X =$ two-simplex, $X_s = \Delta_s$ its codimension-one faces.

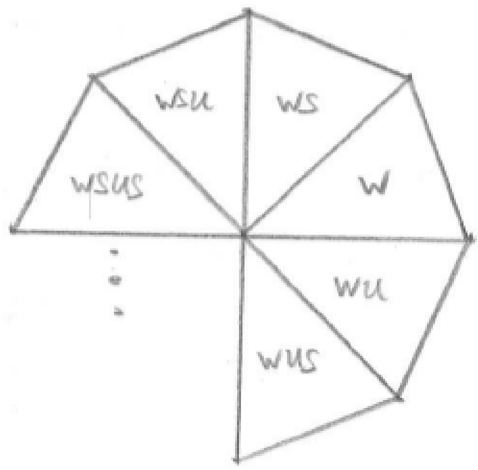



Figure 4.7: For $W = \langle s, t, u \,|\, s^2 = t^2 = u^2 = 1, (st)^3 = 1 \rangle$ the Coxeter complex is not locally finite.

LECTURE 5

# GEOMETRIC REFLECTION GROUPS AND THE DAVIS COMPLEX

06.04.2016

## 5.1 Geometric Reflection Groups

**Theorem 5.1** (this includes Theorem 1.5)**.** *Let $X = P^n$ be a simple convex polytope in $\mathbb{X}^n$ $(n \geq 2)$, with codimension-one faces $\{F_i\}_{i \in I}$, such that if $i \neq j$ and $F_i \cap F_j \neq \emptyset$, then the dihedral angle between them is $\frac{\pi}{m_{ij}}$ where $m_{ij} \in \{2, 3, 4, \ldots\}$ is finite. Put $m_{ii} = 1$ and $m_{ij} = \infty$ if $F_i \cap F_j = \emptyset$.*

*Let $(W, S)$ be the abstract Coxeter system with Coxeter matrix $(m_{ij})_{i,j \in I}$.*

*Define a mirror structure on $X$ by $X_{s_i} = F_i$. For each $i \in I$, let $\bar{s}_i \in Isom(\mathbb{X}^n)$ be the reflection in $F_i$. Let $\overline{W}$ be the subgroup of $Isom(\mathbb{X}^n)$ generated by the $\bar{s}_i$.*

*Then:*

1. *there is an isomorphism $\varphi : W \to \overline{W}$ induced by $s_i \mapsto \bar{s}_i$;*

2. *the induced map $\mathcal{U}(W, P^n) \to \mathbb{X}^n$ is a homeomorphism;*

3. *the Coxeter group $W$ acts properly discontinuously on $\mathbb{X}^n$ with strict fundamental domain $P^n$, so $W$ is a discrete subgroup of $Isom(\mathbb{X}^n)$ and $\mathbb{X}^n$ is tiled by copies of $P^n$.*

*Proof.*

**To 1:** First we show $s_i \mapsto \bar{s}_i$ induces a homomorphism $W \to \overline{W}$.

Each $s_i$ has order 2 in $W$ and each $\bar{s}_i$ has order 2 in $\overline{W}$.

Also:

$$m_{ij} \text{ is finite } \iff F_i \cap F_j \neq \emptyset \text{ and meet at dihedral angle } \frac{\pi}{m_{ij}}$$
$$\iff \bar{s}_i \bar{s}_j \text{ has order } m_{ij}.$$

Hence we have a homomorphism $\varphi : W \to \overline{W}$.

**To 2:** Since $\overline{W}$ acts by isometries on $\mathbb{X}^n$, also $W$ acts by isometries on $\mathbb{X}^n$.

In the $W$-action, each $s_i$ fixes (at least) the faces $F_i$. So by the universal property, the inclusion $f : P \to \mathbb{X}^n$ induces the (unique) $W$-equivariant map

$$\tilde{f} : \mathcal{U}(W, P) \to \mathbb{X}^n.$$

The injectivity of $\varphi$ and 3 follow from the next claim.

**Claim:** $\tilde{f}$ is a homeomorphism.

*Proof.* We will prove the claim via a quite complicated induction scheme on the dimension $n$. Let us introduce some notation first.

Notation:

- $(s_n)$ is the claim when $\mathbb{X}^n = \mathbb{S}^n$ and $P^n = \sigma^n$ is a spherical simplex with dihedral angles $\frac{\pi}{m_{ij}}$ $(n \geq 2)$.
- $(c_n)$ is the claim when $\mathbb{X}^n$ is replaced by $B_x(r)$, the open ball of radius $r$ about a point $x \in \mathbb{X}^n$, and $P^n$ is replaced by $C_x(r)$, the open simplicial cone of radius $r$ about $x$ with dihedral angles $\frac{\pi}{m_{ij}}$.
- $(t_n)$ is the claim in dimension $n$.

We will prove $(c_2)$ and show that $\forall n \geq 2$, $(c_n) \implies (t_n)$ and $(s_n) \implies (c_{n+1})$. Then as $(s_n)$ is a special case of $(t_n)$, we get

$$(c_2) \implies (t_2) \implies (s_2) \implies (c_3) \implies (t_3) \implies \ldots \implies (t_n) \implies \ldots$$

*Proof of $(c_2)$:* In $\mathbb{X}^2$ let

$$W = \langle s_1, s_2 \,|\, s_i^2 = 1, (s_1 s_2)^{m_{12}} = 1 \rangle = D_{2m_{12}}.$$

The basic construction $\mathcal{U}(W, C_x(r))$ is $|W| = 2m_{12}$ copies of $C_x(r)$ glued along mirrors. This is homeomorphic to $B_x(r)$; see Figure 5.1.

*Proof that $(s_n) \implies (c_{n+1})$:* Let $S_x(r)$ be the sphere of radius $r$ about $x$ in $\mathbb{X}^{n+1}$. Regard $\mathbb{S}^n$ (unit-sphere) as living in $T_x\mathbb{X}^{n+1}$. Then the exponential map $\exp : T_x\mathbb{X}^{n+1} \to \mathbb{X}^{n+1}$ induces a homeomorphism from $\mathbb{S}^n \to S_x(1)$.

Let $\sigma^n \subset \mathbb{S}^n$ be the spherical simplex, such that $\exp(\sigma^n) = S_x(1) \cap \overline{C_x(1)}$. Then $\sigma^n$ has dihedral angles $\frac{\pi}{m_{ij}}$, so the Coxeter group $W$ associated to $\sigma^n$ is the same as the one associated to the simplicial cone $C_x(1)$; see Figure 5.2.

Since $(s_n)$ holds,

$$\begin{aligned} &\mathcal{U}(W, \sigma^n) \to \mathbb{S}^n \text{ is a homeomorphism} \\ &\implies \mathcal{U}\left(W, S_x(1) \cap \overline{C_x(1)}\right) \to S_x(1) \text{ is a homeomorphism} \\ &\implies \mathcal{U}(W, \overline{C_x(1)}) \to \overline{B_x(1)} \text{ is a homeomorphism} \\ &\implies \mathcal{U}(W, C_x(1)) \to B_x(1) \text{ is a homeomorphism} \\ &\implies \mathcal{U}(W, C_x(r)) \to B_x(r) \text{ is a homeomorphism.} \end{aligned}$$

This proves $(c_{n+1})$.

*Proof that* $(c_n) \implies (t_n)$*:*

**Definition 5.2.** A $n$-dimensional topological manifold $M^n$ has an $\mathbb{X}^n$*-structure*, if it has an atlas of charts $\{\psi_\alpha : U_\alpha \to \mathbb{X}^n\}_{\alpha \in A}$ where $(U_\alpha)_{\alpha \in A}$ is an open cover of $M^n$, each $\psi_\alpha$ is a homeomorphism onto its image, and for all $\alpha, \beta \in A$ the map

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$$

is the restriction of an element of $\mathrm{Isom}(\mathbb{X}^n)$; see Figure 5.3. In particular an $\mathbb{X}^n$ structure turns $M^n$ into a (smooth) Riemannian manifold.

Facts:

- An $\mathbb{X}^n$-structure on $M^n$ induces one on its universal cover $\widetilde{M}^n$.
- There is a *developing map* $D : \widetilde{M}^n \to \mathbb{X}^n$ given by analytic continuation along paths.
- If $M^n$ is metrically complete, $D$ is a covering map.

Let $x \in P^n \subset \mathbb{X}^n$. Let $r = r_x > 0$ be the distance from $x$ to the nearest $F_i$ which does not contain $x$. Let $C_x(r) = B_x(r) \cap P^n$ be the open simplicial cone in $\mathbb{X}^n$ with vertex $x$.

Let $\mathcal{U}_x = \mathcal{U}(W_{S(x)}, C_x(r))$ where $S(x) = \{s_i \mid x \in F_i\}$. Then $\mathcal{U}_x$ is an open neighbourhood of $[1, x]$ in $\mathcal{U}(W, P^n)$. By $(c_n)$, the map $\mathcal{U}_x \to B_x(r)$ is a homeomorphism. By equivariance, for all $w \in W$ the map

$$w\mathcal{U}_x \to \varphi(w) B_x(r)$$

is also a homeomorphism. Now $\varphi(w)$ is an isometry of $\mathbb{X}^n$, so $M^n = \mathcal{U}(W, P^n)$ has an $\mathbb{X}^n$-structure.

The $W$-action on $\mathcal{U}(W, P^n)$ is cocompact, so by a standard argument $\mathcal{U}(W, P^n)$ is metrically complete. Hence the developing map $D : \widetilde{\mathcal{U}(W, P^n)} \to \mathbb{X}^n$ is a covering map.

The map $D$ is locally given by $\tilde{f}$, and since $\mathcal{U}(W, P^n)$ is connected and $\tilde{f}$ is globally defined, $\tilde{f}$ is also a covering map. But $\mathbb{X}^n$ is simply connected so $\tilde{f} = D$ is a homeomorphism.

□

This finishes the proof of the theorem.

□

## 5.2 The Davis complex – a first definition

Recall: if $X$ has mirror structure $(X_s)_{s\in S}$, then $\mathcal{U}(W,X)$ is ...

- ... connected;
- ... locally finite $\iff$ $W_{S(x)}$ is finite $\forall x \in X$;
- ... the point stabilisers are given by:

$$\mathrm{stab}_W([w,x]) = wW_{S(x)}w^{-1};$$

- ... the $W$-action is properly discontinuous $\iff$ $W_{S(x)}$ is finite $\forall x \in X$.

The *Davis complex* $\Sigma = \Sigma(W,S)$ is $\mathcal{U}(W,K)$ where the chamber $K$ has mirror structure $(K_s)_{s\in S}$ such that $\forall x \in K$, $W_{S(x)}$ is finite.

In order to define $K$: A subset $T \subseteq S$ is *spherical* if $W_T$ is finite; we say $W_T$ is a *spherical special subgroup.*

Consider

$$\{T \subseteq S \,|\, T \neq \emptyset, T \text{ is spherical}\}.$$

This collection is an abstract simplicial complex: if $\emptyset \neq T' \subseteq T$, and $W_T$ is finite, then $W_{T'}$ is finite. Also $\{s\}$ is spherical for all $s \in S$.

This simplicial complex is called the *nerve* of $(W,S)$, denoted by $L = L(W,S)$. Concretely: $L$ has vertex set $S$, and a simplex $\sigma_T$ spanning each $T \subseteq S$ such that $T \neq \emptyset$ and $W_T$ is finite.

**Example 5.3.**

1. If $W$ is finite, the nerve $L$ is the full simplex on $S$.
2. If $W \cong D_\infty = \langle s,t \,|\, s^2 = t^2 = 1\rangle$, the nerve $L$ consists exactly of the two vertices $s$ and $t$.
3. If $W$ is the $(3,3,3)$-triangle group,

$$W = \langle s,t,u \,|\, s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1\rangle,$$

   then the nerve $L$ is a triangle with vertices $s,t,u$ but not filled in as $W$ is not finite.
4. If $W$ is a geometric reflection group with fundamental domain $P$ then $L$ can be identified with the boundary of $P^*$, the dual polytope of $P$. (This needs proof!)
5. If $(W,S)$ is a reducible Coxeter system with

$$(W,S) = (W_1 \times W_2, S_1 \sqcup S_2),$$

   then $T \subseteq S$ is spherical $\iff$ $T = T_1 \sqcup T_2$, with $T_i = T \cap S_i$, and both $T_1$ and $T_2$ are finite. Then $L(W,S)$ is the join of $L(W_1,S_1)$ and $L(W_2,S_2)$. See for example Figure 5.4.

6. (Right-angled Coxeter groups) Let $\Gamma$ be a finite simplicial graph with vertex set $S = V(\Gamma)$ and edge set $E(\Gamma)$. The associated Coxeter group is

$$\begin{aligned} W_\Gamma &= \langle S \mid s^2 = 1\, \forall s \in S, \quad st = ts \iff \{s,t\} \in E(\Gamma)\rangle \\ &= \langle S \mid s^2 = 1\, \forall s \in S, \quad (st)^2 = 1 \iff \{s,t\} \in E(\Gamma)\rangle \end{aligned}$$

Then $\langle s,t\rangle$ is finite if and only if $s$ and $t$ are adjacent in $\Gamma$. Hence the nerve $L(W_\Gamma, S)$ has 1-skeleton equal to $\Gamma$.

**Definition 5.4.** A simplicial complex $L$ is called a *flag complex* if each finite, non-empty set of vertices $T$ spans a simplex in $L$ if and only if any two elements of $T$ span an edge/1-simplex in $L$.

A flag simplicial complex is completely determined by its 1-skeleton.

**Lemma 5.5.** If $(W,S)$ is a right-angled Coxeter system, then $L(W,S)$ is a flag complex.

*Proof.* Suppose $T \subseteq S$, $T \neq \emptyset$ and any two vertices in $T$ are connected by an edge in $L$. Then $W_T \cong (C_2)^{|T|}$ is finite, so $T$ is spherical and $\sigma_T$ is in $L$. □

Now we can define $K$ and its mirror structure $(K_s)_{s\in S}$. Let $L = L(W,S)$ be the nerve of the Coxeter system $(W,S)$ and let $L'$ be its barycentric subdivision.

We define

$$K = \mathrm{Cone}(L').$$

For each $s \in S$, define $K_s$ to be the closed star in $L'$ of the vertex $s$. (The closed star of $s$ is the union of the closed simplices in $L'$ which contain $s$.)

Then $(K_s)_{s\in S}$ is a mirror structure on $K$. We have:

$$\begin{aligned} &\text{Two mirrors } K_s \text{ and } K_t \text{ intersect} \\ &\iff \text{there is an edge of } L \text{ between } s \text{ and } t \\ &\iff \langle s,t\rangle \text{ is finite.} \end{aligned}$$

Similarly,

$$\begin{aligned} \bigcap_{t\in T} K_t \neq \emptyset &\iff T \subseteq S \text{ is a non-empty spherical subset} \\ &\iff W_T \text{ is finite and non-trivial.} \end{aligned}$$

Hence $\forall x \in X$, $S(x) = \{s \in S \mid x \in K_s\}$ is spherical.

**Example 5.6.** 1. If $W$ is finite and the nerve $L$ is a simplex $\Delta$ on $|S|$ vertices then $K = \mathrm{Cone}(L')$ is a simplex of dimension one higher. So $\Sigma$ will be the cone on a tesselation of the sphere induced by the $W$-action.

2. If $W$ is a geometric reflection group with fundamental domain $P$, then $L = \partial P^*$ and so $L' = (\partial P^*)' = \partial P'$. Thus $K$ is the cone on the barycentric subdivision of $\partial P$, so $K$ is the barycentric subdivision of $P$ and $\Sigma$ is the barycentric subdivision of a tesselation of $\mathbb{X}^n$.

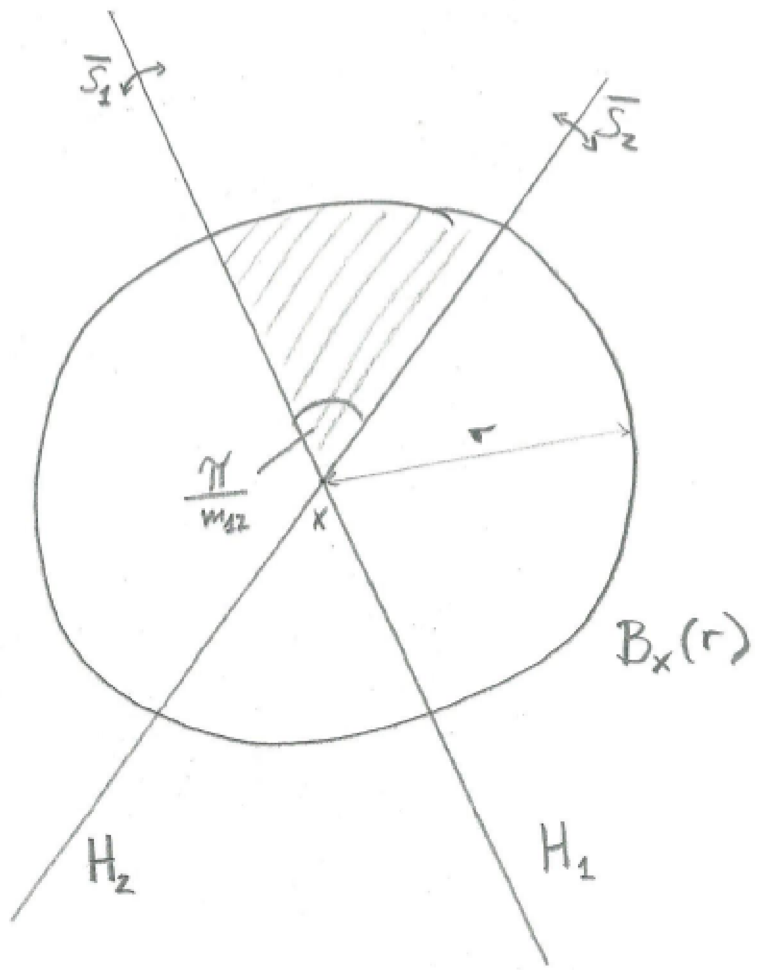



Figure 5.1: The basic construction $\mathcal{U}(W, C_x(r))$ for $W = \langle s_1, s_2 \,|\, s_i^2 = 1, (s_1 s_2)^{m_{12}} = 1\rangle = D_{2m_{12}}$ in dimension $n = 2$.

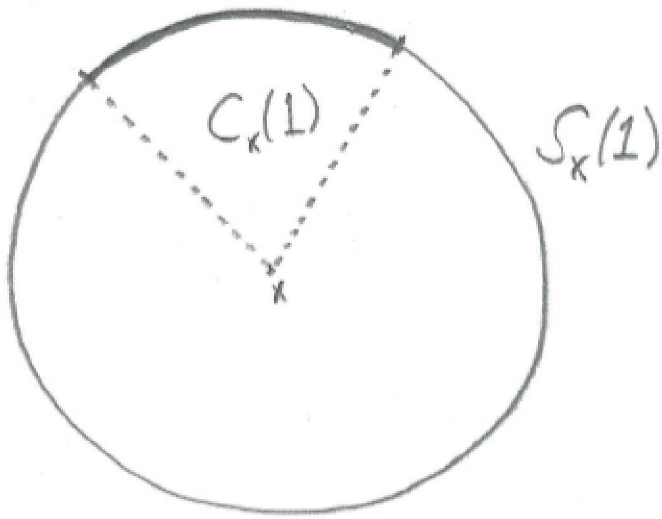



Figure 5.2: The simplicial cone $C_x(1)$.

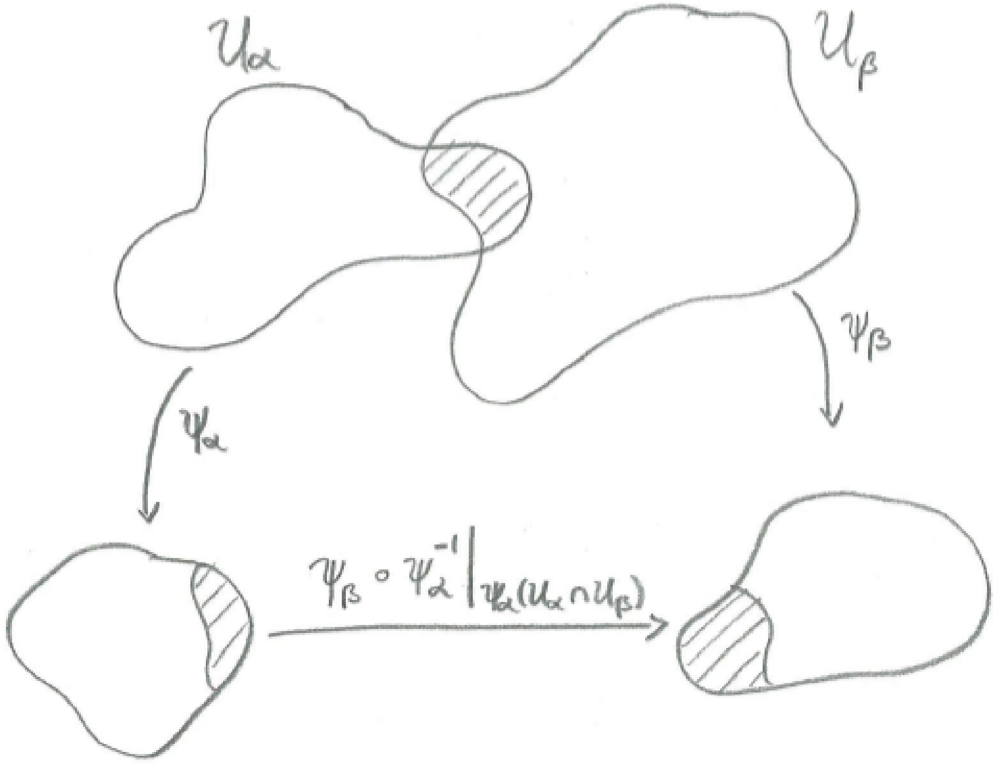



Figure 5.3: The transition map $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$.

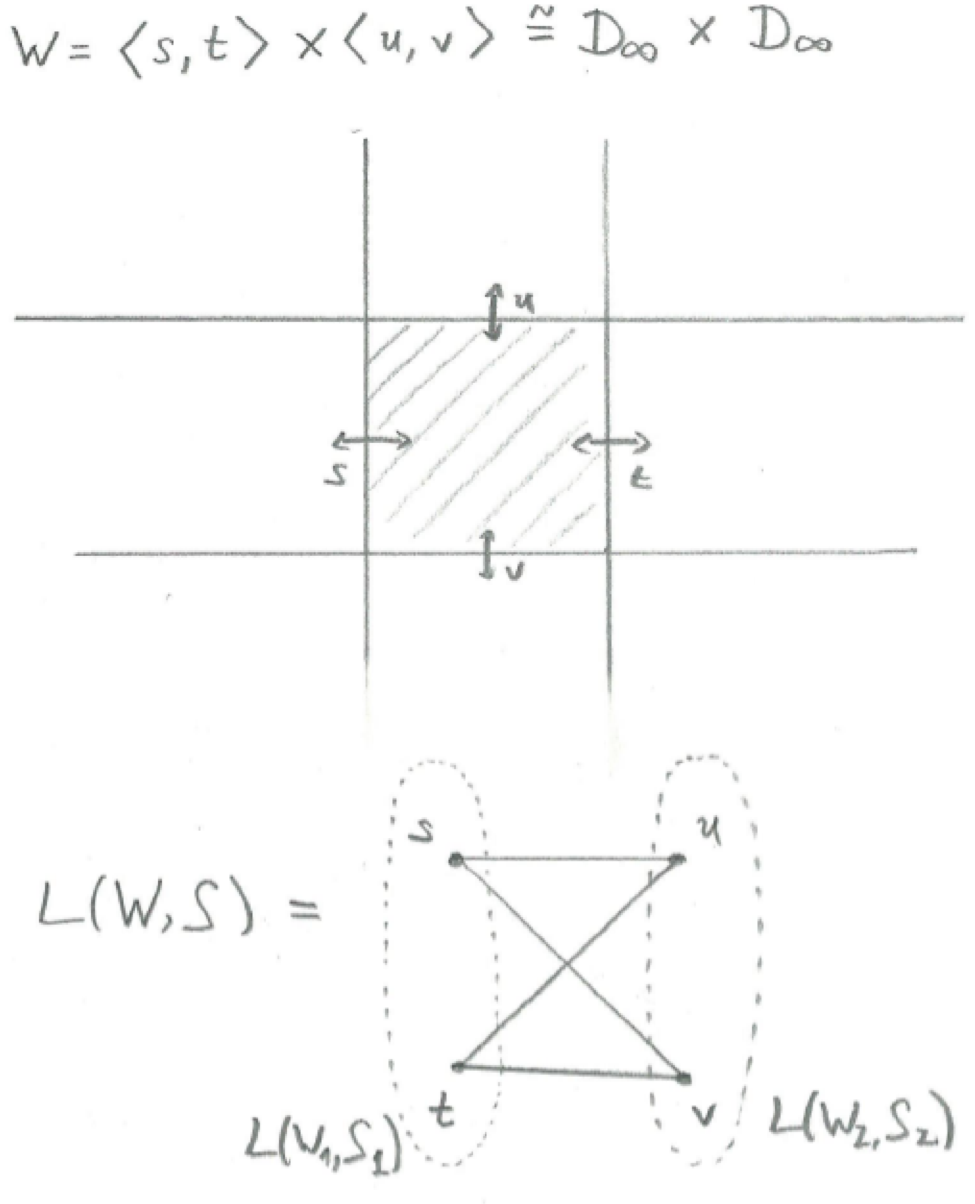



Figure 5.4: The Coxeter system $(W, S)$ with $W = \langle s,t\rangle \times \langle u,v\rangle \cong D_\infty \times D_\infty$ and $S = \{s,t,u,v\}$ is reducible. Its nerve $L = L(W,S)$ is the join of $L(W_1,S_1)$ and $L(W_2,S_2)$ where $W_1 = \langle s,t\rangle$, $S_1 = \{s,t\}$, $W_2 = \langle u,v\rangle$, $S_2 = \{u,v\}$.

LECTURE 6

# TOPOLOGY OF THE DAVIS COMPLEX

13.04.2016

For the rest of this lecture let $(W, S)$ be a Coxeter system with corresponding Davis complex $\Sigma = \Sigma(W, S)$ given by $\Sigma = \mathcal{U}(W, K)$. Recall that:

- $L$ is the nerve of $(W, S)$, i.e. the simplicial complex with vertex set $S$, and a simplex $\sigma_T$ spanned by $\emptyset \neq T \subseteq S$ is contained in $L$ if and only if $W_T$ is finite;
- $L'$ denotes the barycentric subdivision of $L$;
- $K = \text{Cone}(L')$ is called the chamber, and has mirrors $\{K_s\}_{s \in S}$ where $K_s$ is the star of the vertex $s$ in $L'$;
- Given $x \in K$, define $S(x) = \{s \in S \,|\, x \in K_s\}$. Then $\Sigma = \mathcal{U}(W, K) = (W \times K)/\sim$ where
$$(w, x) \sim (w', x') \iff x = x' \text{ and } w^{-1}w' \in W_{S(x)}.$$
For example, chambers $wK$ and $wsK$ are glued together along the mirror $K_s$.

**Example 6.1.** Figures 6.1 to 6.4 illustrate the Davis complexes for certain Coxeter systems $(W, S)$.

## 6.1 Contractibility of the Davis complex

In the last lecture we have seen that:

- $\Sigma$ is connected and locally finite;

- the $W$-action $W \curvearrowright \Sigma$ is properly discontinuous and cocompact, and

$$\mathrm{stab}_W([w,x]) = wW_{S(x)}w^{-1}$$

is a finite group for every $w \in W$, $x \in K$.

Today we will prove that $\Sigma$ is contractible.

**Theorem 6.2** (Davis). *$\Sigma$ is contractible.*

### 6.1.1 Some combinatorial preliminaries

For $w \in W$ define

$$\mathrm{In}(w) = \{s \in S \,|\, \ell(ws) < \ell(w)\},$$
$$\mathrm{Out}(w) = \{s \in S \,|\, \ell(ws) > \ell(w)\}.$$

Since $\ell(ws) = \ell(w) \pm 1$, we have $S = \mathrm{In}(w) \sqcup \mathrm{Out}(w)$.

Recall that we get by the Exchange Condition: if $\ell(ws) < \ell(w)$ and $(s_{i_1}, \dots, s_{i_k})$ is a reduced expression for $w$, then $ws = s_{i_1} \dots \hat{s}_{i_j} \dots s_{i_k}\hat{s}$ for some $j$. Hence $w = s_{i_1} \dots \hat{s}_{i_j} \dots s_{i_k} s$, so there is a reduced expression for $w$ which ends in $s$. So

$$\mathrm{In}(w) = \{s \in S \,|\, \text{a reduced expression for } w \text{ can end in } s\}.$$

**Example 6.3.** If $W = \langle s,t,u \,|\, s^2 = t^2 = u^2, (st)^2 = 1\rangle$ (the right-angled Coxeter group) then

$$\mathrm{In}(ust) = \{t,s\}, \qquad \mathrm{Out}(ust) = \{u\},$$
$$\mathrm{In}(us) = \{s\}, \qquad \mathrm{Out}(us) = \{u,t\}.$$

**Proposition 6.4.** For all $w \in W$, $\mathrm{In}(w)$ is a spherical subset, i.e. $W_{\mathrm{In}(w)}$ is finite.

*Proof.* A sufficient condition for a Coxeter group $W$ to be finite is the following:

**Lemma 6.5.** If there is a $w_0 \in W$ such that $\ell(w_0 s) < \ell(w_0)$ for all $s \in S$, then $W$ is finite.

*Proof.* Use the Exchange Condition to show by induction that for every reduced expression $(s_{i_1}, \dots, s_{i_k})$ there is a reduced expression for $w_0$ which ends in $(s_{i_1}, \dots, s_{i_k})$.

Then for any $w \in W$, we get

$$\ell(w_0) = \ell(w_0 w^{-1}) + \ell(w)$$

by ending a reduced expression for $w_0$ with a reduced expression for $w$. So $\ell(w_0) \geq \ell(w)$ for every $w \in W$.

Hence $W$ is finite. □

We will also need:

**Lemma 6.6.** Let $T \subseteq S$ be a subset and suppose $w$ is a minimal length element in the left coset $wW_T$. Then any $w' \in wW_T$ can be written as $w' = wa'$ where $a' \in W_T$ and $\ell(w') = \ell(w) + \ell(a')$.

Also $wW_T$ has a unique minimal length element.

*Proof.* Existence of length additive factorisation: Deletion Condition.

Uniqueness of minimal length coset representative: Suppose $w_1, w_2$ are both minimal length elements in $w_1 W_T = w_2 W_T$. Then $w_1 = w_2 a'$ with $a' \in W_T$ and

$$\ell(w_1) = \ell(w_2) + \ell(a').$$

On the other hand $\ell(w_1) = \ell(w_2)$, so $a' = 1$ thus $w_1 = w_2$. □

**To prove the proposition:** Let $T = \mathrm{In}(w)$ and let $u$ be a minimal length element in $wW_T$. By Lemma 6.6, $w$ can be written uniquely as $w = ua'$ with $a' \in W_T$ and

$$\ell(w) = \ell(u) + \ell(a').$$

Let $s \in \mathrm{In}(w) = T$, so $\ell(ws) < \ell(w)$. Now $a's \in W_T$ so $ws = ua's$ and by Lemma 6.6 again,

$$\ell(ws) = \ell(u) + \ell(a's).$$

Hence $\ell(a's) < \ell(a')$ for every $s \in \mathrm{In}(w)$. By Lemma 6.5 with $a' = w_0$, $W_{\mathrm{In}(w)}$ is finite.

This finishes the proof of the proposition. □

### 6.1.2 Proof of Theorem 6.2

Enumerate the elements of $W$ as $w_1, w_2, w_3, \ldots$ such that $\ell(w_k) \leq \ell(w_{k+1})$. For $n \geq 1$ let $U_n = \{w_1, \ldots, w_n\} \subseteq W$, so

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots \subseteq W$$

and $W = \bigcup_{n=1}^{\infty} U_n$. Further let

$$P_n = \bigcup_{w \in U_n} wK = \bigcup_{i=1}^{n} w_i K \subseteq \Sigma,$$

so $P_1 \subseteq P_2 \subseteq \ldots$ and $\Sigma = \bigcup_{n=1}^{\infty} P_n$.

Now $P_n$ is obtained from $P_{n-1}$ by gluing on a copy of $K$ along some mirrors. To be precise: $P_n = P_{n-1} \cup w_n K$ where $w_n K$ is glued to $P_{n-1}$ along the union of mirrors $\{K_s \mid \ell(w_n s) < \ell(w_n)\}$. That is, $w_n K$ is glued to $P_{n-1}$ along the union of its mirrors of types $s \in \mathrm{In}(w)$.

By Proposition 6.4, $\mathrm{In}(w)$ is spherical. The theorem then follows from the next lemma.

**Lemma 6.7.**

1. $K$ is contractible;
2. for all spherical $T \subseteq S$, the union of mirrors

$$K^T = \bigcup_{t \in T} K_t$$

is contractible.

*Proof.*

**To 1:** $K$ is a cone.

**To 2:** We have a bijection

$$\{\text{simplices of } L\} \longleftrightarrow \{T \subseteq S \,|\, T \neq \emptyset, T \text{ is spherical}\}.$$

So

$$\{\text{vertices of } L'\} \longleftrightarrow \{T \subseteq S \,|\, T \neq \emptyset, T \text{ is spherical}\}.$$

Hence

$$\{\text{vertices of } K\} \longleftrightarrow \{T \subseteq S \,|\, T \text{ is spherical}\},$$

by identifying the cone point with $\emptyset$.

Moreover we can orient the edges in $K$ by inclusion of types of their endpoints.

Let $\sigma'_T \subset L'$ be the barycentric subdivision of $\sigma_T \subseteq L$. Since $K^T$ is the union of closed stars in $L'$ of the vertices of $\sigma_T$, $K^T$ is the first derived neighbourhood of $\sigma'_T$ in $L'$. Since $\sigma'_T$ is contractible, it is enough to construct a deformation retraction $r : K^T \to \sigma'_T$.

Define $r$ by sending a simplex of $K^T$ with vertex types $\{T'_0, \ldots, T'_k\}$ to the simplex with vertex types $\{T'_0 \cap T, \ldots, T'_k \cap T\}$. Check that this works. □

*Remark* 3. As pointed out to us by Nir Lazarovich, the proof of Theorem 6.2 has a Morse-theoretic interpretation, and part 2 of Lemma 6.7 can be viewed as showing that the down-links are contractible.

### 6.1.3 Second definition of $\Sigma$

Let $P$ be any poset (partially ordered set). A *chain* is a totally ordered subset of $P$. We can associate a simplicial complex $\Delta(P)$, called the *geometric realisation of* $P$ via

$$\text{finite chain with } (n+1) \text{ elements} \quad \longrightarrow \quad n\text{-simplex};$$

see for example Figure 6.5.

Check: $K$ is the geometric realisation of the poset $\{T \subseteq S \,|\, T \text{ spherical}\}$ ordered by inclusion; or equivalently, $K$ is the geometric realisation of the poset $\{W_T \subseteq S \,|\, T \text{ spherical}\}$ ordered by inclusion.

The vertex types in $K$ are preserved by the gluing which gives $\Sigma$. Also the $W$-action on $\Sigma$ is type-preserving, and transitive on each type of vertex.

Note that for the action $W \curvearrowright \Sigma$ each vertex of type $T$ has stabiliser a conjugate of $W_T$. Thus we can identify $\Sigma$ with the geometric realisation of

$$\{wW_T \mid w \in W,\ W_T \text{ is spherical}\},$$

ordered by inclusion.

Cf.: The Coxeter complex is the geometric realisation of

$$\{wW_T \mid w \in W, T \subseteq S\}.$$

## 6.2 Applications to $W$

In the following denote

$$K^{\mathrm{Out}(w)} = \bigcup_{s \in \mathrm{Out}(w)} K_s.$$

If $T$ is spherical, $W^T = \{w \in W \mid \mathrm{In}(w) = T\} \subseteq W$, and

$$W = \bigsqcup \{W^T \mid T \subseteq S, T \text{ spherical } \}.$$

Here $\mathbb{Z}W^T$ denotes the free abelian group with basis $W^T$.

**Theorem 6.8** (Davis)**.**

$$\begin{aligned} H^i(W; \mathbb{Z}W) &\cong \bigoplus_{w \in W} H^i\left(K, K^{Out(w)}\right) \\ &\cong \bigoplus \left\{ \left(\mathbb{Z}W^T \otimes H^i\left(K, K^{S-T}\right)\right) \mid T \subseteq S, T \text{ spherical } \right\} \\ &\cong \bigoplus \left\{ \left(\mathbb{Z}W^T \otimes \overline{H^{i-1}}\left(L - \sigma_T\right)\right) \mid T \subseteq S, T \text{ spherical } \right\} \end{aligned}$$

Theorem 6.8 is used for, e.g.:

- ends of $W$;
- determining when $W$ is virtually free;
- virtual cohomological dimension of $W$;
- showing that any $W$ is the fundamental group of a tree of groups with finite or 1-ended special subgroups as vertex groups, and finite special subgroups as edge groups.

**Definition 6.9.** Let $G$ be any group. A *classifying space* for $G$, denoted by $BG$, is an aspherical CW-complex with fundamental group $G$ (also called an Eilenberg-MacLane space or a $K(G, 1)$). Its universal cover, denoted by $EG$, is called the *universal space* for $G$.

Fact: $BG$ is unique up to homotopy equivalence.
We can define the cohomology of $G$ with coefficients in any $\mathbb{Z}G$-module $A$ by

$$H^*(G;A) = H^*(BG;A),$$

where the latter is cellular cohomology.
Problem: If $G$ has torsion, then no $BG$ is finite dimensional!

**Definition 6.10** (tom Dieck 1977)**.** Let $G$ be a discrete group. A CW-complex $X$ together with a proper, cocompact, cellular $G$-action is a *universal space for proper $G$-actions*, denoted by $\underline{E}G$, if for all finite subgroups $F$ of $G$, the fixed set $X^F$ is contractible.

Note:

- taking $F = \{1\}$ yields that $X$ must be contractible;
- if $H \leq G$ is infinite, $X^H$ is empty since the $G$-action is proper.

**Theorem 6.11.** *$\underline{E}G$ exists and is unique up to $G$-homotopy, and*

$$H^*(G;\mathbb{Z}G) = H^*_c(\underline{E}G),$$

*where the latter is cohomology with compact support, i.e. "cohomology at infinity" of $\underline{E}G$.*

We will prove next time that $\Sigma$ is a (finite dimensional) $\underline{E}W$.
In order to prove Theorem 6.8 we use the following proposition:

**Proposition 6.12** (Brown)**.** If a a discrete group $G$ acts properly discontinuously and cocompactly on an acyclic CW-complex $X$ then

$$H^*(G;\mathbb{Z}G) = H^*_c(X).$$

*Proof of Theorem 6.8 (sketch).* Enumerate the elements of $W$ as $w_1, w_2, w_3, \dots$ such that $\ell(w_k) \leq \ell(w_{k+1})$ and let

$$P_n = \bigcup_{i=1}^{n} w_i K.$$

Then $P_1 \subseteq P_2 \subseteq \dots$ is an increasing sequence of compact subcomplexes of $\Sigma$ so

$$H^*_c(\Sigma) = \varinjlim H^*(\Sigma, \Sigma - P_n).$$

If we write $\hat{P}_n = \bigcup_{i \geq n+1} w_i K$, i.e. $\hat{P} = \bigcup\{wK \mid w \notin \{w_1, \dots, w_n\}\}$ then $\hat{P}_1 \supseteq \hat{P}_2 \supseteq \dots$ and $H^*_c(\Sigma) = \varinjlim H^*(\Sigma, \hat{P}_n)$.
By considering the triples $(\Sigma, \hat{P}_{n-1}, \hat{P}_n)$, we get an exact sequence in cohomology

$$\cdots \longrightarrow H^*(\Sigma, \hat{P}_{n-1}) \longrightarrow H^*(\Sigma, \hat{P}_n) \longrightarrow H^*(\hat{P}_{n-1}, \hat{P}_n) \longrightarrow \cdots$$

By construction we have

$$H^*(\hat{P}_{n-1}, \hat{P}_n) \cong H^*(w_n K, w_n K^{\mathrm{Out}(w_n)}) \cong H^*\left(K, K^{\mathrm{Out}(w_n)}\right).$$

One can now show that the above sequence splits and we hence get

$$H^*(\Sigma, \hat{P}_n) \cong \bigoplus_{i=1}^{n} H^*\left(K, K^{\mathrm{Out}(w_i)}\right).$$

□

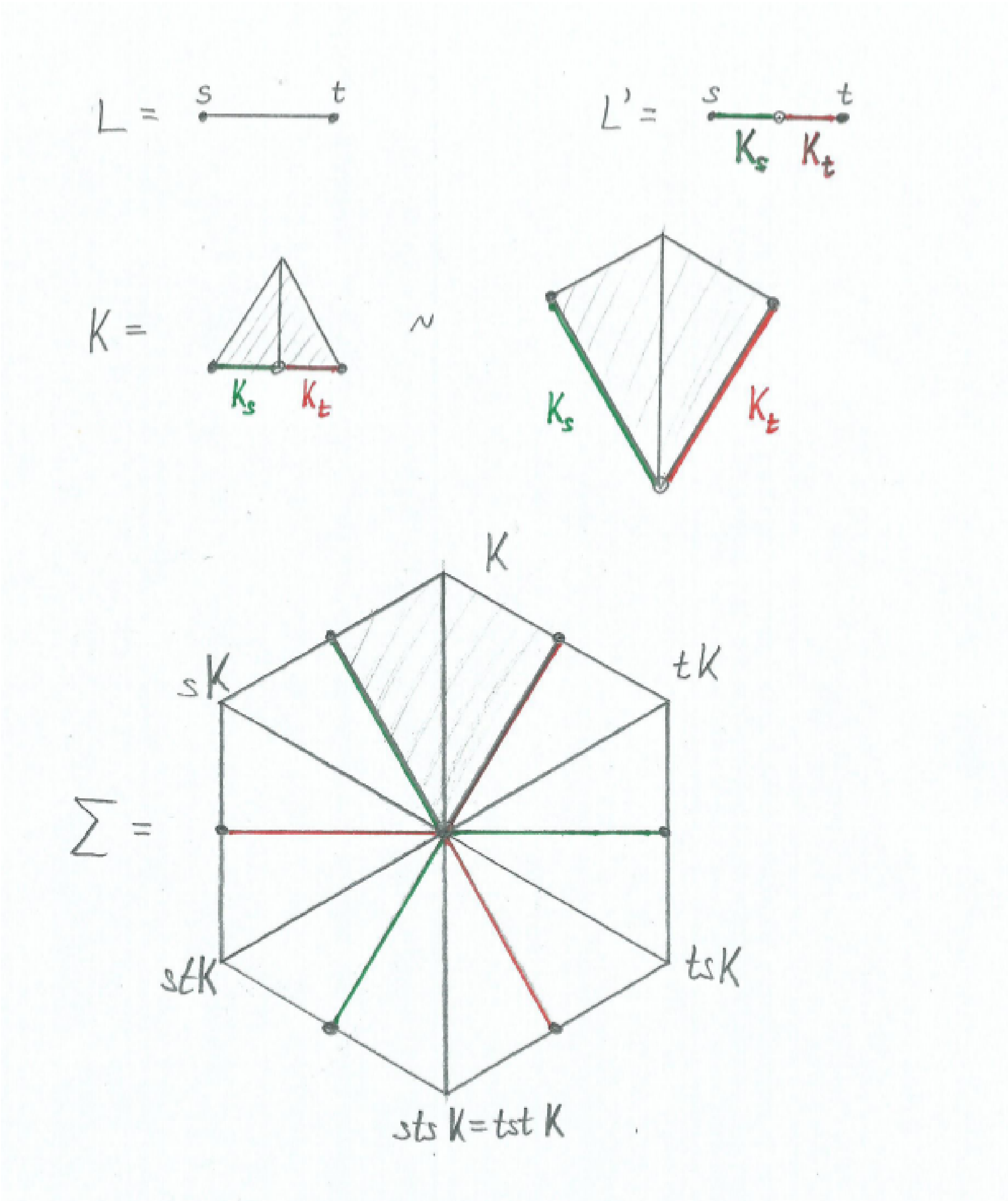



Figure 6.1: Davis complex for $W = \langle s, t \,|\, s^2 = t^2 = 1, (st)^3 = 1 \rangle \cong D_6$. Note that $wK$ and $wsK$ are glued along the $s$-mirror $K_s$.

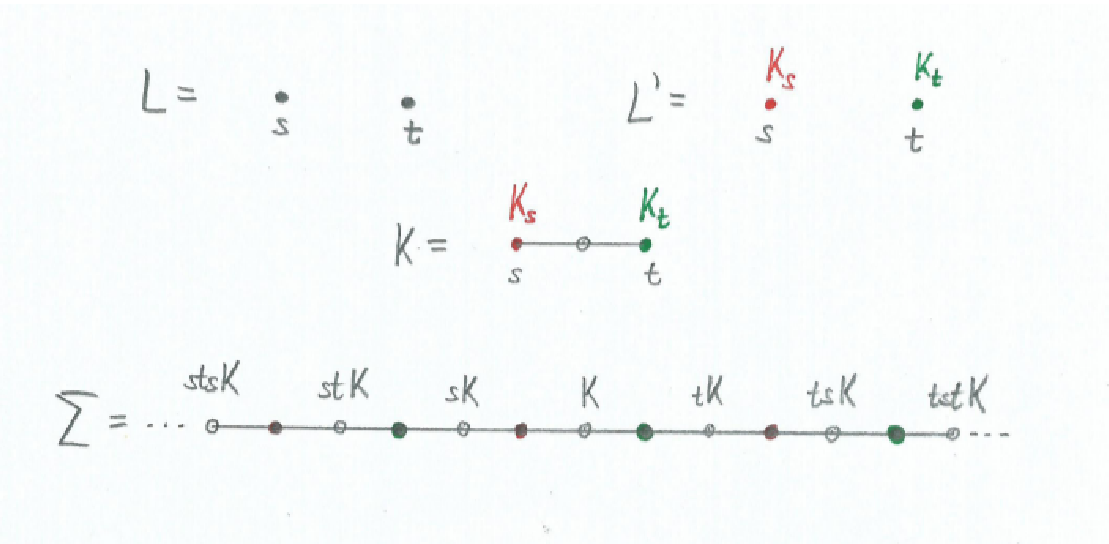



Figure 6.2: Davis complex for $W = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong D_\infty$.

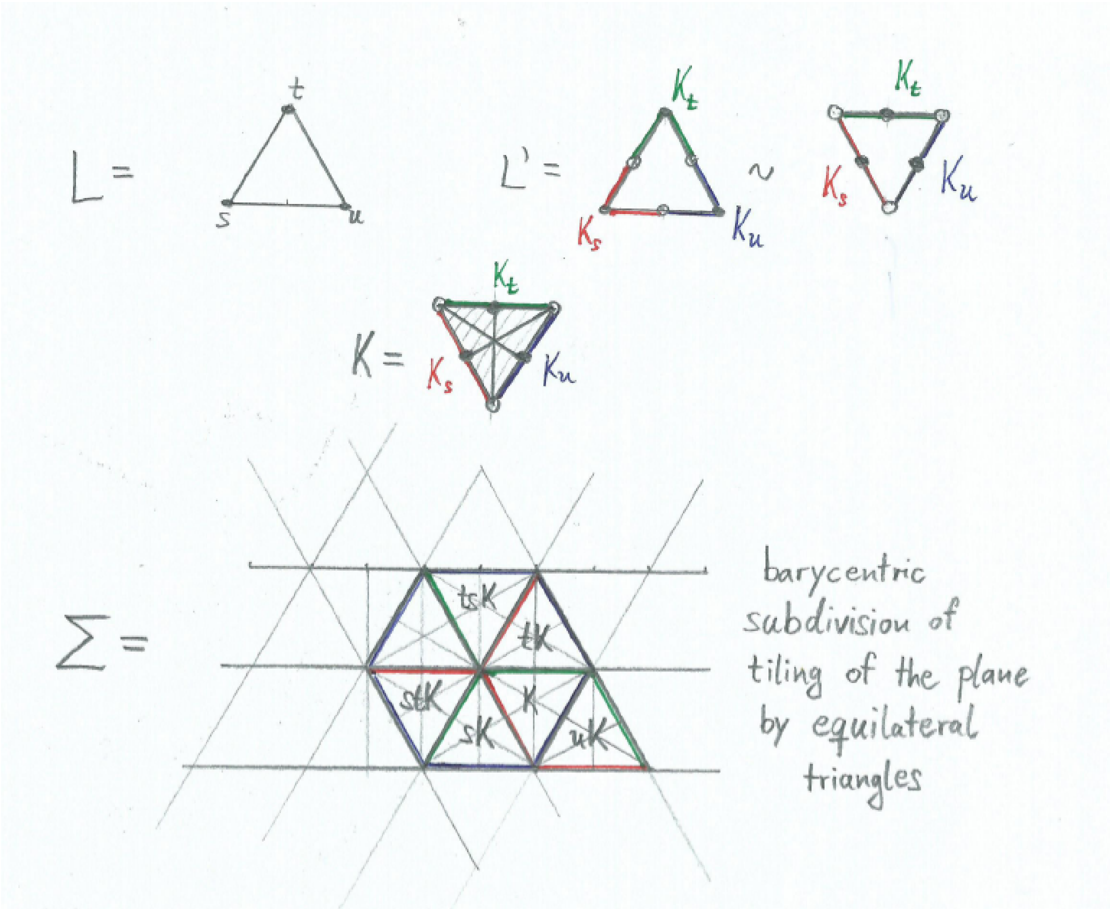



Figure 6.3: Davis complex for $W = \langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle$, i.e. the $(3, 3, 3)$-triangle group.

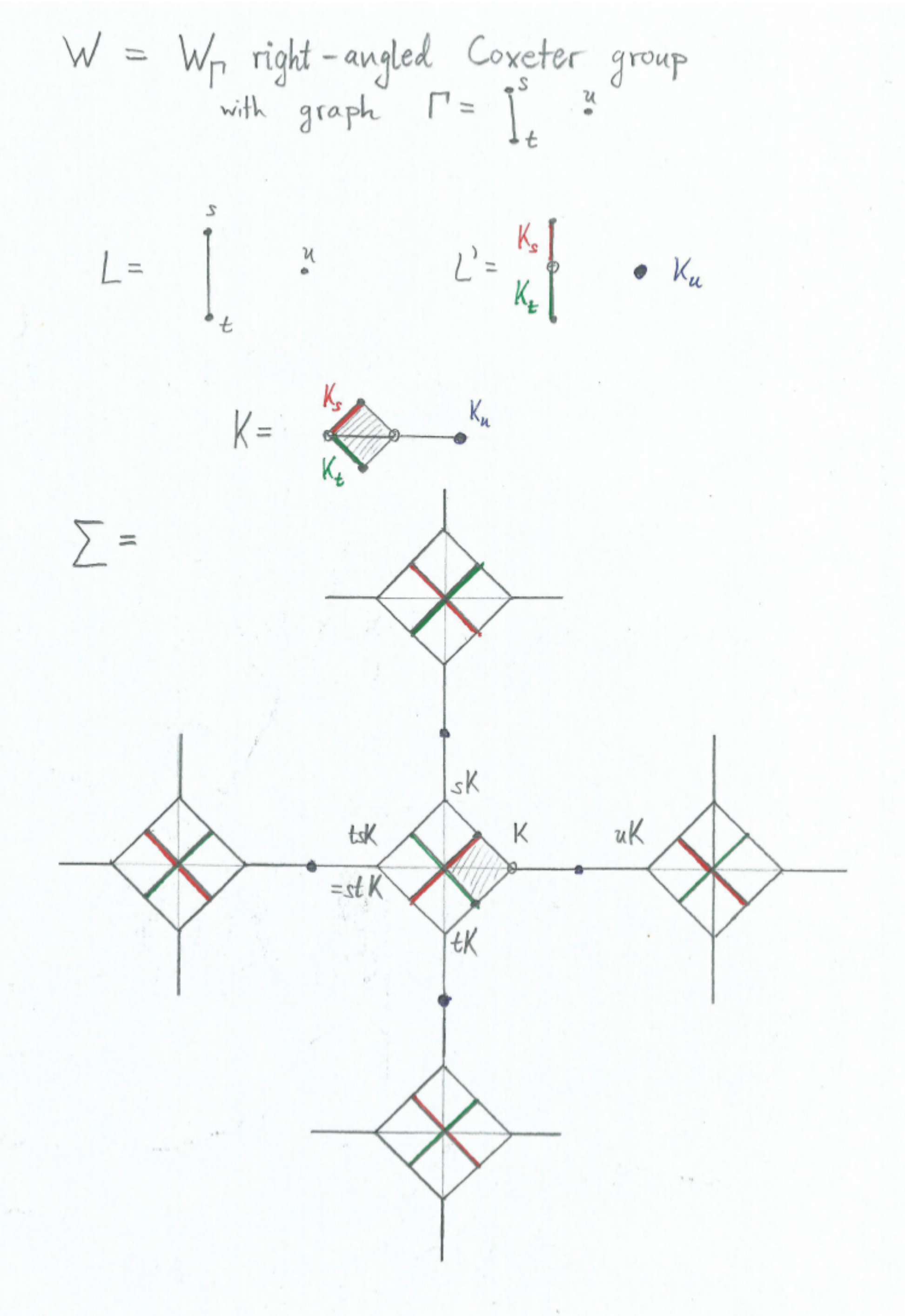



Figure 6.4: Davis complex for the right-angled Coxeter group $W = \langle s, t, u \,|\, s^2 = t^2 = u^2 = 1, (st)^2 = 1\rangle$ with corresponding graph $\Gamma$. Note that this yields a tree-like structure, since $u$ does not commute with $s$ and $t$.

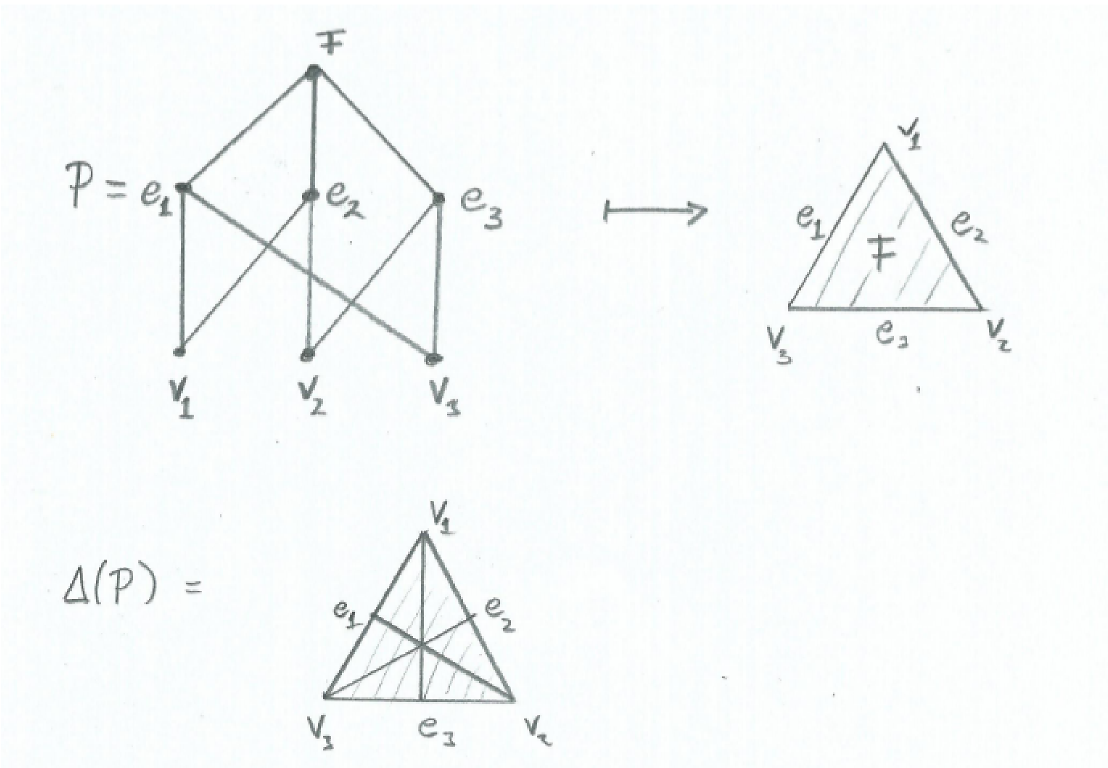



Figure 6.5: The poset $P$ of cells, ordered by inclusion. $\Delta(P)$ is the barycentric subdivision of the corresponding cell complex.

# LECTURE 7

# GEOMETRY OF THE DAVIS COMPLEX

20.04.2016

In the following let $(W, S)$ be a Coxeter system and $\Sigma = \Sigma(W, S)$ be the associated Davis complex with chambers $wK$ $(w \in W)$.

Recall that we have the following bijections:

$$\{\text{vertices of } K\} \longleftrightarrow \{W_T \,|\, T \subseteq S, W_T \text{ is finite}\},$$
$$\{\text{vertices of } \Sigma\} \longleftrightarrow \{wW_T \,|\, T \subseteq S, W_T \text{ is finite}, w \in W\},$$

and the $n$-simplices in $K$ (resp. $\Sigma$) correspond to $(n+1)$-chains in the corresponding posets, ordered by inclusion. Figure 7.1 and Figure 7.2 illustrate this.

## 7.1 Re-cellulation of $\Sigma$

We shall now equip $\Sigma$ with a new cellular structure and denote the resulting $CW$-complex by $\Sigma_{\text{new}}$. The vertices of the new cellulation $\Sigma_{\text{new}}$ are the cosets $wW_{\emptyset}$, i.e. the cosets of the trivial group. Thus the vertices of $\Sigma_{\text{new}}$ are in bijection with the elements of $W$.

The edges of $\Sigma_{\text{new}}$ are spanned by the cosets $wW_{\{s\}}$ $(w \in W, s \in S)$. Now $wW_{\{s\}} = \{w, ws\}$, so the 1-skeleton of $\Sigma_{\text{new}}$ is $\text{Cay}(W, S)$.

In general a subset $U \subseteq W$ is the vertex set of a cell in $\Sigma_{\text{new}}$ $\iff$ $U = wW_T$ where $w \in W$, $W_T$ finite; see for example Figures 7.3 to 7.6. This eliminates the "topologically unimportant" additional cells coming from the barycentric subdivision in the previous definition of $\Sigma = \mathcal{U}(W, K)$.

So a third definition of $\Sigma$ is that it is $\text{Cay}(W, S)$ with all cosets of finite special subgroups "filled in". From now on, we work with $\Sigma_{\text{new}}$ and write $\Sigma$ for it.

**Lemma 7.1.** $\Sigma$ is simply-connected.

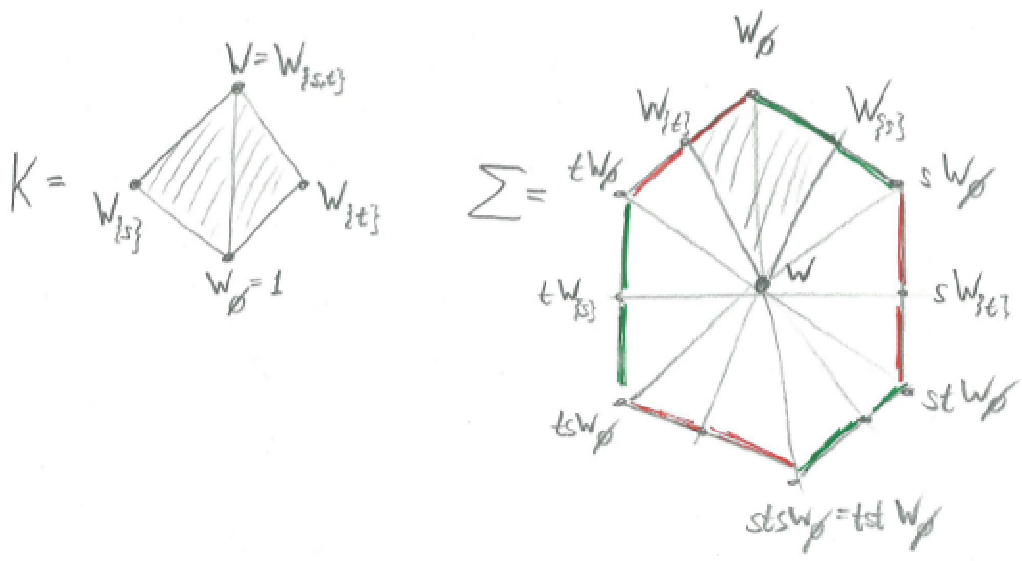



Figure 7.1: The Davis complex constructed from posets for $W = \langle s, t \,|\, s^2 = t^2 = 1, (st)^3 = 1\rangle \cong D_6$.

*Proof.* It is sufficient to consider the 2-skeleton $\Sigma^{(2)}$ and show that any loop in $\Sigma^{(1)} = \mathrm{Cay}(W, S)$ is null-homotopic in $\Sigma^{(2)}$.

The 2-cells of $\Sigma$ have vertex sets $wW_{\{s,t\}}$ with $W_{\{s,t\}}$ a finite dihedral group. This 2-cell has boundary word $(st)^m$ where $W_{\{s,t\}} \cong D_{2m}$. That is, any loop in $\Sigma^{(1)}$ can be filled in by conjugates of relators in the presentation of $(W, S)$. So $\Sigma^{(2)}$ is simply-connected. □

## 7.2 Coxeter polytopes

Recall: if $W$ is finite, $|S| = n$, then a Coxeter polytope is the convex hull of a generic $W$-orbit in $\mathbb{R}^n$; see for example Figure 7.7 and Figure 7.8.

In $\Sigma = \Sigma_{\text{new}}$, the cell with vertex set $wW_T$ is cellularly isomorphic to any Coxeter polytope for $W_T$.

Today we will metrise $\Sigma$ by making each cell $wW_T$ isometric to a (fixed) Coxeter polytope for $W_T$.

## 7.3 Polyhedral complexes

**Definition 7.2.** A *polyhedral complex* is a finite-dimensional CW-complex in which each $n$-cell is metrised as a convex polytope in $\mathbb{X}^n$ (the same $\mathbb{X}^n$ for each $n$-dimensional cell), and the attaching maps are isometries on codimension-one faces.

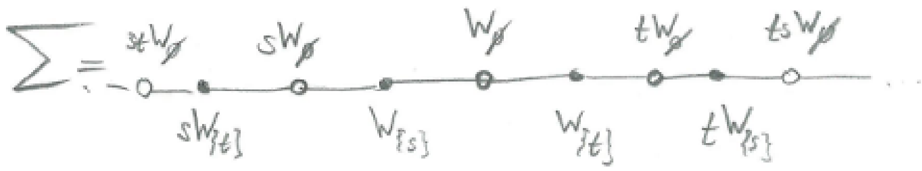



Figure 7.2: The Davis complex constructed from posets for $W = \langle s, t \mid s^2 = t^2 = 1\rangle \cong D_\infty$.

**Theorem 7.3** (Bridson)**.** *If a polyhedral complex $X$ has finitely many isometry types of cells, then $X$ is a geodesic metric space.*

Hence if we use the same Coxeter polytope for each coset of $W_T$, $\Sigma$ is a piecewise Euclidean geodesic metric space.

### 7.3.1 Metrisation of $\Sigma$

Pick $\underline{d} = (d_s)_{s\in S}$, $d_s > 0$. For $W_T$ finite, let $\rho : W_T \to O(n, \mathbb{R})$, $n = |T|$ be the Tits representation.For $t \in T$ the fixed set of $\rho(t)$ is the hyperplane $H_t$ with unit normalvector $e_t$, and the hyperplanes $H_t$, $H_{t'}$ meet at dihedral angle $\frac{\pi}{m}$ where $\langle t, t'\rangle \cong D_{2m}$; see for example Figure 7.9.

Let $C$ be the chamber $\{x \in \mathbb{R}^n \mid \langle x, e_t\rangle \geq 0 \quad \forall t \in T\}$. Then there is a unique $x \in \operatorname{int}(C)$ such that $d(x, H_t) = d_t > 0$ for all $t \in T$. We metrise each $wW_T$ as a copy of the Coxeter polytope which is the convex hull of the $W_T$-orbit of this $x$.

**Example 7.4.** If $W = W_\Gamma$ is right-angled, then each finite $W_T$ is $(C_2)^m$ so we are filling in right-angled euclidean polytopes.

### 7.3.2 Nonpositive curvature

**Theorem 7.5.** *When equipped with this piecewise Euclidean metric, $\Sigma$ is CAT$(0)$.*

**Definition 7.6.** A metric space $X$ is CAT(0) if $X$ is geodesic and geodesic triangles in $X$ are "no fatter" than triangles in $\mathbb{E}^2$.

That means: if $\Delta = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$ is a geodesic triangle in $X$ with respective edge lengths $l_1, l_2, l_3$ then there is a so called *comparison triangle* $\overline{\Delta} = \{[\overline{x}_1\overline{x}_2], [\overline{x}_2\overline{x}_3], [\overline{x}_3\overline{x}_1]\}$ in $\mathbb{E}^2$ with the same respective edge lengths, i.e. $d_X(x_i, x_j) = d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j)$. (Here $[xy]$ denotes some geodesic segment from a point $x$ to a point $y$.) Now $\Delta$ should not be "fatter" than the comparison triangle $\overline{\Delta}$, i.e. we must have

$$d_X(p, q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q}),$$

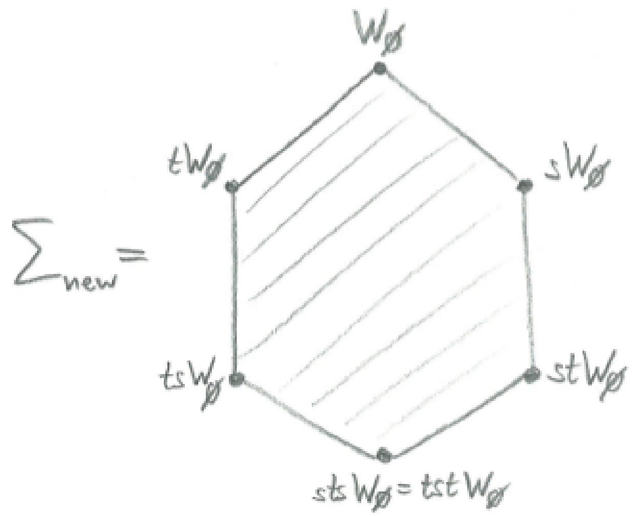



Figure 7.3: The new cellulation of the Davis complex for $W = \langle s, t \,|\, s^2 = t^2 = 1, (st)^3 = 2\rangle \cong D_6$.

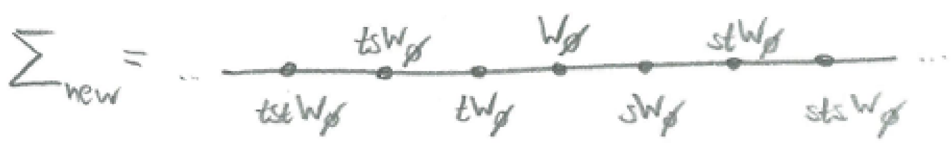



Figure 7.4: The new cellulation of the Davis complex for $W = \langle s, t \,|\, s^2 = t^2 = 1\rangle \cong D_\infty$.

where $p, q$ are arbitrary points on the sides of $\Delta$ and $\overline{p}, \overline{q}$ the corresponding points on $\overline{\Delta}$; see Figure 7.10.

Similarly, a geodesic metric space $X$ is CAT$(-1)$ if geodesic triangles in $X$ are "no fatter" than comparison triangles in $\mathbb{H}^2$.

A metric space $X$ is CAT(1) if all points in $X$ at distance $< \pi$ are connected by geodesics, and all triangles in $X$ with perimeter $\leq 2\pi$ are "no-fatter" than comparison triangles in a hemisphere of $\mathbb{S}^2$.

**Example 7.7.** If $X$ is a metric graph, then $X$ is CAT(1) if and only if each embedded cycle has length $\geq 2\pi$.

*Remark* 4. The motivation for CAT$(\kappa)$ is to give a notion of curvature which applies to symmetric spaces, buildings and many other (possibly) singular spaces.

The next proposition summarises some properties of CAT(0) spaces.

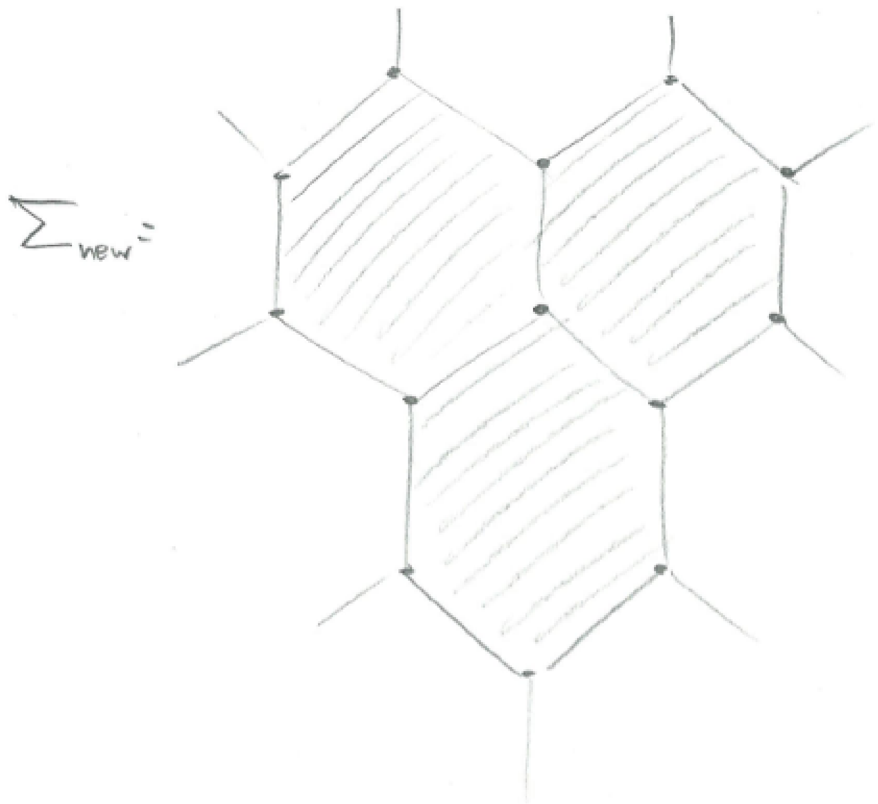



Figure 7.5: The new cellulation of the Davis complex for the $(3,3,3)$-triangle group $W$.

**Proposition 7.8.** Let $X$ be a complete CAT(0) space. Then:

1. $X$ is uniquely geodesic.

2. $X$ is contractible.

3. If $G$ acts on $X$ by isometries and $H$ is a subgroup of $G$ then $X^H$, the fixed set of $H$ in $X$, if non-empty, is convex. In particular convex subsets of CAT(0) spaces are CAT(0), so every fixed set $X^H$ is contractible by 2.

4. (Bruhat-Tits Fixed Point Theorem). If $G$ acts on $X$ by isometries and $G$ has a bounded orbit, then $X^G \neq \emptyset$. In particular for every finite subgroup $H \leq G$, we have $X^H \neq \emptyset$.

5. If a group $G$ acts properly and cocompactly by isometries on $X$ then the "word problem" and the "conjugacy problem" are both solvable for $G$.

We will only give sketch proofs for some of the above results.

*Proof.*

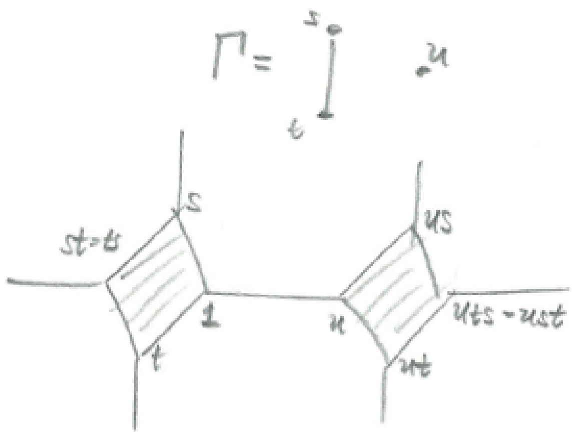



Figure 7.6: The new cellulation of the Davis complex for the right-angled Coxeter group $W = W_\Gamma$ with graph $\Gamma$ as depicted.

**To 1:** Let $x, y \in X$ and let $\gamma = [xy]$ be some geodesic from $x$ to $y$ in $X$. Suppose $\gamma'$ is another geodesic from $x$ to $y$ and let $z$ be a point on $\gamma$. Then $\gamma \cup \gamma'$ forms a geodesic triangle with vertices $x, y, z$. However, a comparison triangle in $\mathbb{E}^2$ is degenerate; see Figure 7.11. Because $X$ is CAT(0) and hence Euclidean comparison triangles are not fatter than geodesic triangles in $X$, the point $z$ has to be in $\gamma'$. Since the point $z$ was arbitrarily chosen, we get that every point of $\gamma$ is a point of $\gamma'$; hence $\gamma = \gamma'$.

**To 2:** By 1, $X$ is uniquely geodesic. With a bit of work, one may now show that a contraction of $X$ is given by sliding each point along its unique geodesic towards some point $x_0$ in $X$.

**To 3:** Let $x, y \in X^H$ be fixed by the $H$-action and let $\gamma$ be the (unique) geodesic from $x$ to $y$. Because $H$ acts by isometries and isometries map geodesics to geodesics, also $h\gamma$ is a geodesic from $hx = x$ to $hy = y$. By uniqueness, we get $h\gamma = \gamma$ pointwise, i.e. $\gamma \subseteq X^H$; see Figure 7.12. Hence $X^H$ is (geodesically) convex.

**To 4:** If $G$ has a bounded orbit $Gx$ in $X$ then we can consider the convex hull of the points in $Gx$. The barycenter is then a fixed point of $G$.

□

These, in combination with Theorem 7.5, prove:

1. $\Sigma$ is a finite-dimensional $\underline{E}W$.

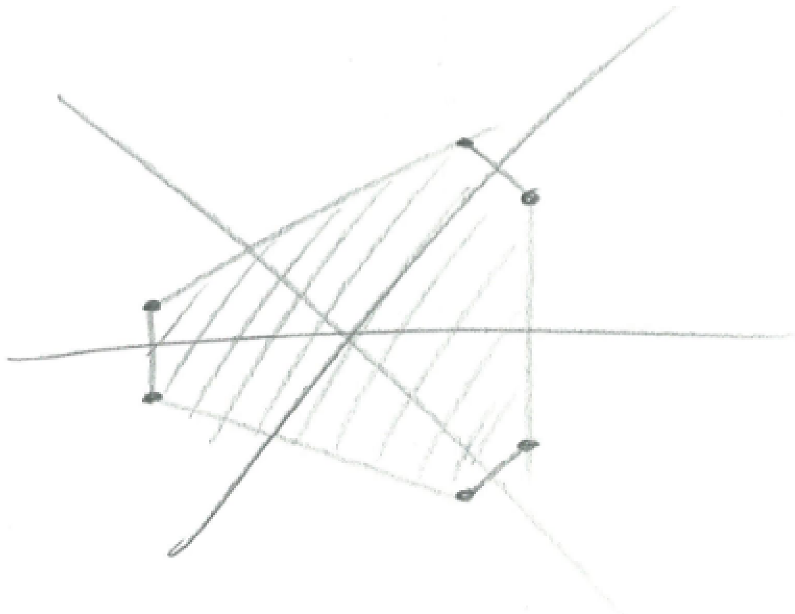


Figure 7.7: A Coxeter polytope for $W = \langle s, t \,|\, s^2 = t^2 = 1, (st)^3 = 2 \rangle \cong D_6$.

2. If $H \leq W$ is finite then there is an element $w \in W$ and a spherical subset $T \subseteq S$, such that $H \leq wW_T w^{-1}$. (This was already earlier proved by Tits.)

3. The "conjugacy problem" for $W$ is solvable. The "isomorphism problem" for Coxeter groups is still open.

## 7.4 Proof of Theorem 7.5

Let us now give a proof of Theorem 7.5. We will need the following "Cartan-Hadamard Theorem for CAT(0) spaces" due to Gromov:

**Theorem 7.9** (Gromov)**.** *Let $X$ be a complete, connected geodesic metric space. If $X$ is locally CAT(0) then the universal cover of $X$ is CAT(0).*

Since $\Sigma$ is complete, connected and simply-connected, it is enough to show that $\Sigma$ is locally CAT(0). For that we use Gromov's Link Condition:

**Theorem 7.10** (Gromov Link Condition)**.** *If $X$ is a piecewise Euclidean polyhedral complex then $X$ is locally CAT(0) if and only if for every vertex $v$ of $X$, the link of $v$ is CAT(1).*

Before we proceed, let us give an example of an $X$ as above which is not CAT(0) and for which the Link Condition does not hold.

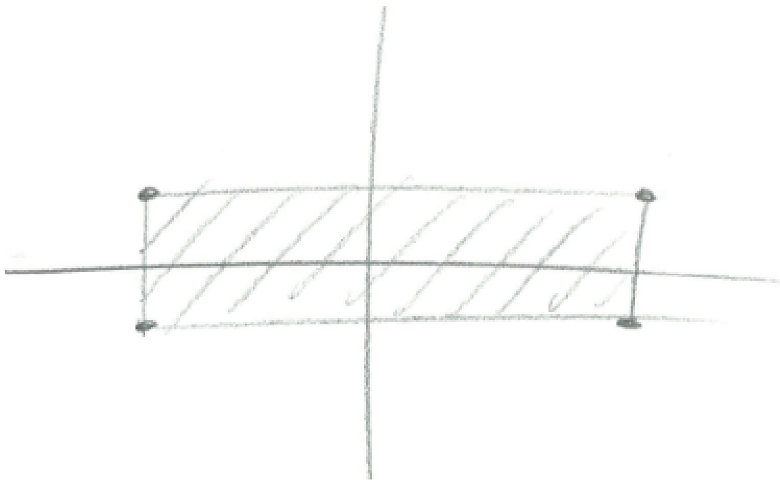


Figure 7.8: A Coxeter polytope for $W \cong (C_2)^n$.

**Example 7.11.** Consider $X$ to be the 2-skeleton of a cube. The link of a vertex is depicted in Figure 7.13. Each arc of it has length $\frac{\pi}{2}$, so $\mathrm{lk}(v, X)$ is not CAT(1).

If we wanted to make $X$ a CAT(0) space, we we would need to fill in the cube.

Therefore we need to investigate the links in $\Sigma$. Because $W$ acts transitively on the vertices of $\Sigma$ (by definition) it is enough to consider the link of the vertex $v = W_\emptyset = 1$. For each $W_T$ finite, $\mathrm{lk}(v, \Sigma)$ contains a spherical simplex $\sigma_T$ which is the link of $v$ in the corresponding Coxeter polytope.

In the Coxeter polytope of Figure 7.14, $\sigma_T$ is the spherical simplex with vertex set the unit normal vectors $\{-e_t\}_{t\in T}$. So we identify $\sigma_T$ with the simplex with vertex set $\{e_t\}_{t\in T}$.

**Corollary 7.12.** In $\Sigma$, the link of $v = 1$ is $L$, the finite nerve, with each simplex $\sigma_T$ of $L$ metrised as the simplex in $\mathbb{S}^{|T|-1}$ with vertex set $\{e_t\}_{t\in T}$.

**Example 7.13.** Figure 7.15 gives an example of a CAT(1) link.

Hence we will be done, if we can show that $L$, with this piecewise spherical structure, is CAT(1). In the special case that $W = W_\Gamma$ is right-angled, $L$ is the flag complex with 1-skeleton $\Gamma$; see Figure 7.16. This motivates the following lemma:

**Lemma 7.14** (Gromov)**.** Suppose all simplices of a simplicial complex $\Delta$ are metrised as spherical simplices with edge lengths $\frac{\pi}{2}$. Then $\Delta$ is CAT(1) if and only if $\Delta$ is flag.

**Corollary 7.15.** If $W_\Gamma$ is right-angled, $\Sigma$ can be metrised as a CAT(0) cube complex (with proper, cocompact $W_\Gamma$-action).

In general a simplicial complex $\Delta$ with an assignment of edge lengths is a *metric flag complex* if a pairwise connected subset of vertices spans a simplex in $\Delta$ if and only if there is a spherical simplex with these edge lengths.

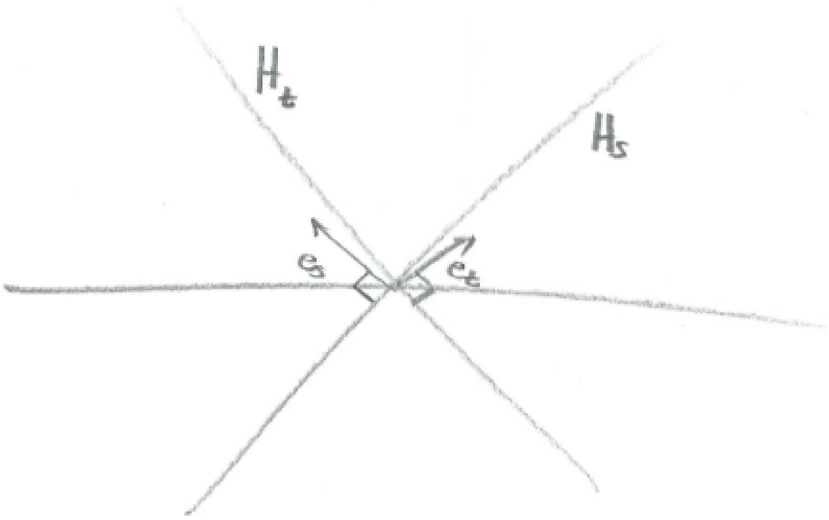



Figure 7.9: Two hyperplanes $H_s = e_s^\perp$ and $H_t = e_t^\perp$ in the Tits representation corresponding to a subgroup $\langle t, s\rangle \cong D_{2m}$.

**Lemma 7.16** (Moussong)**.** Suppose a simplicial complex $\Delta$ is metrised as a spherical simplicial complex so that all edge lengths are $\geq \frac{\pi}{2}$.

Then $\Delta$ is CAT(1) if and only if $\Delta$ is a metric flag complex.

**Corollary 7.17.** Since all edge lengths in $L$ are $\pi - \frac{\pi}{m}$ with $m \geq 2$, $\Sigma$ is CAT(0).

This finishes the proof of Theorem 7.5.

□

One may now ask the question: When is $\Sigma$ not only CAT(0) but actually CAT(−1)? More precisely: When can we equip $\Sigma$ with a piecewise hyperbolic metric such that it is CAT(−1)?

If $W_T$ is finite then $W_T$ acts by isometries on $\mathbb{H}^n$ with $n = |T|$, so we can also define a Coxeter polytope for $W_T$ in $\mathbb{H}^n$. It is important to note that the dihedral angles in any hyperbolic Coxeter polytope will be strictly less than the dihedral angles in a Euclidean Coxeter polytope.

**Example 7.18.** A hyperbolic Coxeter polytope for $W_T \cong C_2 \times C_2$ is depicted in Figure 7.17. Note that the dihedral angles will be strictly less than $\frac{\pi}{2}$.

As before, $\Sigma$ equipped with this piecewise hyperbolic structure is CAT(−1) if and only if the link of every vertex is CAT(1).

**Theorem 7.19** (Moussong)**.** *This piecewise hyperbolic structure on $\Sigma$ is CAT$(-1)$ if and only if there is no subset $T \subseteq S$ such that either:*

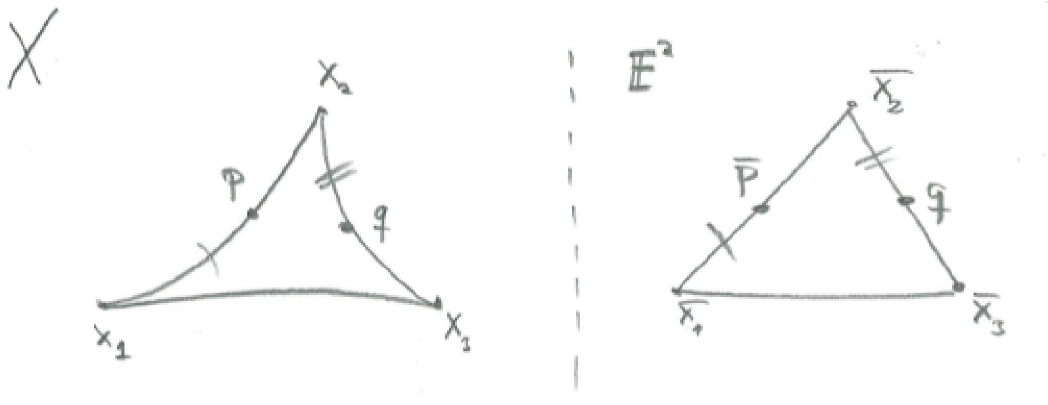



Figure 7.10: A geodesic triangle $\Delta = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$ in a CAT(0) space $X$ with corresponding comparison triangle $\overline{\Delta} = \{[\overline{x}_1\overline{x}_2], [\overline{x}_2\overline{x}_3], [\overline{x}_3\overline{x}_1]\}$ in $\mathbb{E}^2$.

1. *$W_T$ is an Euclidean geometric reflection group of dimension $\geq 2$ (i.e. $(W_T, T)$ is affine of rank 2);*

2. *$(W_T, T)$ is reducible with $W_T = W_{T'} \times W_{T''}$ and both $W_{T'}$ and $W_{T''}$ are infinite.*

*Idea of Proof.* If 1 or 2 hold then the link of a vertex in $\Sigma$, with its Euclidean structure, contains an isometric copy of $\mathbb{S}^{n-1}$ $(n \geq 2)$. Hence we cannot shrink the angles and retain the CAT(1) condition.

In all other cases, one can show that $L$ is "extra-large" so there is enough scope to reduce the angles and retain the CAT(1) condition on the links. □

**Corollary 7.20.** The following are equivalent:

1. $W$ is word hyperbolic;

2. $W$ does not contain a $\mathbb{Z} \times \mathbb{Z}$ subgroup;

3. neither 1 or 2 in the previous theorem hold;

4. $\Sigma$ admits a piecewise hyperbolic metric which is CAT($-1$).

In particular:

**Corollary 7.21.** If $W = W_\Gamma$ is right-angled then $W_\Gamma$ is word hyperbolic, if and only if $\Gamma$ has no "empty squares", i.e. each "square" in $\Gamma$ has at least one of its diagonals; see Figure 7.18.

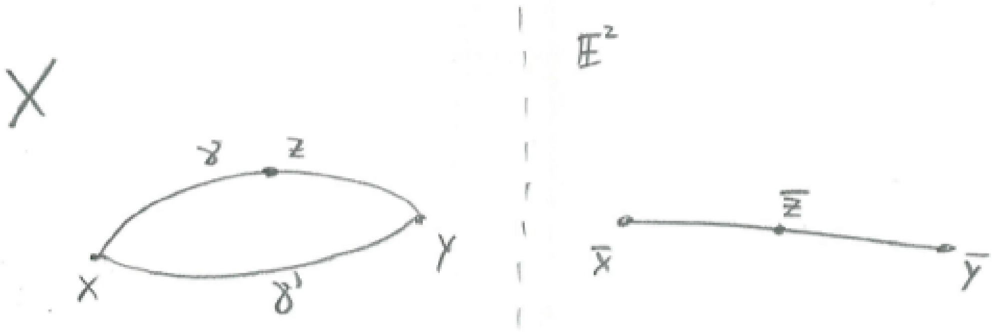



Figure 7.11: Illustration of the proof of assertion 1 in Proposition 7.8.

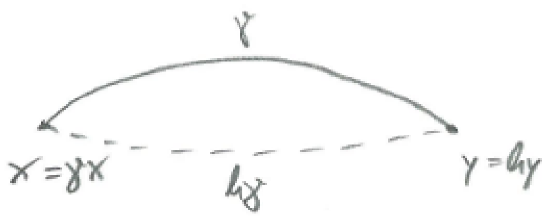



Figure 7.12: Illustration of the proof of assertion 3 in Proposition 7.8.

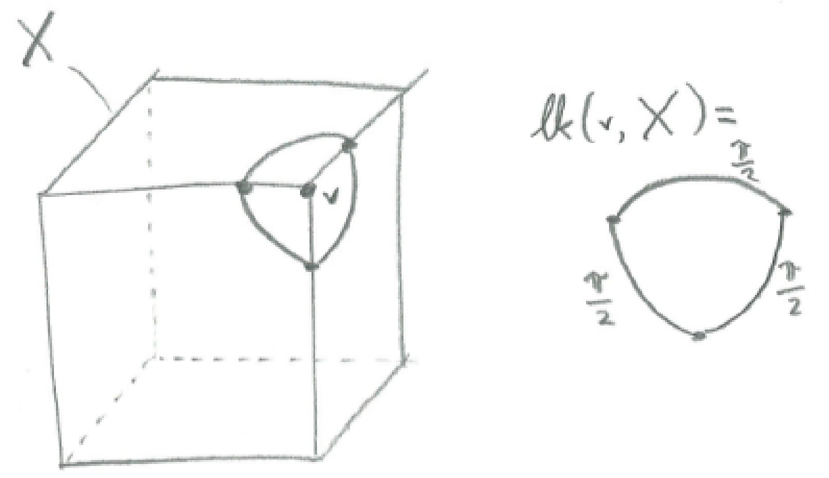



Figure 7.13: If $X$ is the 2-skeleton of a cube its link is not CAT(1).

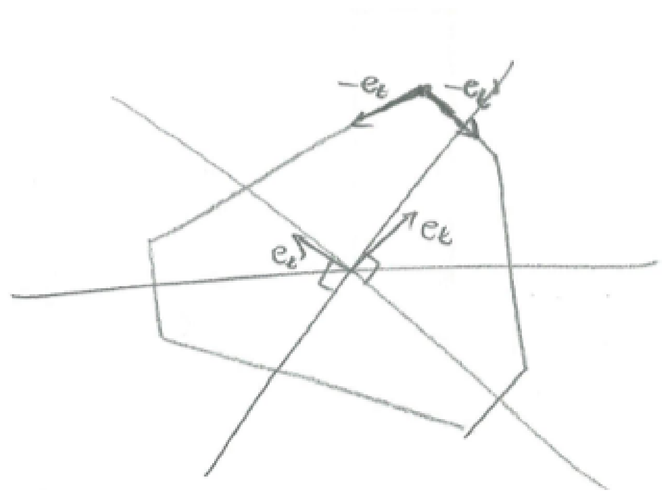



Figure 7.14: In this Coxeter polytope $\sigma_T$ is the spherical simplex with vertex set the unit normal vectors $\{-e_t\}_{t \in T}$.

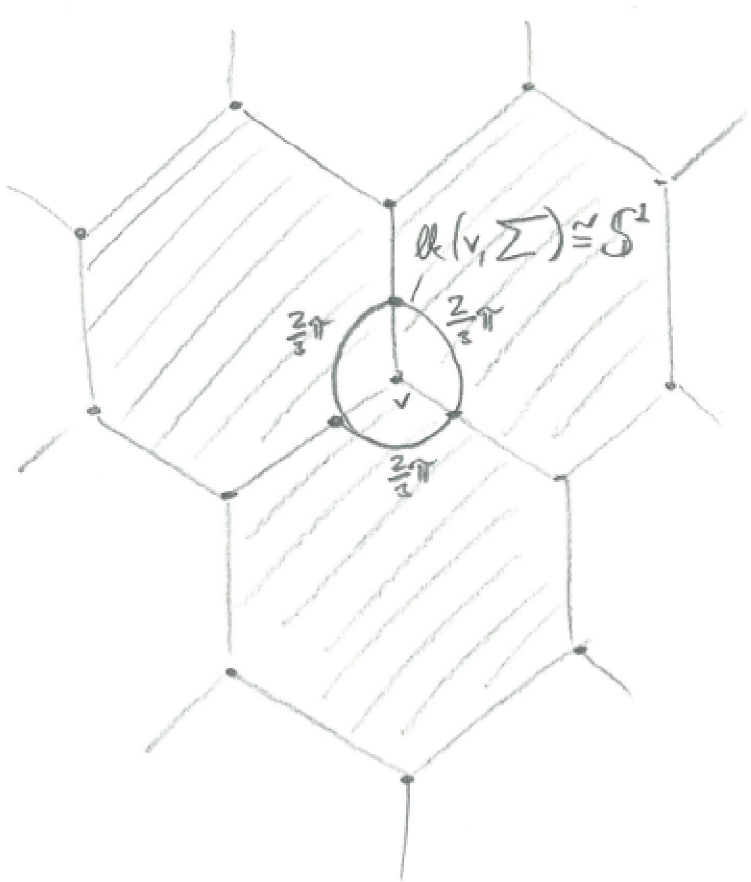



Figure 7.15: If $W$ is the $(3,3,3)$-triangle group then $\mathrm{lk}(v,\Sigma)$ is isometric to $\mathbb{S}^1$.

Figure 7.16: If $W_\Gamma$ is right-angled, each $\sigma_T$ is a right-angled spherical simplex where all edges have length $\frac{\pi}{2}$.

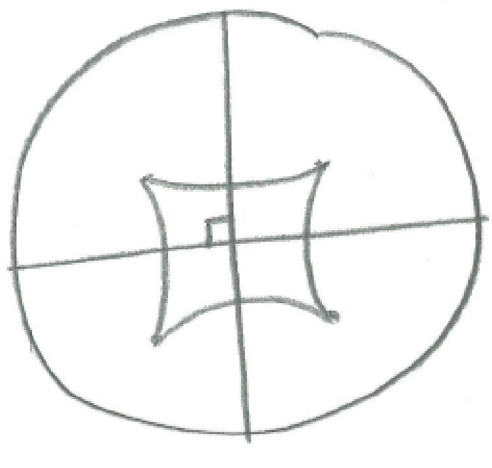


Figure 7.17: A hyperbolic Coxeter polytope for $W_T \cong C_2 \times C_2$.

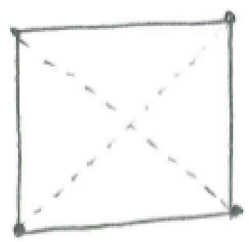


Figure 7.18: The two “diagonals” in a “square” of a graph.

# LECTURE 8

# BOUNDARIES OF COXETER GROUPS

27.04.2016

## 8.1 The visual boundary $\partial X$

Let $X$ be a complete CAT(0) metric space. The *visual boundary* (or *Gromov boundary*) of $X$, denoted $\partial X$, as a set is:

$$\{\text{geodesic rays in } X\}/\sim \quad = \quad \{\text{geodesic rays from some } x_0 \in X\}/\sim$$

where $\gamma \sim \gamma'$ if their images are at (uniform) bounded distance from each other. Sometimes we denote the equivalence class of a geodesic ray $\gamma$ by $\gamma(\infty)$.

As for the topology of $\partial X$: a basis of open sets is given by all the

$$U(\gamma, r, \epsilon) = \{\delta \text{ a geodesic ray starting from } x_0 \text{ and passing through the ball } B_\epsilon(\gamma(r))\}$$

where $\gamma$ is a geodesic ray, and $\epsilon, r > 0$; see Figure 8.1. That means that $\gamma(\infty)$ and $\gamma'(\infty)$ are "close" in $\partial X$ if $\gamma, \gamma'$ track each other for a long time.

If a group $G$ acts *geometrically* (properly and cocompactly by isometries) on a complete CAT(0) space $X$ then $G$ is called a *CAT(0) group*, and $\partial X$ is called a *CAT(0) boundary of* $G$. Note that the CAT(0) is not necessarily uniquely determined by the group.

**Example 8.1.**

1. Coxeter groups are CAT(0) groups, and for any choice of $\underline{d} = (d_s)_{s \in S}$ the visual boundary $\partial \Sigma_{\underline{d}}$ is a CAT(0) boundary of $W$. Here $\Sigma_{\underline{d}}$ denotes the Davis complex with the piecewise Euclidean structure determined by $\underline{d}$.

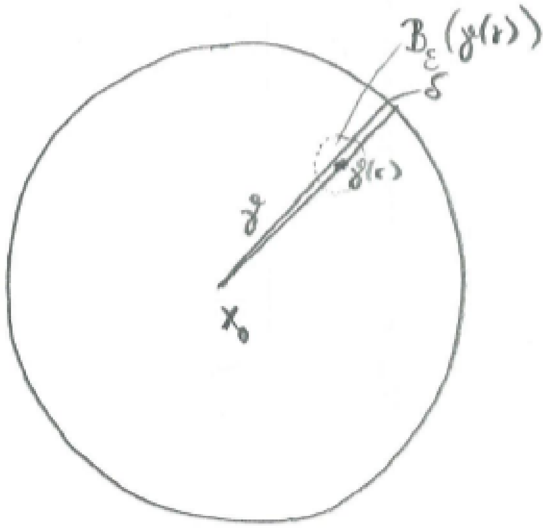



Figure 8.1: The open sets $U(\gamma, r, \epsilon)$ defining the topology of the Gromov boundary $\partial X$.

2. Right-angled Artin groups $A_\Gamma$ are CAT(0) groups: Let $\Gamma$ be a finite simplicial graph with vertex set $S$. Define

$$A_\Gamma = \langle S \mid st = ts \iff \{s,t\} \text{ is an edge in } \Gamma \rangle.$$

Note that we get an epimorphism $A_\Gamma \twoheadrightarrow W_\Gamma$ by adding the relations $s^2 = 1$ for every $s \in S$ in order to obtain the corresponding right-angled Coxeter group $W_\Gamma$.

Now the *Salvetti complex* $S_\Gamma$ for $A_\Gamma$ is the cell complex with:

- one vertex $v$;
- a directed loop for each $s \in S$;
- for each complete subgraph $K_n$ of $\Gamma$, an $n$-torus is glued in along the corresponding loops; see Figure 8.2.

Then $\pi_1(S_\Gamma) \cong A_\Gamma$ and, by considering the links, the universal cover of $S_\Gamma$, denoted by $\tilde{S}_\Gamma$, can be equipped with a piecewise Euclidean CAT(0) metric.

Questions: Suppose $G$ is a CAT(0) group.

1. Are all CAT(0) boundaries of $G$ homeomorphic?
2. If $G$ has CAT(0) boundaries $\partial X$ and $\partial X'$, are $\partial X$ and $\partial X'$ equivariantly homeomorphic?

*Remark* 5.

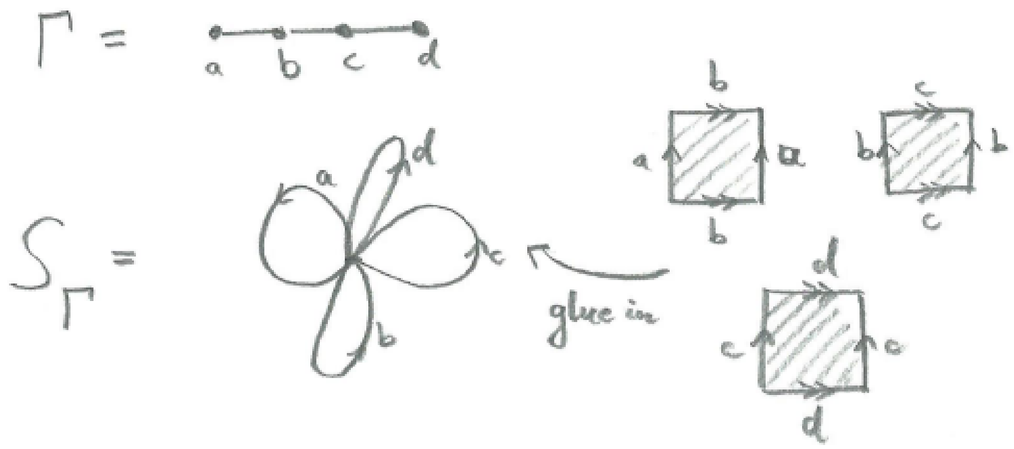



Figure 8.2: The construction of the Salvetti complex $S_\Gamma$.

- If $G$ is word hyperbolic then the answer to both questions is: yes.
- For a right-angled Artin group $A_\Gamma$ we have:

$$\begin{aligned} A_\Gamma \text{ is word hyperbolic } &\iff \Gamma \text{ has no edges} \\ &\iff A_\Gamma \text{ is a free group} \end{aligned}$$

**Theorem 8.2** (Croke-Kleiner, 2000)**.** *Let* $\Gamma$ *be the simplicial graph in Figure 8.2. Then* $A_\Gamma$ *has* 2 *non-homeomorphic* $CAT(0)$ *boundaries, obtained by varying angles in the three tori.*

Question 1 is open for Coxeter groups. For Question 2 there is the following result by Qing:

**Theorem 8.3** (Qing, 2013)**.** *Let* $\Gamma$ *be the simplicial graph in Figure 8.2. Then there are* $\underline{d} \neq \underline{d}'$ *such that* $\partial\Sigma_{\underline{d}}$ *and* $\partial\Sigma_{\underline{d}'}$ *are not equivariantly homeomorphic.*

*Proof idea.* Each Coxeter polytope is a rectangle. Now vary the edge lengths. □

## 8.2 Relationship between right-angled Artin groups (RAAG) and right-angled Coxeter groups (RACG)

**Theorem 8.4** (Davis-Januszkiewicz)**.** *Every RAAG has finite index in some RACG.*

*Proof.* Given $\Gamma$ we construct graphs $\Gamma'$ and $\Gamma''$ such that $X = \tilde{S}_\Gamma$ is the same cube complex as the Davis complex for $W_{\Gamma'}$, and $W_{\Gamma''} \curvearrowright X$. We use this action to show that both $W_{\Gamma'}$ and $A_\Gamma$ have index $2^n$ in $W_{\Gamma''}$ where $n = |V(\Gamma)|$. See for example Figure 8.3. □

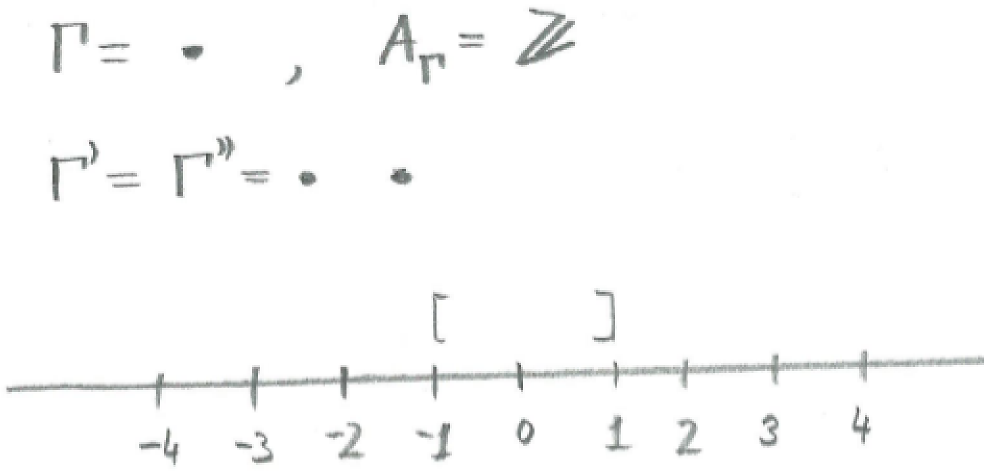



Figure 8.3: $A_\Gamma$ is generated by translations by 1; $W_{\Gamma'} \cong D_\infty$ is generated by reflections in $-1$ and 1; $W_{\Gamma''} \cong D_\infty$ is generated by reflections in 0 and 1.

However, there are RACGs which are not quasi-isometric to any RAAG.

**Example 8.5.**

1. There are no 1-ended word hyperbolic RAAGs, but if you consider the geometric reflection group $W_\Gamma$ of a right-angled pentagon in $\mathbb{H}^2$ (see Figure 1.9) then $W_\Gamma$ is 1-ended and word hyperbolic.

2. By considering a quasi-isometry invariant called divergence, Dani-T constructed an infinite sequence of RACGs $W_d$ which are in distinct quasi-isometry classes and not quasi-isometric to any RAAG.

## 8.3 The Tits boundary $\partial_T X$

Let $X$ be a complete CAT(0) space. The *Tits boundary* $\partial_T X$ is the same set as $\partial X$, but its topology is different. It is the metric topology induced by the *Tits metric* $d_T$.

Idea: Define an $n$-dimensional flat in $X$ as an isometrically embedded copy of $\mathbb{E}^n$. In the Tits boundary $\partial_T X$, the boundary of each $n$-dimensional flat, $n \geq 2$, is isometric to $\mathbb{S}^{n-1}$, i.e. $\partial_T X$ "detects the flats".

*Remark* 6. The Tits boundary of $\mathbb{H}^n$ is discrete. The Tits boundary was defined first for symmetric spaces of non-compact type and for Euclidean buildings – in both cases $\partial_T X$ is a spherical building.

Before we can define the Tits metric we need to introduce the so called *angular metric* $d_\sphericalangle$ on the visual boundary $\partial X$. For every $\xi, \eta \in \partial X$ we set

$$d_\sphericalangle(\xi, \eta) = \sup_{x \in X} \{\text{Alexandrov angle between } \xi \text{ and } \eta \text{ at } x\}.$$

**Example 8.6.** In $\mathbb{H}^2$ any two boundary points $\xi, \eta \in \partial \mathbb{H}^2$ are connected by a (unique) geodesic; hence $d_\sphericalangle(\xi, \eta) = \pi$. See Figure 8.4 for an illustration.

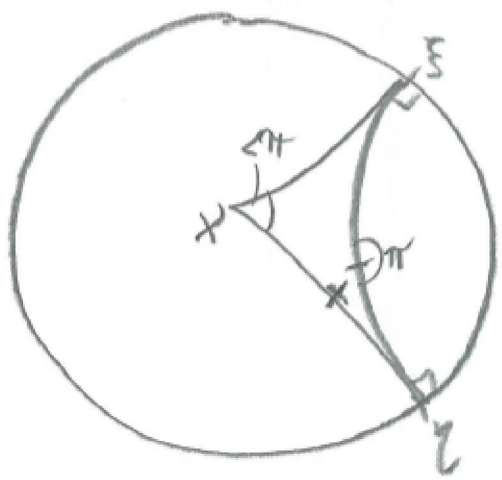



Figure 8.4: The angles between two equivalent geodesic rays in $\mathbb{H}^2$.

Now, for every $\eta, \xi \in \partial_T X$, the Tits metric is defined to be the Alexandrov angle between two geodesic rays $[x_0, \xi)$ and $[x_0, \eta)$, if there is a rectifiable path in $(\partial X, d_\sphericalangle)$ from $\xi$ to $\eta$. If there is no such rectifiable path we set $d_T(\xi, \eta) = \infty$.

## 8.4 Combinatorial boundaries (joint with T. Lam)

An *infinite reduced word* in $W$ is the label on a geodesic ray in $\mathrm{Cay}(W, S)$ starting at 1. We define a partial order on infinite reduced words: $\underline{w} \leq \underline{w}'$ if the set of walls crossed by $\underline{w}$ is contained in the set of walls crossed by $\underline{w}'$.

Each $\underline{w}$ determines a non-empty subset $\partial_T \Sigma(\underline{w})$ of the Tits boundary of the Davis complex $\partial_T \Sigma$.

**Theorem 8.7** (Lam-T)**.** *The sets $\partial_T\Sigma(\underline{w})$ partition $\partial_T\Sigma$; they are path connected, totally geodesic subsets and*

$$\partial_T\Sigma(\underline{w}) = \bigcup_{\underline{w}'\leq \underline{w}} \partial_T\Sigma(\underline{w}')$$

*Remark* 7. The sets $\partial_T\Sigma(\underline{w})$ are the same as elements of the minimal combinatorial compactification due to Caprace-Lcureux.

## 8.5 Limit roots (Hohlweg-Labb-Ripoll 2014, Dyer-Hohlweg-Ripoll 2013)

Let $(W, S)$ be a Coxeter system with Coxeter matrix $M = (m_{ij})$. Recall that the bilinear form associated to the Tits representation $\rho : W \to GL(V)$ where $V$ has basis $\{\alpha_1, \ldots, \alpha_n\}$ is:

$$B(\alpha_i, \alpha_j) = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right), & \text{if } m_{ij} < \infty \\ -1, & \text{if } m_{ij} = \infty. \end{cases}$$

As Vinberg proposed, we can relax this by allowing $B(\alpha_i, \alpha_j) \in (-\infty, -1]$.

**Example 8.8.** Let $W = \langle s, t \,|\, s^2 = t^2 = 1\rangle \cong D_\infty$ and let $\{\alpha, \beta\}$ denote a basis for $V$.

1. $B(\alpha, \beta) = -1$:

   We can define the *roots* $\Phi$ to be the $W$-orbit of $\{\alpha, \beta\}$. In our situation this amounts to

   $$\Phi = \{\pm(n\alpha + (n+1)\beta), \pm((n+1)\alpha + n\beta) \,|\, n \in \mathbb{Z}\};$$

   see Figure 8.5.

   We can intersect $V$ with the following affine hyperplane

   $$E = \{v \in V \,|\, v = \lambda\alpha + \lambda\beta, \lambda + \mu = 1\}.$$

   The *limit roots* are the accumulation points in $E$ of normalized roots, i.e. $E \cap \{\mathbb{R}\gamma \,|\, \gamma \in \Phi\}$.

   Note that all limit roots lie on the *isotropic cone*

   $$\begin{aligned} Q &= \text{span}\{\alpha + \beta\} \\ &= \{v \in V \,|\, B(v, v) = 0\}. \end{aligned}$$

2. $B(\alpha, \beta) < -1$: see Figure 8.6.

**Theorem 8.9** (Hohlweg-Labb-Ripoll)**.** *The limit roots lie on the isotropic cone.*

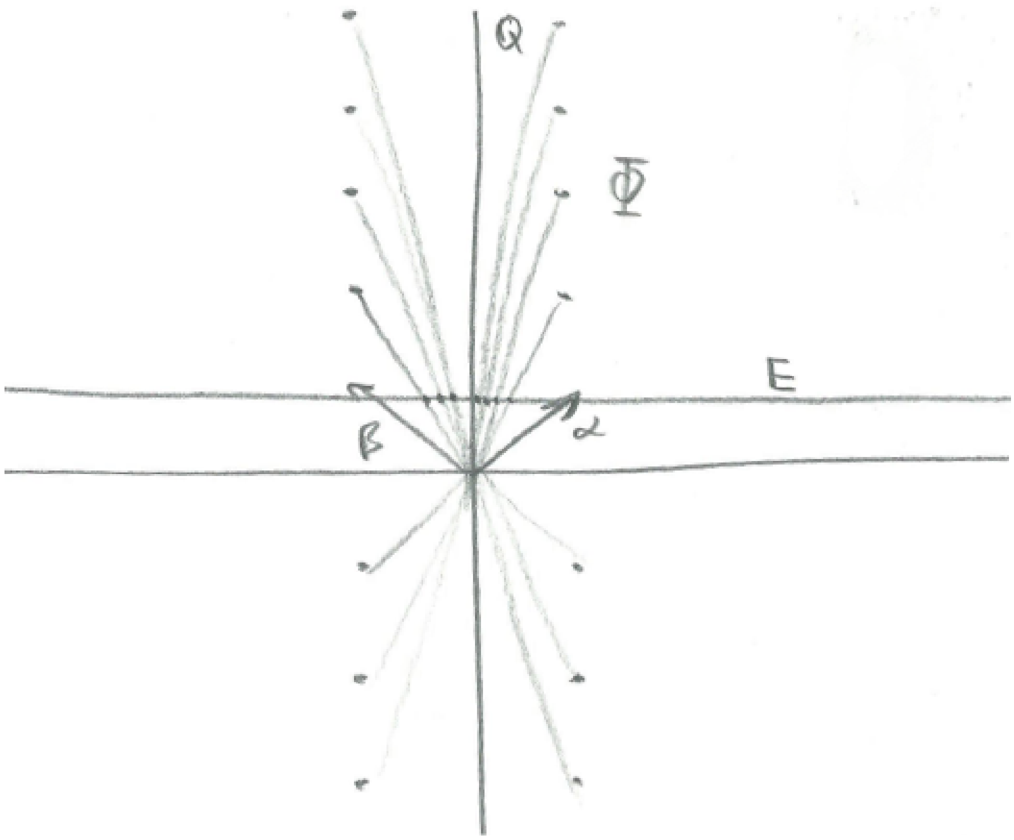



Figure 8.5: The isotropic cone $Q$, the affine hyperplane $E$, the roots $\Phi$. The limit root is the intersection point of $E$ with $Q$.

**Theorem 8.10** (Hohlweg-Praux-Ripoll, 2013)**.** *If $B$ has signature $(n, 1)$ then the set of limit roots equals the limit set of $W$ acting on the hyperbolic plane.*

**Theorem 8.11** (Chen-Labb, 2015)**.** *If $B$ has signature $(n, 1)$ then there is a unique limit root associated to each infinite reduced word which arises through a sequence of initial subwords.*

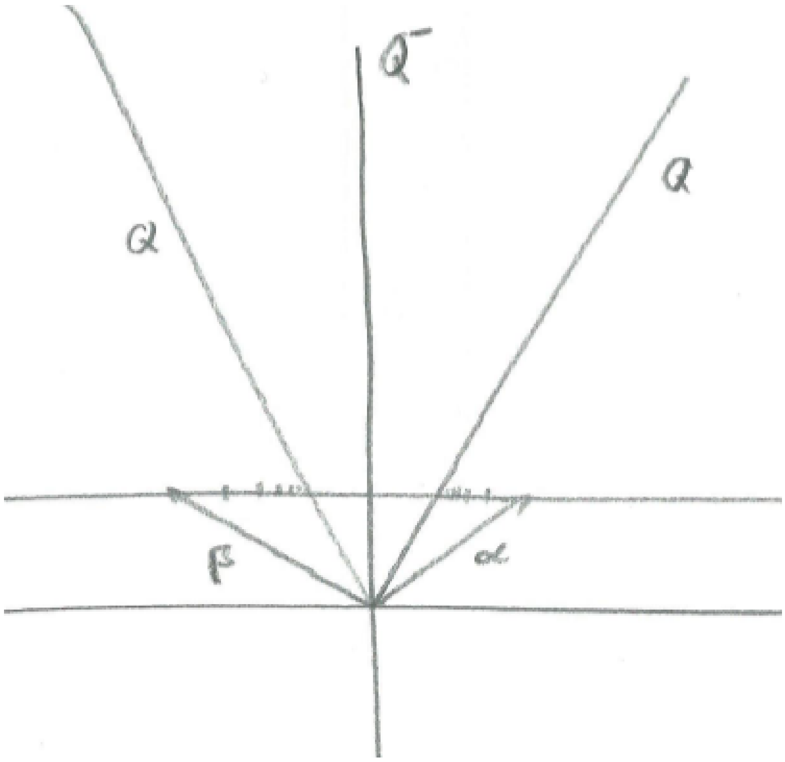



Figure 8.6: For $B(\alpha, \beta) < -1$ the set $Q = \{v \in V \mid B(v, v) = 0\}$ is now two lines. We have exactly two limit roots where $Q$ intersects $E$.

LECTURE 9

# BUILDINGS AS APARTMENT SYSTEMS

04.05.2016

## 9.1 Definition of buildings and first examples

**Definition 9.1** (Tits 1950s)**.** Let $(W, S)$ be a Coxeter System. A *building of type* $(W, S)$ is a simplicial complex $\Delta$ which is the union of subcomplexes called *apartments*, each apartment being a copy of the Coxeter complex for $(W, S)$. The maximal simplices in $\Delta$ are called *chambers* and:

1. any two chambers are contained in a common apartment; and
2. if $A$, $A'$ are two apartments there is an isomorphism $A \to A'$ which fixes $A \cap A'$ pointwise.

Recall the following two descriptions of the Coxeter complex.

1. The Coxeter complex is given by the basic construction $\mathcal{U}(W, X)$ where $X$ is a simplex with codimension-one faces $\{\Delta_s \mid s \in S\}$, and mirrors $X_s = \Delta_s$. So $\mathcal{U}(W, X)$ is $W$-many copies of $X$ with the $s$-mirrors $wX$ and $wsX$ glued together.
2. The Coxeter complex is also the geometric realisation of the poset $\{wW_T \mid T \subseteq S, w \in W\}$ ordered by inclusion.

There are also more recent variations of the above definition:
$\Delta$ could instead be a polyhedral complex with apartments being other geometric realisations of $(W, S)$, e.g. the Davis complex.
In particular if $(W, S)$ is a geometric reflection group on $\mathbb{X}^n$ then one can realise the apartments as copies of $\mathbb{X}^n$ tiled by copies of $P$, the convex polytope which is the fundamental domain for the $W$-action on $\mathbb{X}^n$. These copies of $P$ are then the chambers.

We say that such a building $\Delta$ is respectively *spherical*, *affine* or *Euclidean*, or *hyperbolic* as $\mathbb{X}^n$ is respectively $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$.

**Example 9.2.** A single apartment (a Coxeter complex or a tiling of $\mathbb{X}^n$ or a Davis complex) is a *thin building*.

If the apartment is a Coxeter complex or a tiling of $\mathbb{X}^n$, a *panel* is a codimension-one face of a chamber; if the apartment is a Davis complex $\mathcal{U}(W, K)$, a *panel* is a copy of a mirror.

A building is *thick* if each panel is contained in at least three chambers.

We will now give some examples of buildings of different types.

**Example 9.3.** Consider $W = \langle s \mid s^2 = 1 \rangle \cong C_2$. Thinking of $W$ as acting by reflections on $\mathbb{S}^0$ = two points, an apartment in a building of type $(W, S)$ is just two points; hence a building of type $(W, S)$ is a collection of at least two points.

**Example 9.4.** If $W = \langle s, t \mid s^2 = t^2 = 1, st = ts \rangle \cong C_2 \times C_2$, the action $W \curvearrowright \mathbb{S}^1$ induces a tiling; see Figure 9.1. Thus any building of this type will be a graph which is a union

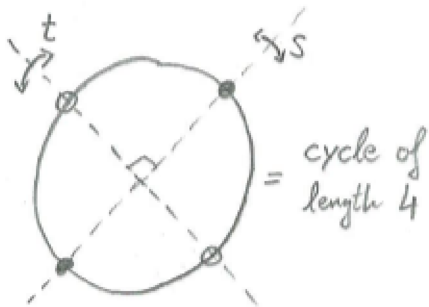



Figure 9.1: The action of $W = C_2 \times C_2$ on $\mathbb{S}^1$ given by reflections in two perpendicular lines.

of 4-cycles.

Let $K_{m,n}$ be the complete bipartite graph on $m + n$ vertices; see for example Figure 9.2.

Claim: $K_{m,n}$ is a building of type $(W, S)$.

It is indeed a union of 4-cycles and each pair of edges is contained in a 4-cycle. Also $\mathrm{Aut}(K_{m,n})$ acts transitively on 4-cycles and using this we can show that axiom 2 holds.

A connected bipartite graph is a *generalised $m$-gon* if it has girth $2m$ and diameter $m$ where the girth of a graph is the length of a shortest circuit and the diameter of a graph is the maximal distance between any two vertices in the graph.

Check: Generalised 2-gons are the same things as complete bipartite graphs.

Check: Every building of type $(W, S)$, $W \cong C_2 \times C_2$, is a generalised 2-gon.


Figure 9.2: The complete bipartite graph on $3 + 4$ vertices $K_{3,4}$.

Figure 9.3: The Coxeter complex for $D_\infty \cong \langle s, t \,|\, s^2 = t^2 = 1\rangle$.

**Example 9.5.** Let $W = \langle s, t \,|\, s^2 = t^2 = 1\rangle \cong D_\infty$. Then the apartments in any building of type $(W, S)$ are as depicted in Figure 9.3 and the chambers are edges. Any tree without valence 1 vertices is a building of type $(W, S)$ and vice versa!

For axiom 1, any two edges are contained in a common line ✓.

Also axiom 2 is easily seen to hold as the isometry $A \to A'$ does not have to be the restriction of a map $\Delta \to \Delta'$ (but usually it will be).

**Example 9.6.** Let $\Gamma$ be a graph as depicted in Figure 9.4 and let $W = W_\Gamma \cong D_\infty \times D_\infty$ be the associated right-angled Coxeter group. Buildings of type $(W, S)$ have apartments

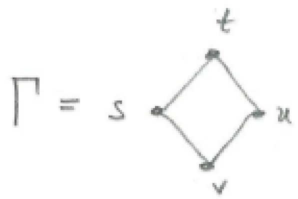



Figure 9.4: The graph $\Gamma$ for the right-angled Coxeter group $W_\Gamma \cong D_\infty \times D_\infty$.

as in Figure 9.5 when the apartments are realised as tilings of $\mathbb{E}^2$ and not as Coxeter

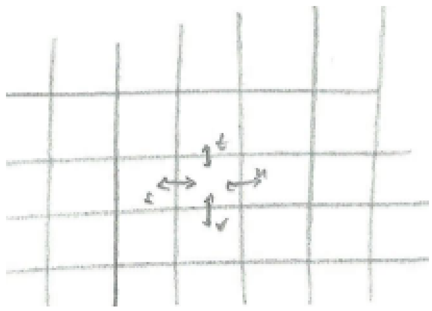


Figure 9.5: A tiling of $\mathbb{E}^2$ induced by a group action of $D_\infty \times D_\infty$.

complexes. These are then nothing but products of two tesselated lines, i.e. products of two copies of apartments for $D_\infty$. Also the chambers are squares, i.e. products of two edges.

Let $T, T'$ be trees without valence 1 vertices.

Claim: $\Delta = T \times T'$ is a building of type $(W, S)$ with $W = D_\infty \times D_\infty$.

$\Delta$ is a union of subcomplexes of the form $\ell \times \ell'$ where $\ell \subset T, \ell' \subset T'$ are lines. These are the apartments of $\Delta$.

Now any two chambers in $\Delta$ have the form $e_1 \times e_1'$ and $e_2 \times e_2'$ where $e_1, e_2$ are edges in $T$ and $e_1', e_2'$ are edges in $T'$. If we choose a line $\ell \subset T$ containing $e_1, e_2$ and a line $\ell' \subset T'$ containing $e_1', e_2'$ then $\ell \times \ell'$ is an apartment of $\Delta$ containing these two chambers.

Similarly one can show that also axiom 2 holds.

The same argument works for all products of buildings: If $\Delta$ is a building of type $(W, S)$ and $\Delta'$ is a building of type $(W', S')$ then $\Delta \times \Delta'$ is a building of type $(W \times W', S \times S')$.

**Example 9.7** (Bourdon's building, a 2-dimensional hyperbolic building)**.** Let $\Gamma$ be a $p$-cycle, $p \geq 5$, and let $q \geq 2$. Then *Bourdon's building* $I_{p,q}$ has as apartments hyperbolic planes tesselated by right-angled $p$-gons. The chambers are the $p$-gons, and each edge is contained in $q$ chambers. The links of vertices are the complete bipartite graphs on $q + q$ vertices $K_{q,q}$. An example is given in Figure 9.6 for $q = 3$.

## 9.2 Links in buildings

Global to local: Suppose $X$ is a Euclidean or hyperbolic building of dimension $n$. Then for all vertices $v$ of $X$, their link $\mathrm{lk}(v, X)$ is a spherical building of dimension $(n-1)$.

There is also a local-to-global theorem for buildings which will allow us to construct examples such as Bourdon's building by geometric methods.

## 9.3 Extended Example: the building for $GL_3(q)$.

A similar method will give the building for $GL_n(K)$ for $n \geq 3$ and any field $K$. We can also replace $GL_n$ by $PGL_n, SL_n$ or $PSL_n$.

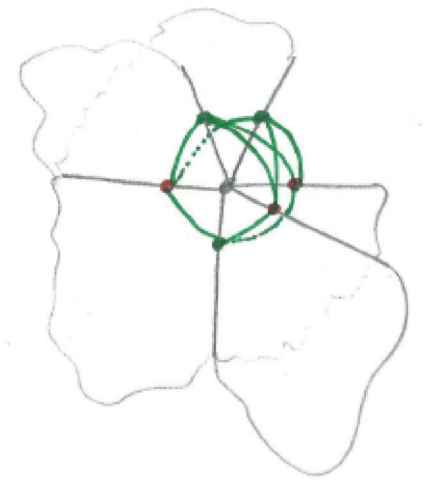


Figure 9.6: The link of a vertex in Bourdon's building $I_{p,q}$ for $q = 3$.

In the following let $V = \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$ the 3-dimensional vector space over $\mathbb{F}_q$.
Let

$$\mathcal{P} = \{\text{1-dimensional subspaces of } V\}$$

(i.e. projective points) and let

$$\mathcal{L} = \{\text{2-dimensional subspaces of } V\}$$

(i.e. projective lines). Then $\mathcal{P} \sqcup \mathcal{L}$ is a projective plane.

The *incidence graph* or *flag graph* or *flag complex* $\Delta$ of this projective plane is the bipartite graph with vertex set $\mathcal{P} \sqcup \mathcal{L}$ and

there is an edge between a point $p \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$
$\iff$ the point $p$ is incident with the line $\ell$
$\iff$ the 1-dimensional subspace $p$ is contained in the 2-dimensional subspace $\ell$.

Further note that $\{p, \ell\}$ is an edge if and only if

$$\{0\} \subsetneq p \subsetneq \ell \subsetneq V,$$

i.e. $p, \ell$ are part of a flag in $V$.

Over $\mathbb{F}_q$ we have

$$\#\mathcal{L} = \#\mathcal{P} = \frac{q^3 - 1}{q - 1} = q^2 + q + 1,$$

so $\Delta$ has $2(q^2 + q + 1)$ vertices. Also, each $p \in \mathcal{P}$ is contained in $\frac{q^2-1}{q-1} = q + 1$ distinct lines, and each line $\ell \in \mathcal{L}$ contains $q + 1$ distinct points. Hence each vertex of $\Delta$ has valence $q + 1$.

**Example 9.8.** For an example for $q = 2$ we refer to Figure 9.7. Here $\Delta$ has $7 + 7$ vertices, each of valence 3.

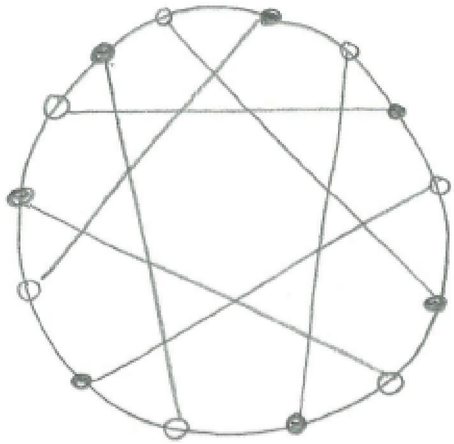


Figure 9.7: The Heawood graph/projective plane of order 2.

From now on we denote $G = \mathrm{GL}_3(q)$. Then $G$ acts on $V = \mathbb{F}_q^3$, preserving $\mathcal{P}$ and $\mathcal{L}$, and preserving incidence; hence $G$ acts on $\Delta$ preserving colours of vertices.

The $G$-action on $\mathcal{P}$ and on $\mathcal{L}$ is transitive, and is also transitive on flags in $V$. Therefore it is transitive on both colours of vertices and on edges of $\Delta$.

What are the stabiliser-subgroups of these actions? Let $\{e_1, e_2, e_3\}$ be the standard basis of $V = \mathbb{F}_q^3$. Then there is an edge between the projective point $\langle e_1\rangle$ and the projective line $\langle e_1, e_2\rangle$ corresponding to the flag

$$\{0\} \subsetneq \langle e_1\rangle \subsetneq \langle e_1, e_2\rangle \subsetneq V.$$

We get for the respective vertex stabilisers:

$$\mathrm{stab}_G(\langle e_1\rangle) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\} =: P_2;$$

$$\mathrm{stab}_G(\langle e_1, e_2\rangle) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\} =: P_1.$$

$P_1$ and $P_2$ are the *standard parabolic subgroups* of $G$.

We now compute the edge stabiliser for the edge between $\langle e_1\rangle$ and $\langle e_1, e_2\rangle$:

$$\mathrm{stab}_G(\{\langle e_1\rangle, \langle e_1, e_2\rangle\}) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\} = P_1 \cap P_2 =: B.$$

The subgroup $B$ is the *standard Borel subgroup* of $G$. The groups $B$, $P_1$, $P_2$ and $G$ form a poset ordered by inclusion as depicted in Figure 9.8.

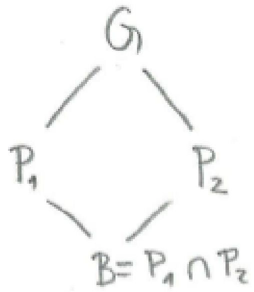



Figure 9.8: The poset given by $B$, $P_1$, $P_2$ and $G$; ordered by inclusion.

By the Orbit-Stabiliser Theorem we have bijections

$$G/P_1 \longleftrightarrow \mathcal{L}, \quad G/P_2 \longleftrightarrow \mathcal{P}, \quad G/B \longleftrightarrow \text{edge set of } \Delta;$$

hence we can label all simplices in $\Delta$ by $G$-cosets of $B$, $P_1$ or $P_2$. Two edges $gB$ and $hB$ are adjacent along a $P_i$-vertex if and only if $g^{-1}h \in P_i$.

Now consider the cycle $A$ in $\Delta$ corresponding to the standard basis; see Figure 9.9. The pointwise stabiliser of $A$ in $G$ is

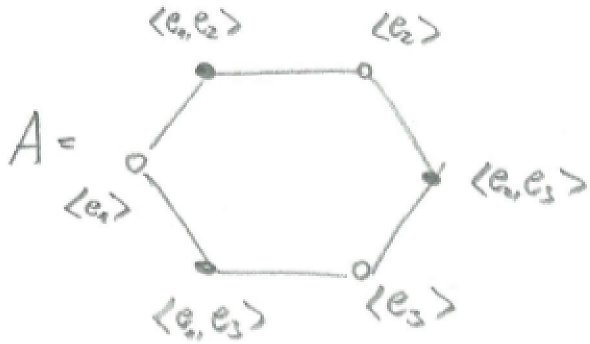



Figure 9.9: The cycle $A$ corresponding to the standard basis $\{e_1, e_2, e_3\}$.

$$\left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in G \right\} = T,$$

the *torus of* $G$. The setwise stabiliser of $A$ in $G$ is the subgroup $N$ of $G$ consisting of monomial matrices, i.e. those matrices which have exactly one non-zero entry in each row and each column. This subgroup is the normaliser of $T$ in $G$.

Note: $T = B \cap N$ and $N/T \cong S_3$ which is isomorphic to $W = \langle s_1, s_2 \mid s_i^2 = 1, (s_1 s_2)^3 = 1 \rangle$ with

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $A$ is an apartment for a building of type $(W, S)$.

Claim: $\Delta$ is a building of type $(W, S)$.

Its apartments are 6-cycles and its chambers are its edges. The key to proving axioms 1 and 2 is that there is a bijection

$$\text{6-cycles in } \Delta \longleftrightarrow \text{unordered bases for } V.$$

We have poset isomorphisms between the three different posets in Figure 9.10.

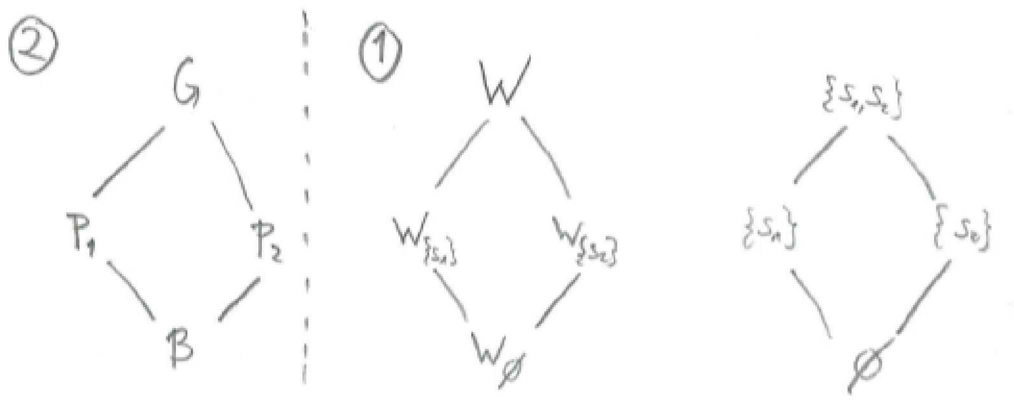



Figure 9.10: The three depicted posets are isomorphic. Note that the one labeled by 2 is used to construct the Coxeter complex/Davis complex for $W$, whereas those labeled by 1 are used to construct $\Delta$.

The group $W$ is the *(spherical) Weyl group* of $G = GL_3(q)$. We have the so called *Bruhat decomposition* which can be seen in Figure 9.11. The pair $B$ and $N$ in this example form a *(spherical) BN-pair* (also called a *Tits system*), since they can be used to construct a spherical building associated to $G$.

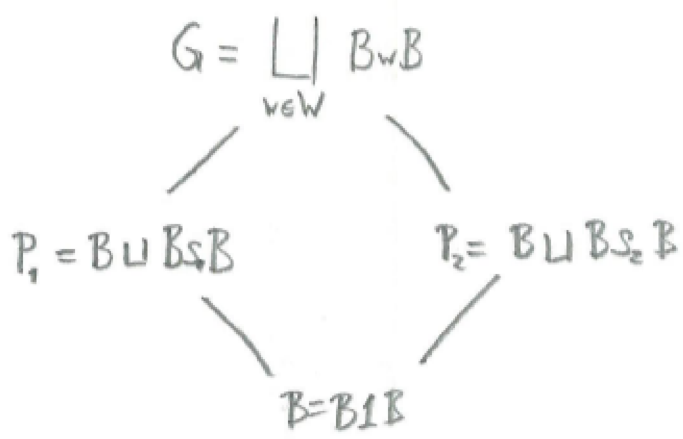



Figure 9.11: The Bruhat decomposition of $G$.

# LECTURE 10

# BUILDINGS AS CHAMBER SYSTEMS

11.05.2016

Let $(W, S)$ be a Coxeter system. A *building of type* $(W, S)$, $\Delta$, is a union of *apartments*, each apartment being one of the following geometric realisations of $(W, S)$ (the same for each apartment):

1. $\mathbb{X}^n$ tiled by copies of polytope $P$, where $W = \langle S \rangle$ with $S = \{$ reflections in codimension-one faces of $P\}$;
2. Coxeter complex;
3. Davis complex.

The *chambers* of $\Delta$ are resp.:

1. copies of $P$;
2. maximal simplices;
3. copies of $K$, where $\Sigma = \mathcal{U}(W, K)$.

Here each apartment is given by the same basic construction $\mathcal{U}(W, X)$ and the chambers are copies of $X$.

The chambers and apartments satisfy two axioms:

(B1) any two chambers are contained in a common apartment;

(B2) given any two apartments $A$, $A'$, there is an isomorphism $A \to A'$ fixing $A \cap A'$.

**Definition 10.1.** A *panel* in $\Delta$ is resp.:

1. a codimension-one face of a chamber,

2. a codimension-one simplex,

3. a copy of a mirror of $K$;

i.e. a copy of a mirror of $X$ where the apartments are $\mathcal{U}(W, X)$.

Important for today: Each panel has a unique *type* $s \in S$. This uses (B2) and the fact that the isomorphism $A \to A'$ is type-preserving on panels.

**Example 10.2** (Product of trees)**.** Let $W = W_\Gamma$ be a right-angled Coxeter group where $\Gamma$ is a graph as in Figure 9.4. In case 1 the apartments and the building are as in Figure 10.1.

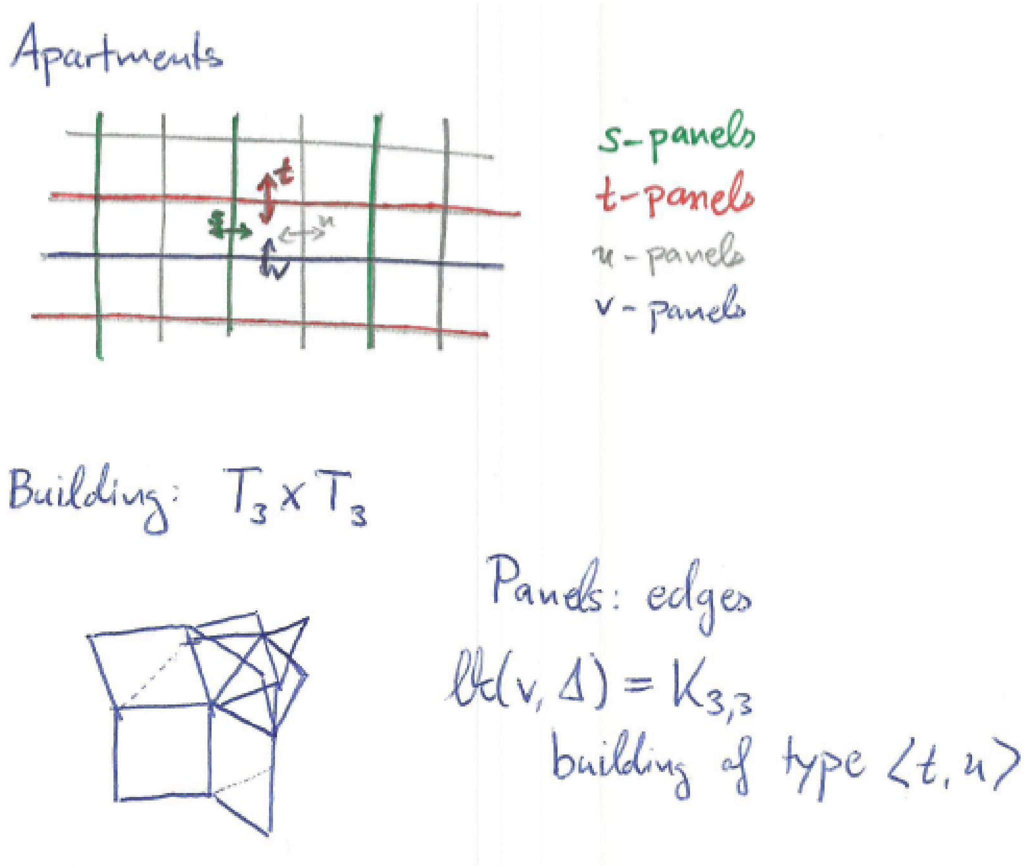



Figure 10.1: Apartments and building for a right-angled Coxeter group $W_\Gamma$.

**Example 10.3.** Let $W = \langle s_1, s_2, s_3 \rangle$ be the $(3, 3, 3)$-triangle group. The apartments and the building are as in Figure 10.2.

Today we will give a second definition of a building due to Tits in the 1980s.

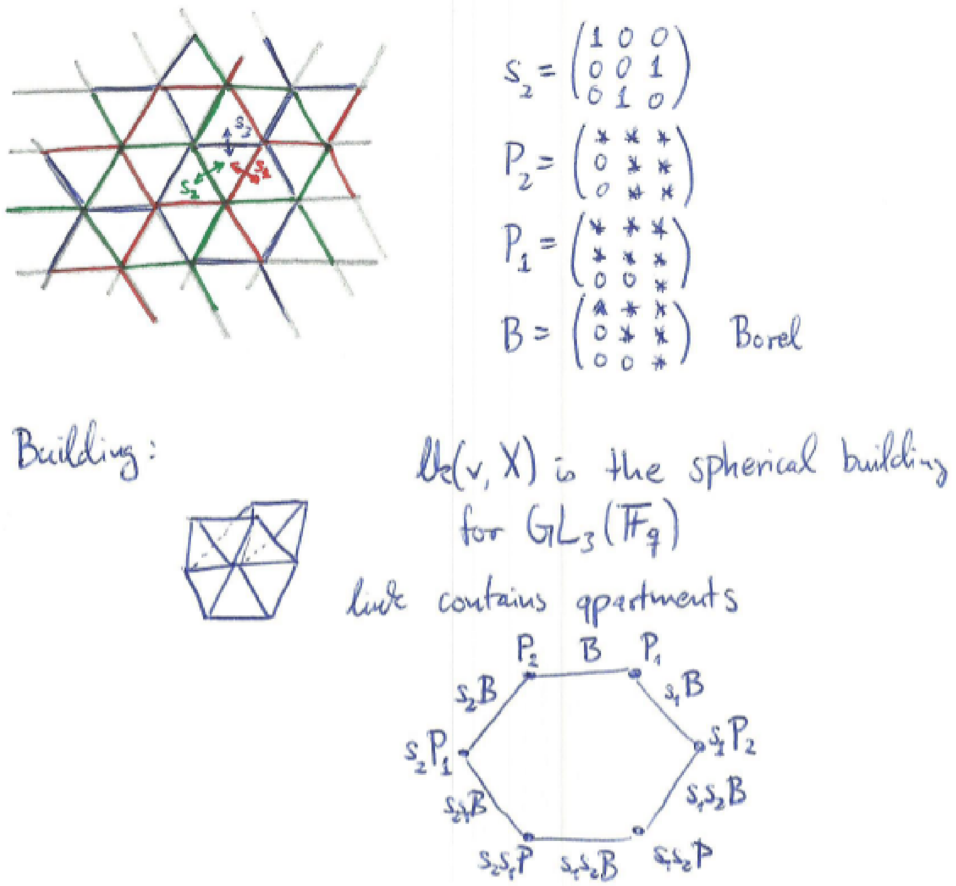



Figure 10.2: Apartments and building for the $(3, 3, 3)$-triangle group.

## 10.1 Chamber systems

Let $I$ be a finite set. A set $C$ of *chambers* is a *chamber system* over $I$ if each $i \in I$ determines an equivalence relation on $C$, denoted by $\underset{i2}{\sim}$. We say two chambers $x$ and $y$ are *$i$-adjacent* if $x \underset{i}{\sim} y$ and are *adjacent* if $x \underset{i}{\sim} y$ for some $i \in I$.

There are two main examples:

**Example 10.4.**

1. Let $(W, S)$ be a Coxeter system with $S = \{s_i \,|\, i \in I\}$. Let $C = W$ and define

$$w \underset{i}{\sim} w' \iff w^{-1}w' \in W_{\{s_i\}} = \langle s_i \rangle,$$

so $w \underset{i}{\sim} w' \iff w = w'$ or $w' = ws_i$.

2. Let $G$ be a group, $B < G$ be a subgroup and for each $i \in I$ let $P_i$ be a subgroup with $B \subsetneq P_i \subsetneq G$. Let $C = \{gB \,|\, g \in G\} = G/B$ be the left cosets of $B$ in $G$.

Define

$$gB \underset{i}{\sim} hB \iff gP_i = hP_i \iff g^{-1}h \in P_i$$

Now each $i$-equivalence class contains $[P_i : B]$ elements.

Note that if we put $G = W$, $B = W_\emptyset = 1$ and $P_i = W_{\{s_i\}}$ then the first example is a special case of the second.

## 10.2 Galleries, residues and panels

Let $C$ be a chamber system. A *gallery* is a sequence of chambers $(c_0, \dots, c_k)$ such that $c_{j-1}$ is adjacent to $c_j$ and $c_{j-1} \neq c_j$ for $1 \leq j \leq k$. The gallery has *type* $(i_1, \dots, i_k)$ where $c_{j-1} \sim_{i_j} c_j$.

A chamber system is *connected* if there is a gallery between any two chambers. Let $J \subseteq I$ be a subset. A *$J$-residue* is a $J$-connected component of $C$, i.e. a maximal subset of $C$ such that each pair of chambers in this subset is connected by a gallery with type in $J$. An $\{i\}$-residue is a *panel*.

Back to the two examples before:

**Example 10.5.**

1. Each gallery in $C = W$ corresponds to a word in $S$ and hence can be identified with a path between vertices in $\mathrm{Cay}(W, S)$:

$$(w_0, w_1, \dots, w_k) \leftrightarrow (s_{i_1}, \dots, s_{i_k}) \text{ where } w_{i_{j+1}} = w_{i_j} s_{i_j}$$

$\leftrightarrow$ 

Since $S$ generates $W$, the chamber system $C = W$ is connected. The $i$-panels are $\{w, ws_i\}$, and the $J$-residues are left cosets of $W_J = \langle s_j \mid j \in J \rangle$.

2. The chamber system in the second example is connected if and only if $G$ is generated by the $P_i$. The $J$-residues are left cosets of $\langle P_j \mid j \in J \rangle$.

## 10.3 $W$-valued distance functions

Let $C$ be a chamber system over $I$ and $(W, S)$ be a Coxeter system with $S = \{s_i \mid i \in I\}$.

A *$W$-valued distance function* is a map

$$\delta : C \times C \to W$$

such that for all reduced words $(s_{i_1}, \dots, s_{i_k})$ and for all $x, y \in C$ the following holds:

$\delta(x, y) = s_{i_1} \cdots s_{i_k}$ if and only if $x$ and $y$ can be joined by a gallery from $x$ to $y$ of type $(i_1, \dots, i_k)$ in $C$.

The idea is that $\delta(x, y)$ gives you a $w \in W$ which tells you how to get from $x$ to $y$ in $C$.

## 10.4 Second definition of a building (Tits 1980s)

A *building of type* $(W, S)$ with $S = \{s_i \mid i \in I\}$ is a chamber system over $I$ which is equipped with a $W$-valued distance function, and is such that each panel has at least two chambers. A building is *thick* if each panel has $\geq 3$ chambers. A building is *thin* if each panel has exactly 2 chambers.

**Example 10.6.** Let $C = W$. Define $\delta : C \times C \to W$ by $\delta(w, w') = w^{-1}w'$. This defines a $W$-valued distance function.

Indeed, let $(s_{i_1}, \dots, s_{i_k})$ be a reduced word. Then the following holds:

$$\text{There is a gallery from } w \text{ to } w' \text{ of type } (i_1, \dots, i_k) \iff ws_{i_1} \cdots s_{i_k} = w' \\ \iff s_{i_1} \cdots s_{i_k} = w^{-1}w' = \delta(w, w');$$

hence $\delta$ is a $W$-valued distance function and $C = W$ is a (thin) building.

Note that the word metric $d_S$ satisfies:

$$d_S(w, w') = \ell_S(w^{-1}w') = \ell_S(\delta(w, w')).$$

Why did we restrict ourselves to <u>reduced</u> words in the definition of a $W$-valued distance function? Well, suppose $x \underset{i}{\sim} y \underset{i}{\sim} z$ with $x \neq y$ and $y \neq z$. Then we have a gallery $(x, y, z)$ of type $(i, i)$. Now $(s_i, s_i)$ is not reduced as $s_i s_i = 1$, but we could have either $x = z$ or $x \neq z$. In this situation $\delta$ does not act like an actual metric in the way that $\delta(x, z) = 1 \iff x = z$.

**Proposition 10.7.** These are some properties of a building $\Delta$:

1. $\Delta$ is connected.
2. $\delta$ maps onto $W$.
3. $\delta(x, y) = \delta(y, x)^{-1}$.
4. $\delta(x, y) = s_i \iff x \underset{i}{\sim} y$ and $x \neq y$.
5. If $x \neq y$, and $x \underset{i}{\sim} y$ and $x \sim_j y$ then $i = j$.
6. If $(s_{i_1}, \dots, s_{i_k})$ is reduced in $(W, S)$ then for all chambers $x$ and $y$ there is at most one gallery of this type from $x$ to $y$.

*Proof.* Exercise. □

**Definition 10.8.** A gallery in $\Delta$ is *minimal* if there is no shorter gallery between its endpoints.

**Lemma 10.9** (Key result for buildings)**.** A gallery of type $(i_1, \dots, i_k)$ is minimal if and only if the word $(s_{i_1}, \dots, s_{i_k})$ is reduced.

Using this one can show, e.g.:

**Proposition 10.10.** If $J \subseteq I$ then every $J$-residue is a building of type $(W_J, J)$.

Cf: in the geometric examples above, the links are buildings of type $\langle s_i, s_j \rangle$ where $m_{ij}$ is finite.

**Theorem 10.11.** *Definition 1 and Definition 2 of buildings are equivalent, in the sense we will explain in the proof.*

*Proof.*

**Defn 1 $\implies$ Defn 2:** Let $C$ be the set of chambers of $\Delta$. First, we turn $C$ into a chamber system by defining $c \underset{i}{\sim} c' \iff c \cap c'$ is a panel of type $s_i$. Now we need a $W$-valued distance function

$$\delta : C \times C \to W.$$

Let $A$ be an apartment with chambers $C(A)$. Then two chambers $c$ and $c'$ in $A$ have the form $c = wX$, $c' = w'X$ with $w, w' \in W$. Define

$$\delta_A : C(A) \times C(A) \to W$$

by $\delta_A(wX, w'X) = w^{-1}w'$. This gives a $W$-valued distance function on $C(A)$. Then if $c$, $c'$ are chambers in $\Delta$, axiom (B1) says there is an apartment $A$ in $\Delta$ containing both. Define $\delta(c, c') = \delta_A(c, c')$.

Is this well-defined? We will use (B2) to show this. Let $A$, $A'$ be two apartments containing both $c$ and $c'$, and $\varphi : A \to A'$ be an isomorphism fixing $A \cap A'$. So $\varphi(c) = c$ and $\varphi(c') = c'$. Since $\delta_A$ is a $W$-valued distance function there is a gallery $\gamma$ from $c$ to $c'$ in $A$ of type $(i_1, \ldots, i_k)$ where $(s_{i_1}, \ldots, s_{i_k})$ is a reduced word for $\delta_A(c, c')$. Then $\varphi(\gamma)$ is a gallery of the same type from $\varphi(c) = c$ to $\varphi(c') = c'$ and $\varphi(\gamma)$ is contained in $A'$. So $\delta'_A(c, c') = s_{i_1} \cdots s_{i_k} = \delta_A(c, c')$ and $\delta$ is well-defined.

To show that $\delta$ is a $W$-valued distance function, let $(s_{i_1}, \ldots, s_{i_k})$ be a reduced word.

If $\delta(c, c') = s_{i_1} \cdots s_{i_k}$ then $\delta_A(c, c') = s_{i_1} \cdots s_{i_k}$ for any apartment $A$ containing $c$ and $c'$. Thus there is a gallery from $c$ to $c'$ in $A$ of type $(i_1, \ldots, i_k)$ and we have necessarily a gallery in $\Delta$.

If there is a gallery $\gamma$ in $\Delta$ from $c$ to $c'$ of type $(i_1, \ldots, i_k) \ldots$

Induction on $k$: For $k = 1$:

$$c \sim_{i_1} c' \implies \delta(c, c') = s_{i_1}$$

For $k \geq 2$: let $c''$ be the second last chamber in $\gamma$, so $c'' \cap c'$ is a panel in $A$ where $A$ is some apartment containing $c$ and $c'$.

By induction, $\delta(c, c'') = s_{i_1} \cdots s_{i_{k-1}}$ and there is a gallery $\gamma'$ from $c$ to $c''$ of type $(i_1, \ldots, i_{k-1})$ in some apartment $A'$; see Figure 10.3. Let $\varphi : A' \to A$ be an isomorphism fixing $A \cap A'$. Then $\varphi(\gamma')$ goes from $\varphi(c) = c$ to $\varphi(c'') \in C(A)$ and $\varphi$ fixes the panel $c'' \cap c'$.

Now $\varphi(c'') \sim_{i_k} c'$ in $A$, so $(\varphi(\gamma'), c')$ is a gallery in $A$ from $c$ to $c'$ of type $(i_1, \ldots, i_k)$.

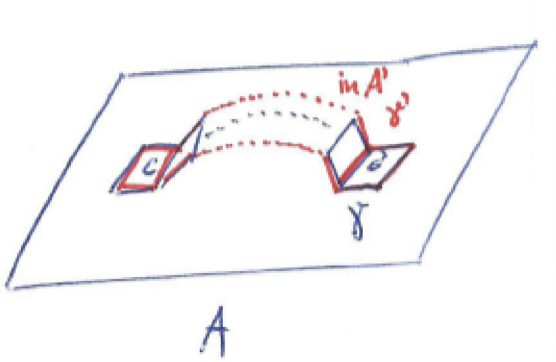



Figure 10.3:

**Defn 2 $\implies$ Defn 1:** Where are the apartments? They will be images of $W$, in the following sense:

**Definition 10.12.** For any subset $X \subseteq W$, a map $\alpha : X \to \Delta$ (where $\Delta$ is now a chamber system) is a $W$-isometric embedding if, $\forall x, y \in X$,

$$\delta(\alpha(x), \alpha(y)) = x^{-1}y.$$

An *apartment* is any image of $W$ under a $W$-isometric embedding.

**Proposition 10.13.** Any $W$-isometric embedding $\alpha : X \to \Delta$ where $X \subsetneq W$, extends to all of $W$.

*Proof.* By Zorn's Lemma, it is enough to extend $\alpha$ to a strictly larger subset of $W$. If $X = \emptyset$ we are done, so assume that $X \neq \emptyset$. Then there is $x_0 \in X$ and $s_i \in S$ such that $x_0 s_i \notin X$. We can precompose $\alpha$ by left-multiplying by $x_0^{-1}$, so we may assume without loss of generality that $x_0 = 1$ and $s_i \notin X$. We will define $\alpha(s_i)$.

Case 1: $\ell(s_i x) > \ell(x) \, \forall x \in X$. This is the case where, in Cay$(W, S)$, all elements of $X$ lie on the same side of the wall $H_{s_i}$ as 1.

We define $\alpha(s_i)$ to be any chamber of $\Delta$ which is $i$-adjacent to $\alpha(1)$, but not equal to $\alpha(1)$; see Figure 10.4.

Case 2: $\exists x_1 \in X$ such that $\ell(s_i x_1) < \ell(x_1)$.

By the Exchange Condition there is a reduced word for $x_1$ starting with $s_i$. Define $\alpha(s_i)$ to be $y$, the second chamber in the corresponding gallery from $\alpha(1)$ to $\alpha(x_1)$ in $\Delta$.

Now check that in both cases $\delta(\alpha(s_i), \alpha(x)) = s_i^{-1}x = s_i x \, \forall x \in X$. This is combinatorics in $W$. □

**Corollary 10.14.** With this definition of apartments, axioms (B1) and (B2) hold.

□

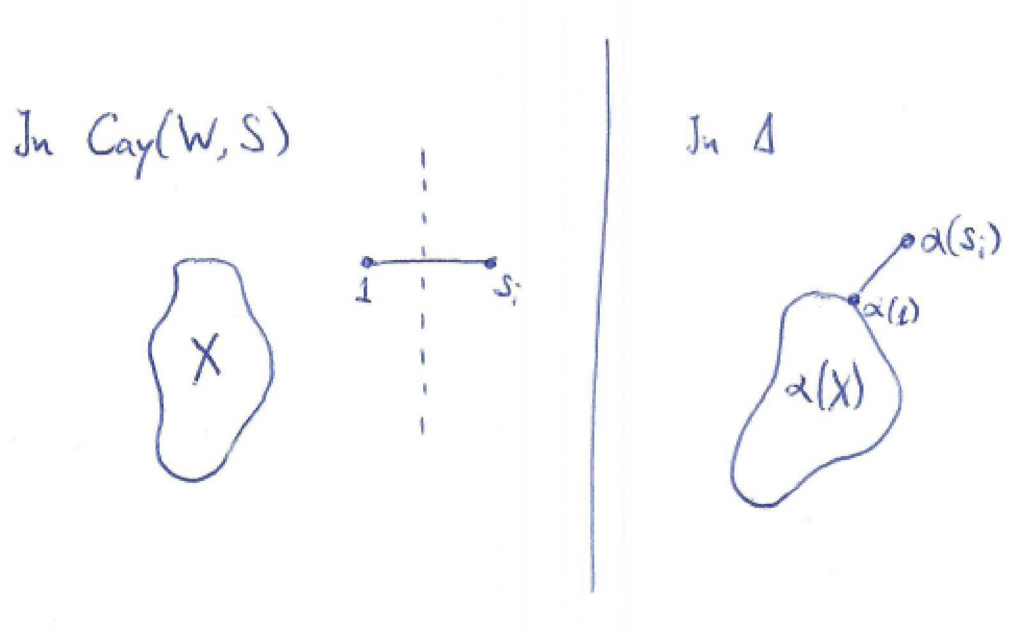



Figure 10.4:

# LECTURE 11

# COMPARING THE TWO DEFINITIONS, RETRACTIONS, $BN$-PAIRS

18.05.2016

## 11.1 Comparing the definitions

Let $(W, S)$ be a Coxeter system, $S = \{s_i \mid i \in I\}$. A building of type $(W, S)$ is:

Defn 1 (loosely): a union of apartments tiled by chambers such that axioms (B1) and (B2) hold.

Defn 2: a chamber system over $I$ (i.e. a set $\Delta$ with equivalence relations $\underset{i}{\sim}$) equipped with $W$-valued distance function (i.e. $\delta : \Delta \times \Delta \to W$ such that for every reduced word $(s_{i_1}, \ldots, s_{i_k})$ and for all $x, y \in \Delta$:

$$\delta(x, y) = s_{i_1} \cdots s_{i_k} \iff \text{there is a gallery of type } (i_1, \ldots, i_k) \text{ from } x \text{ to } y)$$

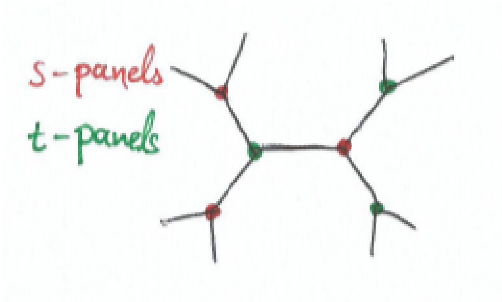
such that each panel (i.e. $i$-equivalence class) has $\geq 2$ chambers.

The building $\Delta$ is *thin* if each panel has exactly two chambers, and is *thick* if each panel has $\geq 3$ chambers.

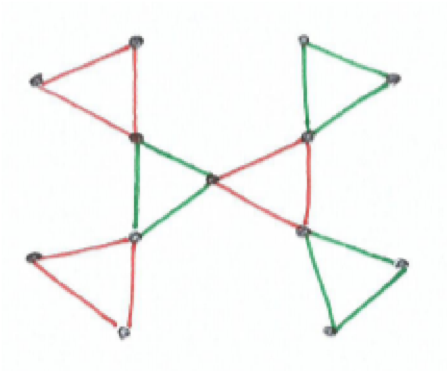
Defn 1 can be viewed as giving a geometric realisation of Defn 2. However there is also another geometric realisation: Form a graph with vertex set the chambers, and an undirected edge labelled $i$ between two vertices/chambers $x$ and $y \iff x \underset{i}{\sim} y$ and $x \neq y$.

When the chamber system is a building, this graph is connected, edge-coloured by $|I|$ colours, the apartments are the copies of $\mathrm{Cay}(W, S)$, and each vertex is adjacent to at least one edge of each colour. An $i$-panel with $n$ chambers is a complete subgraph on $n$ vertices, each edge having colour $i$.

**Example 11.1.** If $\Delta$ (Defn 1) is $T_3$




its egdes are the chambers and the chamber system graph is



One advantage of Defn 2 is that the apartments are not part of the definition.

In fact a building may have more than one collection of apartments satisfying (B1) and (B2). It is a theorem of Tits that $\Delta$ has a unique maximal system of apartments.

However, Defn 1 has better geometric and topological properties:

**Theorem 11.2.** *Let $\Delta$ be a building (Defn 1)of type $(W, S)$.*

- *If $\Delta$ is Euclidean/affine (resp. hyperbolic), i.e. its apartments are $\mathbb{E}^n$ (resp. $\mathbb{H}^n$) tiled by a $W$-action, then $\Delta$ is $CAT(0)$ (resp. $CAT(-1)$).*
- *If the apartments are Davis complexes then $\Delta$ can be equipped with a piecewise Euclidean metric such that it is a $CAT(0)$ space.*
- *If $W$ is word hyperbolic, then $\Delta$ (with apartments Davis complexes) can be equipped with a piecewise hyperbolic metric such that $\Delta$ is $CAT(-1)$.*

*Outline of a proof.* If $(W, S)$ is irreducible and affine, this is classical (Bruhat-Tits). Otherwise a combination of Tits, Gaboriau-Panlin, Charney-Lytschak, Davis.

For $CAT(0)$ and $CAT(-1)$: First map $\Delta$ to an apartment, using a retraction. Then we use the Davis complex being CAT(0) or CAT(−1). We will talk more about retractions in the last section of this lecture. □

### 11.1.1 Right-angled buildings (Davis)

A building is *right-angled* if its type $(W, S)$ is a right-angled Coxeter system. For any right-angled $(W, S)$, we will construct a building of this type as a chamber system, i.e. using Defn 2.

**Definition 11.3.** Let $\Gamma$ be a finite simplicial graph with vertex set $S$. For each $s \in S$, let $G_s$ be a group of order $\geq 2$. The *graph product* of this family $(G_s)_{s \in S}$ over $\Gamma$ is:

$$G_\Gamma = \langle G_s, s \in S \mid \text{ relations in each } G_s, [G_s, G_t] = 1 \iff \{s, t\} \in E(\Gamma) \rangle.$$

Special cases:

1. If each $G_s = \langle s \mid s^2 = 1 \rangle$ then $G_\Gamma = W_\Gamma$ the right-angled Coxeter group.

2. If each $G_s = \langle s \rangle \cong \mathbb{Z}^2$ then $G_\Gamma = A_\Gamma$ the right-angled Artin group.

Let $g \in G_\Gamma \backslash \{1\}$. Check: we can write $g = g_{i_1} \cdots g_{i_k}$ where $g_{i_j} \in G_{s_{i_j}} \backslash \{1\}$ and $(s_{i_1}, \ldots, s_{i_k})$ is reduced in $(W_\Gamma, S)$. Such an expression is a *reduced expression* for $g$.

**Theorem 11.4** (Green)**.** *If $g = g_{i_1} \cdots g_{i_k}$ and $g' = g'_{i_1} \cdots g'_{i_k}$ are reduced expressions for $g, g' \in G_\Gamma \backslash \{1\}$ then $g = g' \iff$ one can get from one expression to the other by "shuffling", i.e. using $[g_{i_j}, g_{i_{j+1}}] = 1$.*

Chamber system: Now the set of chambers is $G_\Gamma$ and we define $g \underset{i}{\sim} g' \iff g^{-1}g' \in G_{s_i}$ (check: this is an equivalence relation). Note: $|G_{s_i}| \geq 2$ so each panel has $\geq 2$ chambers.

Building: Define $\delta : G_\Gamma \times G_\Gamma \to W_\Gamma$ by $\delta(g, g') = s_{i_1} \cdots s_{i_k}$ where $g^{-1}g' \in G_\Gamma$ has reduced expression $g_{i_1} \cdots g_{i_k}$. By the theorem, this is well-defined. Check: $\delta$ is a $W_\Gamma$-valued distance function.

So we have a building $\Delta$ of type $(W_\Gamma, S)$. Each $s_i$-panel has cardinality $|G_{s_i}|$. The group $G_\Gamma$ acts freely and transitively on the set of chambers. The associated graph is the Cayley graph for $G_\Gamma$ with respect to the generating set $\bigcup_{s \in S} G_s$.

Some particular geometric realisations for right-angled buildings include:

1. If $W_\Gamma = D_\infty \times D_\infty$, $\Delta$ can be realised as a product of trees.

2. If $W_\Gamma$ is a hyperbolic geometric reflection group, then one can realise $\Delta$ as a hyperbolic building, e.g. Bourdon's building $I_{p,q}$: the apartments are $\mathbb{H}^2$ tiled by right-angled $p$-gons ($p \geq 5$). This is $\Delta$ when $\Gamma =$$p$-cycle and each $|G_s| = q$ and the links are $K_{q,q}$.

3. For all $W_\Gamma$, the building $\Delta$ can be realised as a CAT(0) cube complex, which is $\delta$-hyperbolic $\iff$ $W_\Gamma$ is word hyperbolic $\iff$ $\Gamma$ has no "empty squares".

## 11.2 Retractions

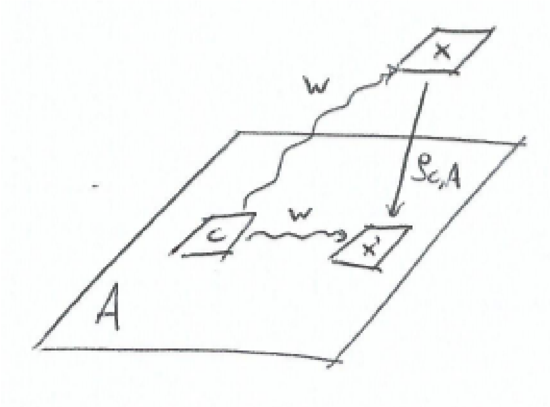
Let $\Delta$ be a building of type $(W, S)$. Fix an apartment $A$ and a chamber $c$ in $A$. The *retraction onto $A$ with centre $c$* is the map

$$\rho_{c,A} : \Delta \to A$$

such that for any chamber $x$ of $\Delta$, $\rho_{c,A}(x)$ is the unique chamber $x'$ of $A$ such that

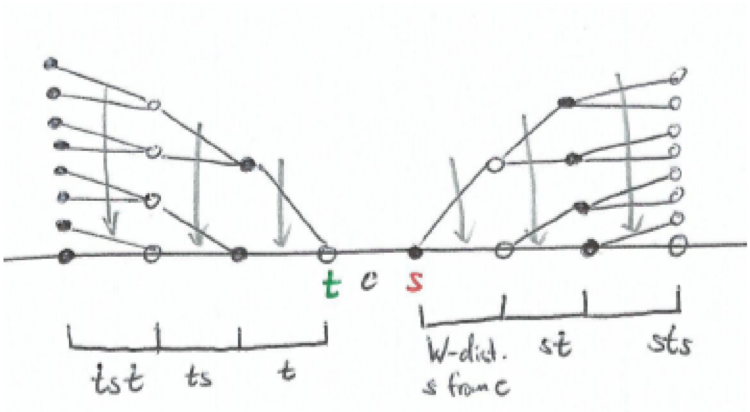
$$\delta(c, x) = \delta_A(c, x'),$$

where the first is the $W$-distance in $\Delta$ and the second is the $W$-distance in $A$:




Note: If $x \in A$ then $\rho_{c,A}(x) = x$, so $\rho_{c,A}$ fixes $A$.
If $\Delta$ has geometry/topology, $\rho_{c,A}$ is a retraction in the usual sense.

**Example 11.5.** Here the tree $T_3$ is drawn to show the fibres of $\rho_{c,A}$:




Retractions are key tools for studying buildings. After applying a retraction $\rho_{c,A}$, we can use properties of the apartment $A$.

A key property of retractions is that they are distance non-increasing in the following sense:

**Proposition 11.6.** Let $\rho = \rho_{c,A}$ be a retraction of a building $\Delta$ of type $(W, S)$. Then for all chambers $x, y$ of $\Delta$

$$\ell_S(\delta(\rho(x), \rho(y))) \leq \ell_S(\delta(x, y)).$$

If $\Delta$ is a metric space then for all points $x, y \in \Delta$

$$d(\rho(x), \rho(y)) \leq d(x, y).$$

*Proof.* Induction on $\ell_S(w)$ where $w = \delta(x, y)$. If $\delta(x, y) = s_i$ then $\rho(x) = \rho(y) \iff \delta(c, x) = \delta(c, y)$, and otherwise $\rho(x) \underset{i}{\sim} \rho(y)$ with $\rho(x) \neq \rho(y)$. □

Under a retraction $\rho_{c,A}$:

- a minimal gallery in $\Delta$ can be sent to a "folded" or a "stuttering" gallery in $A$, i.e. a gallery with a repeated chamber.
- an apartment $A'$ can be sent to either all of $A \iff c$ is in $A$, or to "half" of $A$ else.

### 11.2.1 Two applications of retractions

1. Apartments are convex, i.e. if $x$ and $y$ are chambers, and $A$ is any apartment containing both $x$ and $y$ then, if $\gamma$ is a minimal gallery from $x$ to $y$, $\gamma$ is contained in $A$.
2. Gate property/projections to residues: Recall that for $J \subseteq I$, a $J$-residue is $J$-connected component of $\Delta$. Let $R$ be any residue. Then for all chambers $x$ in $\Delta$, there is a unique chamber $c$ of $R$ closest to $x$. This chamber $c$ is called the *gate* of $R$ with respect to $x$ or $\text{proj}_R(x)$.

## 11.3 $BN$-pairs

These give an "algebraic construction" of buildings $\Delta$ together with a highly transitive group action.

Recall: $G = GL_3(q)$ has

- (standard) Borel subgroup

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\};$$

- torus

$$T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in G \right\};$$

- normaliser of the torus

$$N = \{\text{monomial matrices in } G\};$$

- Weyl group $W = N/T \cong S_3$ generated by images of

$$n_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

  i.e. $(W, S)$ is a Coxeter system with $S = \{s_1, s_2\}$, where $s_i = n_i T$.

- (standard) parabolic subgroups

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}, P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\},$$

Exercise: $P_i = B \sqcup Bs_iB = B \sqcup Bn_iB$ since $s_i = n_iT$ and $T < B$.

**Definition 11.7.** A *$BN$-pair* or a *Tits system* is a group $G$ with a pair of subgroups $(B, N)$ such that

(BN0) $G$ is generated by $B$ and $N$;

(BN1) $T = B \cap N$ is normal in $N$ and $W = N/T$ is a Coxeter system with distinguished generators $S = \{s_1, \ldots, s_n\} = \{s_i \mid i \in I\}$;

(BN2) For all $w \in W$ and all $s_i \in S$

$$BwB \cdot Bs_iB = BwBs_iB \subseteq BwB \cup Bws_iB;$$

(BN3) $s_iBs_i^{-1} = s_iBs_i \neq B \ \forall i \in I$.

*Remark* 8. 1. (BN2) and (BN3) are well-defined: each $w \in W$ is $w = nT$ for some $n \in N$ and $T \leq B$.

2. Taking inverses in (BN2) yields

$$Bs_iBwB \subseteq BwB \cup Bs_iwB.$$

**Lemma 11.8** (Bruhat decomposition)**.** If $G$ has a $BN$-pair then

$$G = \bigsqcup_{w \in W} BwB.$$

*Proof.* First, we show $G = \bigcup_{w \in W} BwB$. Let $g \in G$ then by (BN0),

$$g = b_1 n b_2 n_2 \dots b_k n_k b_{k+1}$$

where $b_i \in B$ and $n_i \in N$. So $g \in Bn_1Bn_2B \cdots Bn_kB = Bw_1Bw_2B \cdots Bw_kB$ where $w_i = n_iT$. Now apply (BN2) and Remark 8 to get $g = \bigcup_{w \in W} BwB$.

Disjoint union: combinatorics on words in $(W, S)$. □

Exercise: The $B, N$ in $GL_3(q)$ as above are a $BN$-pair. Similarly, one could take $B$ to be upper-triangular matrices, and $N$ the monomial matrices in $SL_3(q), PGL_3(q), PSL_3(g)$.

More examples will be presented in the next lecture.

### 11.3.1 Strongly transitive actions

Let $\Delta$ be a building. Write $\mathrm{Aut}_C(\Delta)$ for the group of chamber system automorphisms of $\Delta$, i.e. bijections $\varphi$ on the set of chambers such that $\varphi(x) \underset{i}{\sim} \varphi(y) \iff x \underset{i}{\sim} y$.

**Example 11.9.** If $\Delta = W$, i.e. $\Delta$ is a thin building, then $\mathrm{Aut}_C(\Delta) = W$, but the full automorphism group $\mathrm{Aut}(\Delta)$ could be much bigger.

**Definition 11.10.** Given an apartment system on $\Delta$, a subgroup $G \leq \mathrm{Aut}_C(\Delta)$ is *strongly transitive* if the $G$-action is transitive on pairs $\{(c, A) \mid c$ is a chamber of the apartment $A\}$.

*Remark* 9. $G$ is strongly transitive

$\iff$ $G$ is chamber transitive, and for all chambers $c$, $\mathrm{stab}_G(c)$ acts transitively on apartments containing $c$

$\iff$ $G$ is transitive on apartments, and for every apartment $A$, $\mathrm{stab}_G(A)$ is transitive on chambers in $A$, i.e. $\mathrm{stab}_G(A)$ induces $W$ on $A$.

Next time we will see that $BN$-pairs are "the same thing" as strongly transitive actions on buildings.

In the construction, $\Delta$ will have $B$ as a chamber stabiliser and $N$ as an apartment stabiliser. Then $BwB$ is the set of chambers in the same $B$-orbit as $wB$.

If $A$ is the apartment stabilised by $N$, then the fibres of the retraction $\rho_{B,A}$ are of the form $BwB$, i.e. $\rho_{B,A}^{-1}(wB) = BwB$.

Then (BN2) can be interpreted as concatenating galleries.

# LECTURE 12

# STRONGLY TRANSITIVE ACTIONS

25.05.2016

Let $\Delta$ be a building with apartment system $\mathcal{A}$. A group $G \leq \mathrm{Aut}_C(\Delta)$ (chamber system automorphisms of $\Delta$) is *strongly transitive* with respect to $\mathcal{A}$ if $G$ is transitive on pairs:

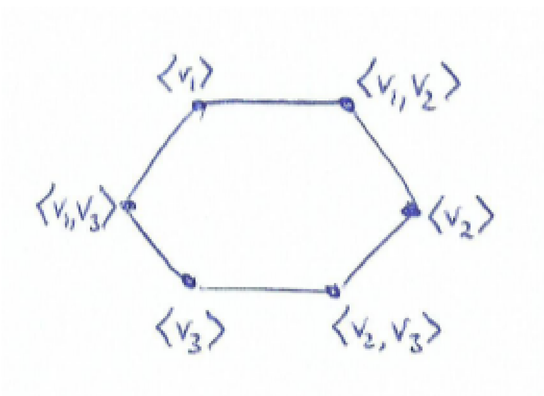
$$\{(c, A) \,|\, c \text{ chamber of an apartment } A \in \mathcal{A}\}$$

**Example 12.1.**

1. $G = \mathrm{GL}_3(F)$ is strongly transitive on $\Delta$, the spherical building associated to $G$.

   Chambers: edges

   Apartments: 6-cycles

   where $\{v_1, v_2, v_3\}$ is a basis for $V = F^3$.

2. If $\Delta$ is a tree $T$, then $\mathrm{Aut}_C(T)$ is strongly transitive $\iff$ $T$ is either regular or biregular.

3. If $\Delta$ is the right-angled building associated to the graph product $G_\Gamma$ of groups $(G_s)_{s\in V(\Gamma)}$ then $\mathrm{Aut}_C(\Delta)$ is strongly transitive (Bourdon-Kubena-T).

The next two theorems, due to Tits, say that $BN$-pairs are essentially the same thing as strongly transitive actions. The proofs use the gate property, and a lot of word combinatorics in Coxeter systems.

**Definition 12.2.** Suppose $G$ has a $BN$-pair. For each $i \in I$ define

$$P_i = B \sqcup Bs_iB.$$

By (BN2), $P_i$ is a subgroup of $G$.

**Theorem 12.3.** *Let $G$ be a group with subgroups $B$ and $N$ such that axioms (BN0)-(BN2) hold. Then there is a building $\Delta = \Delta(B, N)$ with chambers $\{gB \mid g \in G\}$, i-adjacency given by*

$$gB \underset{i}{\sim} hB \iff g^{-1}h \in P_i,$$

*and a $W$-valued distance function*

$$\delta(gB, hB) = w \iff g^{-1}h \in BwB.$$

*Now let $c_0 = B$, $A_0 = \{wc_0 \mid w \in W\}$, and define $\mathcal{A} = \{gA_0 \mid g \in G\}$. Then $G$ acts strongly transitively with respect to $\mathcal{A}$ on the building $\Delta$, with $B$ the stabiliser of $c_0$, and $N$ the (setwise) stabiliser of $A_0$. If also (BN3) holds then $\Delta$ is thick.*

**Theorem 12.4.** *Let $(W, S)$ be a building with apartment system $\mathcal{A}$, and let $G$ be a strongly transitive group of automorphisms of $\Delta$. Choose a chamber $c_0$ and an apartment $A_0$ containing $c_0$. Define*

$$B = stab_G(s_0) \text{ and } N = stab_G(A_0),$$

*where the last stabiliser is to be understood setwise. Then $(B, N)$ satisfy (BN0)-(BN2) and for all chambers $c$,*

$$\delta(c_0, c) = w \iff gB \subseteq BwB,$$

*where $c = gB$. Additionally, if $\Delta$ is thick then also (BN3) holds.*

## 12.1 Parabolic subgroups

Suppose $G$ has Tits system $(B, N)$. Write $W_J$ for the special subgroup $W_J = \langle s_j \mid j \in J\rangle$. Define for $J \subseteq I$

$$P_J = \bigsqcup_{w\in W_J} BwB.$$

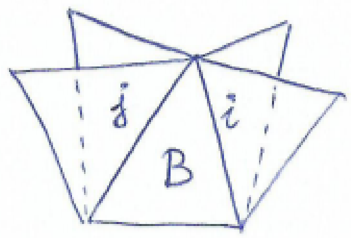
Then by (BN2), $P_J$ is a subgroup of $G$, called a *(standard) parabolic subgroup.*

The group $P_i$ above is $P_{\{i\}}$, and $B$ is $P_\emptyset = B1B = B$. The subgroup $B$ is called the *(standard) Borel subgroup.* A *parabolic subgroup* is any conjugate of a $P_J$ in $G$, and a *Borel subgroup* is any conjugate of $B$.

Using the $BN$-pair axioms and the theorems above one can show:

**Theorem 12.5.**

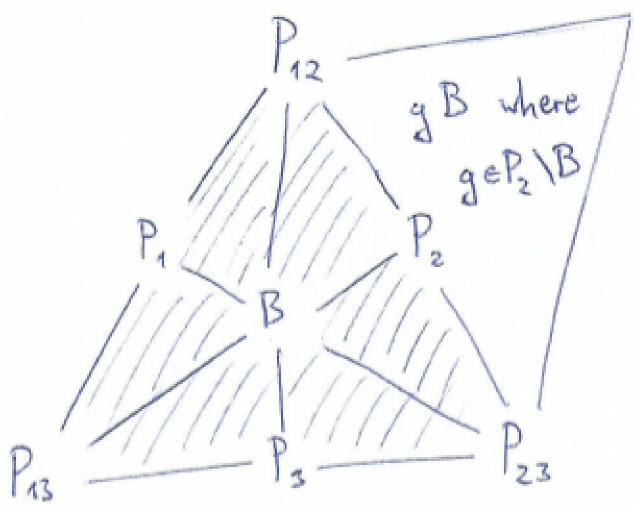
1. *If $B \subseteq P \subseteq G$ then $P = P_J$ for some $J \subseteq I$.*
2. *$P_J \cap P_K = P_{J\cap K}$ and $\langle P_J, P_K \rangle = P_{J\cup K}$.*
3. *$P_J$ is the stabiliser of the $J$-residue of $\Delta(B,N)$ containing $B$. In particular, $P_i$ stabilises the $i$-panel containing $B$.*




Consider the poset ordered by inclusion

$$\{gP_j \,|\, g \in G, J \subsetneq I\}.$$

The building $\Delta = \Delta(B,N)$ can be realised as the simplicial complex which is the geometric realisation of this poset, e.g. if $I = \{1,2,3\}$, base chamber $c_0 = B$ is




In this realisation, the apartments are Coxeter complexes, realised from the poset

$$\{wW_J \,|\, w \in W, J \subsetneq I\}.$$

The building $\Delta$ also has a *Davis realisation* corresponding to the poset

$$\{gP_J \mid g \in G, J \subsetneq I, W_J \text{ finite}\}.$$

Each apartment is a Davis complex.

LECTURE 13

# $BN$-PAIRS INCL. KAC-MOODY, GEOMETRIC CONSTRUCTIONS OF BUILDINGS

01.06.2016

## 13.1 Examples of $BN$-pairs

1. (Spherical buildings)

   For simple matrix groups over arbitrary fields (Bruhat, Chevalley, Tits, Borel) there are 4 classical families of types

   $$A_n, B_n, C_n, D_n$$

   and 5 exceptional groups of types

   $$E_6, E_7, E_8, F_4, G_2.$$

   The "type" here is the type of the associated Weyl group, which is a finite Coxeter group, e.g. type $A$ is special linear groups, type $B$ is orthogonal groups, type $C$ is symplectic groups, type $D$ is unitary groups.

   Over a finite field, these are "finite groups of Lie type" or "Chevalley groups", and (after perhaps quotienting) these are most of the finite simple groups.

   Over any field, the Borel subgroup $B$ is a maximal connected solvable subgroup of $G$; the torus $T$ is a maximal, connected, abelian subgroup of $G$ chosen such that $T \subseteq B$, and then define $N$ to be the normaliser of $T$ in $G$.

   **Theorem 13.1.** *This gives a Tits system, with $W$ the Weyl group.*

   **Theorem 13.2** (Tits)**.** *Suppose $\Delta$ is a building of type $(W, S)$ where $(W, S)$ is an irreducible Coxeter system and $W$ is finite. If $|S| \geq 3$ then $\Delta = \Delta(B, N)$ is the* *spherical building for some classical or algebraic group $G(F)$. Moreover $Aut(\Delta)$ is $G \rtimes Aut(F)$.*

*Remark* 10. The case $|S| = 2$ includes the classification of all projective planes, which is wide open.

2. (Affine buildings)

   For the same groups, when over a field with a discrete valuation, we have a second $BN$-pair such that the associated building is affine, i.e. is of type $(W, S)$, a Euclidean reflection group (Bruhat-Tits theory).

   Let $F$ be any field. A *discrete valuation* is a surjective homomorphism of groups

$$v : F^* \to (\mathbb{Z}, +)$$

such that $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in F^*$ with $x + y \neq 0$. We extend this to $F$ by putting $v(0) = +\infty$.

**Example 13.3.** For $p$ prime, the *p-adic valuation* on $\mathbb{Q}$ is

$$v_p\left(\frac{a}{b}\right) = n, \quad \text{where } \frac{a}{b} = p^n \frac{a'}{b'} \text{ with } p \nmid a', p \nmid b'.$$

The *valuation ring* in $F$ is $A = \{x \in F \mid v(x) \geq 0\}$. The units in $A$ are $A^* = \ker(v)$. A *uniformiser* is an element $\pi \in A$ with $v(\pi) = 1$. The principal ideal

$$\pi A = \{x \in F \mid v(x) > 0\}$$

is maximal in $A$ so $A/\pi A$ is a field $k$, called the *residue field.*

**Example 13.4.** If $v = v_p$ on $\mathbb{Q}$, $A = \{\frac{a}{b} \mid p \nmid b\}$, $\pi = p$, then $k = \mathbb{F}_p$.

A discrete valuation $v$ on $F$ induces an $\mathbb{R}$-valued absolute value on $F$

$$|x| = e^{-v(x)}.$$

Properties:

$$|xy| = |x + y|, \quad |x| = |-x|, \quad |x + y| \leq \max\{|x|, |y|\},$$

where the last inequality is sometimes called the *non-Archimedean inequality.* This gives us a distance function on $F$:

$$d(x, y) = |x - y|.$$

We can then form the completion $\hat{F}$ of $F$ with respect to $d$. Then $\hat{F}$ is a field with discrete valuation $v$, valuation ring $\hat{A}$ (often denoted $\mathcal{O}$ and called the *ring of integers*), residue field $\hat{A}/\pi\hat{A} \cong A/\pi A = k$.

A *non-Archimedean field* is a field with discrete valuation which is complete with respect to the induced distance (cf. the local Archimedean fields are $\mathbb{R}$ or $\mathbb{C}$). A topological field is *local* if it is locally compact.

$\hat{F}$ is locally compact $\iff$ $\hat{A}$ is compact $\iff$ $k$ is finite.

**Example 13.5.**

a) The completion of $\mathbb{Q}$ with respect to the $p$-adic valuation is $\mathbb{Q}_p$ with ring of integers $\mathbb{Z}_p$. We can write elements of $\mathbb{Q}_p$ as

$$\sum_{n\geq N} a_n p^n,$$

where $a_n \in \{0, 1, \ldots, p-1\}$, $a_N \neq 0$, $N \in \mathbb{Z}$. Such an element has valuation $N$. Then

$$\mathbb{Z}_p = \left\{ \sum_{n\geq 0} a_n p^n : a_n \in \{0, \ldots, p-1\} \right\}.$$

b) If $\mathbb{F}_q((t)) = \{\sum_{n\geq N} a_n t^n \mid a_n \in \mathbb{F}^q, a_n \neq 0\}$, $q = p^d$, $p$ prime, where the valuation of $\sum_{n\geq N} a_n t^n$ is $N$, then the valuation ring is $\mathcal{O} = \mathbb{F}_q[[t]]$, a uniformiser is $\pi = t$ and $k = \mathcal{O}/\pi\mathcal{O} = \mathbb{F}_q$. One could similarly consider $F((t))$ for any field $F$ and would still get as a residue field $F$ back.

c) All local non-Archimedean fields are:
   - $\mathbb{Q}_p$ or a finite extension of $\mathbb{Q}_p$, in characteristic 0;
   - $\mathbb{F}_q((t))$ or a finite extension of $\mathbb{F}_q((t))$ in characteristic $p$.

Now to the $BN$-pair:

Let $F$ be a field with discrete valuation, valuation ring $\mathcal{O}$ and residue field $k = \mathcal{O}/\pi\mathcal{O}$. Let $G$ be a group as in the spherical case, e.g. $SL_n$ (not $GL_n$), with $B, N, T$ as above.

The surjection $\mathcal{O} \twoheadrightarrow k = \mathcal{O}/\pi\mathcal{O}$ induces a surjection $G(\mathcal{O}) \twoheadrightarrow G(k)$. The *Ivahori subgroup* $I$ of $G(F)$ is the preimage of the Borel subgroup $B(k)$ of $G(k)$, under the natural map $G(\mathcal{O}) \to G(k)$. If $F$ is local non-Archimedean then $G(\mathcal{O})$ is a maximal compact subgroup of $G(F)$, and $I$ has finite index in $G(\mathcal{O})$ and is hence also compact. We often write $K$ for $G(\mathcal{O})$.

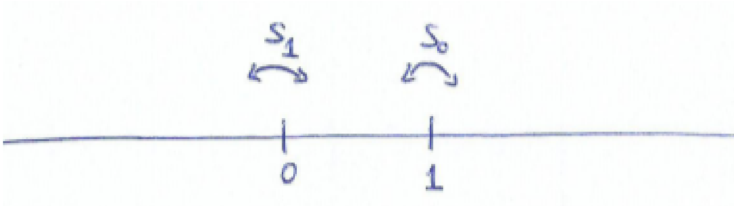
**Example 13.6.** Let $G = SL_3$, $F = \mathbb{Q}_p$. Consider $SL_3(\mathbb{Z}_p) \to SL_3(\mathbb{F}_p)$,

$$\left\{ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \in SL_3(\mathbb{Z}_p) \right\} \twoheadrightarrow \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in SL_3(\mathbb{F}_p) \right\}$$

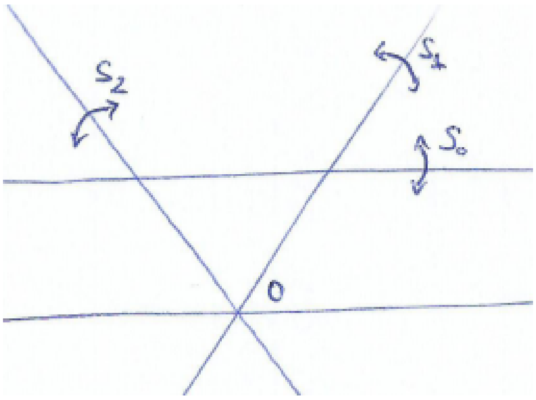
**Theorem 13.7.** *The pair $(I, N(F))$ is an affine Tits system, i.e. a $BN$-pair such that the associated building is affine. The Weyl group for the pair $(I, N(F))$ is $\tilde{W}$ of type $\tilde{X}_n$ where the Weyl group for the spherical $BN$-pair $(B, N)$ is of type $X_n$.*

**Example 13.8.**

a) The group $G = SL_3$ has spherical Weyl group $W$ of type $A_1$, i.e. generated by
$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
affine Weyl group $\tilde{W}$ of type $\tilde{A}_1$.
So $\tilde{W} \cong D_\infty = \langle s_0, s_1 \rangle$ and the affine building for $SL_2(F)$ is a tree.




b) $SL_3$ has spherical Weyl group of type $A_2$, $W \cong S_3$.




The affine Weyl group $\tilde{W} = \langle s_1, s_2, s_3 \rangle$ is the $(3,3,3)$-triangle group and has type $\tilde{A}_2$.

c) In general if $W = \langle s_1, \ldots, s_n \rangle$ then $\tilde{W} = \langle s_0, s_1, \ldots, s_n \rangle$ where $s_1, \ldots, s_n$ are linear reflections in $\mathbb{E}^n$ and $s_0$ is an affine reflection.

We have the affine Bruhat decomposition
$$G(F) = \bigsqcup_{w \in \tilde{W}} IwI.$$
The standard parabolic subgroups are
$$\tilde{P}_J = \bigsqcup_{w \in \tilde{W}_J} IwI,$$
where $\tilde{W}_J$ is a special subgroup of $\tilde{W}$. They are often called *parahorics* (*para*bolic with respect to the Iwa*hori*). The chambers of the affine building are $G(F)/I$ and
$$gI \underset{i}{\sim} hI \iff g^{-1}h \in \tilde{P}_i = I \sqcup I s_i I.$$

Then each $i$-panel has $[\tilde{P}_i : I]$ chambers.

If $F$ is a local non-Archimedean field, this means $\Delta$ is locally finite, and each parahoric is compact (recall if $I \lneq H \lneq G(F)$ then $H = \tilde{P}_J$). In particular $G(\mathcal{O}) = \tilde{P}_J$ where $J = \{1, \dots, n\}$.

Consider the realisation of $\Delta$ with apartments Coxeter complexes. Then simplices in $\Delta$ are of the form $g\tilde{P}_J, g \in G(F)$. So if $F$ is local non-Archimedean, $G(F)$ acts chamber-transitively hence cocompactly on this simplicial complex with compact stabilisers.

*Remark* 11. The group $GL_n(F)$ acts vertex-transitively on the affine building for $SL_n(F)$, the $SL_n(F)$-action is vertex-transtive.

**Theorem 13.9.** *Let $\Delta$ be a building of type $(\tilde{W}, \tilde{S})$ an irreducible affine Coxeter system. If $|\tilde{S}| \geq 4$, i.e.* $\dim(\Delta) \geq 3$*, then $\Delta$ is the affine building coming from an affine Tits system for a classical or algebraic group $G$ over a field $F$ with discrete valuation. Moreover $Aut(\Delta) = G(F) \rtimes Aut(F)$.*

*Proof.* Uses that the Tits boundary of $\Delta$ is the spherical building of the pair $(B(F), N(F))$. If $\dim(\Delta) = n$ then this spherical building has dimension $n-1$. □

3. (infinite non-affine) Some Kac-Moody theory

   *Kac-Moody Lie algebras* are a family of possibly infinite-dimensional Lie algebras over arbitrary fields, which have Weyl groups $(W, S)$. The Coxeter system $(W, S)$ can be anything satisfying the *crystallographic restriction*:

   $$m_{ij} \in \{2, 3, 4, 6, \infty\}, i \neq j,$$

   e.g. could be any right-angled Coxeter system.

   The Kac-Moody algebra is $\infty$-dimensional $\iff$ $(W, S)$ is infinite non-affine. Assume this from now on.

   Tits associated to Kac-Moody algebras groups called *Kac-Moody groups.*

   Kac-Moody groups are infinite but have a presentation similar to Steinberg's presentation for finite groups of Lie type.

   There is a root system

   $$\Phi = \underbrace{\Phi^+}_{\text{pos. roots}} \sqcup \underbrace{\Phi^-}_{\text{neg. roots}}.$$

   The Kac-Moody group is generated by subgroups called *root groups* $U_\alpha \cong (F, +), \alpha \in \Phi$. The relations are commutator relations between root groups.

Compare: In $SL_3(q)$ the positive root groups are given by

$$U_{\alpha_1} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong (F_q, +)$$
$$U_{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$
$$U_{\alpha_3} = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where we have
$$[U_{\alpha_1}, U_{\alpha_2}] \subseteq U_{\alpha_1+\alpha_2}.$$

Similarly the negative root groups have the star below the diagonal.

Tits showed that Kac-Moody groups have twin $BN$-pairs $(B^+, N), (B^-, N)$. Here $B^+$ (resp. $B^-$) is generated by the torus (which has an intrinsic definition) and the positive (resp. negative) root groups. Thus we get twin buildings $\Delta^+ = \Delta(B^+, N), \Delta^- = \Delta(B^-, N)$ with $\Delta^+ \cong \Delta^-$.

**Theorem 13.10.** *If $\Lambda$ is a Kac-Moody group over a finite field $\mathbb{F}_q$ with $q \gg 0$, with infinte non-affine Weyl groups. Then:*

*a) $\Lambda$ is a non-compact lattice in the locally compact group $Aut(\Delta^+ \times \Delta^-)$.*

*b) Let $G$ be the closure of $\Lambda$ in $Aut(\Delta^-)$. Then each parabolic subgroup of $\Lambda$ with respect to $(B^+, N)$ (e.g. $B^+$ itself), is a non-compact lattice in the locally compact group $G$ and in $Aut(\Delta^-)$.*

The affine analogue of these results is:

a) $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$ embeds in $\mathrm{SL}_n(\mathbb{F}_q((t))) \times \mathrm{SL}_n(\mathbb{F}_q((t^{-1})))$ as an arithmetic non-compact lattice.

b) $\mathrm{SL}_n(\mathbb{F}_q[t])$ is an arithmetic non-compact lattice in $\mathrm{SL}_n(\mathbb{F}_q((t^{-1})))$.

## 13.2 Other constructions of buidlings

A key result is the following local-to-global theorem.

Let $(W, S)$ be a geometric reflection group, with $S$ the set of reflections in faces of a convex polytope $P$. Let $X$ be a connected polyhedral complex. A *type function* $\tau : X \to P$ is a morphism of CW-complexes which restricts to an isometry on each maximal cell. We say $X$ is of type $(W, S)$ if there is a type function $\tau : X \to P$. By pulling back, each cell of $X$ has type $T \subseteq S$.

**Theorem 13.11.** *Let $X$ be a connected polyhedral complex of type $(W, S)$. Assume:*

1. *the link of each vertex is $CAT(1)$;*
2. *for each point $x \in X$ of type $T \subseteq S$ with $|T| \leq 3$, through any two points in $lk(x, X)$ there passes an isometrically embedded sphere of dimension $|T| - 1$.*

*(E.g. these hold if all links are spherical buildings.)*
*Then the universal cover of $X$ is a building of type $(W, S)$.*

*Proof.* Uses a theorem of Tits concerning the low rank residues in chamber systems, and a "metric recognition" theorem for buildings by Charney-Lytchak. □

This result has been used to construct "non-classical" buildings by many people, e.g.

1. Tits, Ronan: exotic $\tilde{A}_2$ buildings. Problem: no information on Aut($\Delta$).
2. Cartwright-Steger: exotic $\tilde{A}_2$ buildings with vertex-transitive automorphism groups.
3. Radu 2016: $\tilde{A}_2$ buildings with non-Desarguesian links (i.e. vertex links are incidence graphs of non-Desarguesian projective planes, i.e. not $\mathbb{P}_2(\mathbb{F}_q)$) and cocompact automorphism group.
4. Ballmann-Brin: construction of polygonal complexes with all faces $k$-gons, all links the same graph $L$. So if $L$ is a 1-dimensional spherical building one gets a 2-dimensional building.
5. Vdovina: finite polygonal complexes whose universal cover is a hyperbolic building whose fundamental group acts cocompactly.
6. Polygons of groups: complexes of groups.

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