Zigzag construction of Expander graphs

we construct an infinite sequence of d-regular expander graphs algorithmically via elementary operations.

Two operations: ① Powering ② Zigzag

① Powering: $G \to G^2$

G^2 is the graph on the same set of vertices, and edges for two-step-walks. More accurately

$$A_{G^2} := A_G \cdot A_G . \quad \left(\text{also } M_{G^2} = M_G \cdot M_G ,\ M_G = \frac{1}{d} \cdot A_G\right)$$

G^2 is d^2-regular (a self loop counts as adding 1 to degree)

<u>Claim:</u> G^2 is d^2-regular, has eigenvalues $(\lambda_i^2)_{i=1}^n$.

(we can remove the d self loops : $A_{G^2} - d \cdot I_n$)

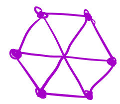
② zigzag : G, H two graphs $G \textcircled{z} H$

Notation: G is an (n,m,α)-graph denotes an m-regular graph on n vertices with $\max(|\lambda_2|, |\lambda_n|) \leq \alpha$.

$\forall i > 1 \quad |\lambda_i| \leq \alpha \qquad 1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n \geq -1$

Theorem [RVW '00]: : Assume G is (n,m,α)-graph H is (m,d,β)-graph. Then $G \textcircled{z} H$ is an $(nm, d^2, \beta + \max(\alpha, \beta^2))$-graph.

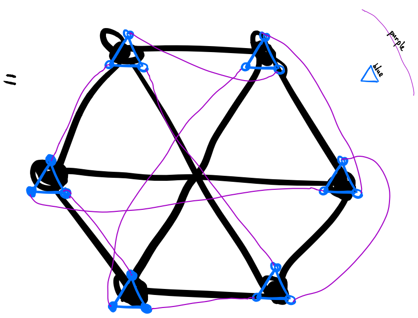
We define the $\textcircled{z}$ operation. First, an example:

$G =$ 

$|V_G| = 6$
$d_G = 3$

$H =$ 

$|V_H| = 3$
$d = 2$

$G \textcircled{z} H =$ 

First,

edges: 

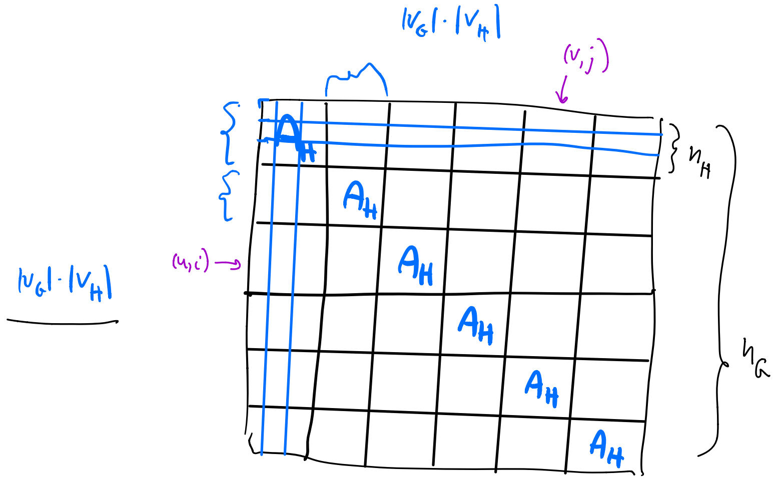
more generally, (1) replace each G vertex by a copy of H
(2) connect G-edges to an available slot in the cloud
(3) New edges in $G \textcircled{z} H$ are H-edge then G-edge then H-edge.

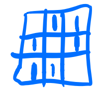
In matrix form:

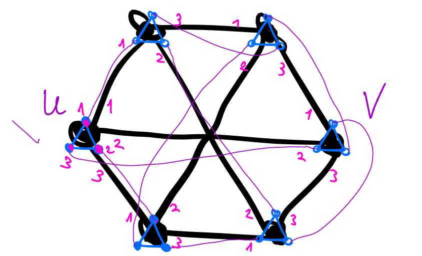
We define $\breve{A}_H$ a matrix for blue steps

also $\breve{A}_G$ a matrix for purple steps

Our final adj matrix for $G \textcircled{z} H$ will be $\breve{A}_H \breve{A}_G \breve{A}_H$



$A_H =$ 



$$\tilde{A}_H = I_{V_G} \otimes A_H$$

i-th edge of u   j-th edge of v

$$\boxed{\tilde{A}_G((u,i),(v,j)) = 1 \iff u \underset{G}{\sim} v \quad e_i(u) = e_j(v)}$$

$e_i(u)$ is an edge in G

$e_1(u) \dots e_d(u)$ are the neighboring edges to u

$$A_{G \circledz H} := \tilde{A}_H \tilde{A}_G \tilde{A}_H \quad \leftarrow \text{adj matrix of zigzag product}$$

Why is this a good expander?

Suppose $S \subset V_{G \circledz H}$. If S splits most clouds $\rightarrow$ inner edges cross

If S is full/empty on most clouds $\rightarrow$ G edges cross

Proof of theorem:

Notation For a d-regular graph we let $M = \frac{1}{d} A_G$ be the normalized adj. matrix.

$$\langle f, g\rangle := \mathop{\mathbb{E}}_{v} f(v)g(v) = \frac{1}{|V|}\cdot \sum_{v} f(v)g(v) \qquad \|f\|^2 = \langle f, f\rangle, \ldots$$

$$\circledast \qquad \lambda = \max_{f \perp \vec{1}} \frac{|\langle Mf, f\rangle|}{\langle f, f\rangle} = \frac{\sum_{i>1} \lambda_i \alpha_i^2}{\sum_{i>1}\alpha_i^2} \quad \Big(\text{where } f = \sum \alpha_i \cdot \underset{\text{ev's}}{V_i}\Big)$$

$$(\lambda = \max |\lambda_2|, |\lambda_n|).$$

$$\circledast \qquad \langle Mf, g\rangle = \mathbb{E}(Mf)(v)\ g(v)$$

$$= \mathop{\mathbb{E}}_{v}\Big[\mathop{\mathbb{E}}_{u\sim v} f(u)\Big] g(v) = \mathop{\mathbb{E}}_{u\sim v} f(u)g(v)$$

(all edges $uv \in E$)

Fix $f : V_{G \circledS H} \longrightarrow \mathbb{R}$ s.t. $\mathop{\mathbb{E}}_{\substack{u\sim V_G \\ i \sim V_H}} f(u,i) = 0$ need: $|\langle Mf, f\rangle| \leq \gamma \langle f, f\rangle$.

Define $f^{\parallel}(u,i) = \mathop{\mathbb{E}}_{j \sim V_H} f(u,j)$

$f = f^{\parallel} + f^{\perp}$

$f^{\parallel} = f_G \otimes \vec{1}_H$

$\mathbb{E} = 0 \quad \mathbb{E} = 0 \quad \ldots$

Define $f^{\perp}(u,i) = f(u,i) - f^{\parallel}(u,i)$

$f^{\parallel}$ – const. on each cloud

$f^{\perp}$ – expectation $= 0$ on each cloud.

$$\langle f^{\perp}, f^{\parallel}\rangle = \mathop{\mathbb{E}}_{(u,i)} f^{\perp}(u,i)\cdot f^{\parallel}(u,i) = \mathop{\mathbb{E}}_{u}\mathop{\mathbb{E}}_{i} \underbrace{f^{\parallel}(u,i)}_{\substack{\text{doesn't} \\ \text{depend on} \\ i}} \cdot f^{\perp}(u,i) = 0$$

$f = f^{\parallel} + f^{\perp}$

$$|\langle Mf, f\rangle| = |\langle Mf^{\parallel}, f^{\parallel}\rangle + \langle Mf^{\perp}, f^{\perp}\rangle + \langle Mf^{\parallel}, f^{\perp}\rangle + \langle Mf^{\perp}, f^{\parallel}\rangle|$$

$$\leq |\langle Mf^{\parallel}, f^{\parallel}\rangle| + |\langle Mf^{\perp}, f^{\perp}\rangle| + 2|\langle Mf^{\parallel}, f^{\perp}\rangle|$$

$$|\langle M f^{\parallel}, f^{\parallel}\rangle| = \left|\left\langle \tilde{A}_G \overbrace{\tfrac{1}{d}\tilde{A}_H f^{\parallel}}^{f^{\parallel}}, \overbrace{\tfrac{1}{d}\tilde{A}_H f^{\parallel}}^{f^{\parallel}} \right\rangle\right|$$

$$= |\langle \underbrace{\tilde{A}_G}_{\frac{1}{d_G}\cdot A_G} f^{\parallel}, f^{\parallel}\rangle| = |\langle M_G \hat{f}, \hat{f}\rangle|$$

where $\hat{f}: V_G \to \mathbb{R}$ $\quad \hat{f}(u) = f^{\parallel}(u,1)$

$$\mathop{\mathbb{E}}_{u\sim V_G} \hat{f}(u) = \mathop{\mathbb{E}}_{\substack{u\sim V_G \\ i\sim V_H}} f^{\parallel}(u,i) = \mathbb{E}\, f(u,i) = 0$$

$$\le \lambda_G \cdot \underbrace{\|\hat{f}\|^2}_{= \|f^{\parallel}\|^2} \le \lambda_G = \alpha$$

$$|\langle M f^{\perp}, f^{\perp}\rangle| = |\langle \tilde{M}_G \tilde{M}_H f^{\perp}, \tilde{M}_H f^{\perp}\rangle| \le$$

normalized adj matrix

$$M = \underbrace{\tfrac{1}{d_H}\tilde{A}_H}_{\tilde{M}_H} \tilde{A}_G \underbrace{\tfrac{1}{d_H}\tilde{A}_H}_{\tilde{M}_H}$$

C.S.

$$\le \|\tilde{M}_G \tilde{M}_H f^{\perp}\| \cdot \|\tilde{M}_H f^{\perp}\| \le \|\tilde{M}_H f^{\perp}\|^2$$

only decreases the norm

$$\le \lambda_H^2 \cdot \|f^{\perp}\|^2$$

$$= \beta^2 \cdot \|f^{\perp}\|^2$$

Recall $\quad \tilde{M}_H = M_H \otimes I_{V_G} \qquad \|\tilde{M}_H f^{\perp}\| \le \lambda_H \cdot \|f^{\perp}\|$

$$\begin{pmatrix} M_H & & \\ & M_H & \\ & & \ddots \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}$$

$\mathbb{E} f_1 = 0$

$\mathbb{E} f_2 = 0$

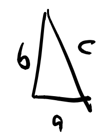
$\vdots$

$$\tilde{M}_H f^{\perp} = \begin{pmatrix} M_H f_1 \\ M_H f_2 \\ \vdots \end{pmatrix} \qquad \lambda_H \cdot \|f_1\|$$

$$2\left|\langle M f^{\perp}, f^{\parallel} \rangle\right| \le \underbrace{2\|M f^{\perp}\|}_{\substack{\le \\ \lambda_H \cdot \|f^{\perp}\|}} \cdot \|f^{\parallel}\| \le 2\beta \cdot \underbrace{\|f^{\perp}\| \cdot \|f^{\parallel}\|}_{\le \frac{\|f\|^2}{2}}$$

$$\|f\|^2 = \|f^{\parallel}\|^2 + \|f^{\perp}\|^2$$

[AMGM : $2xy \le \sqrt{x^2 + y^2}$]



$$|\langle Mf, f \rangle| \le \underbrace{\alpha \cdot \|f^{\parallel}\|^2} + \underbrace{2\beta \|f^{\parallel}\| \cdot \|f^{\perp}\|}_{\substack{\le \text{ AMGM} \\ \beta\left(\|f^{\parallel}\|^2 + \|f^{\perp}\|^2\right) = \beta\|f\|^2 \\ \le \ \beta\|f\|^2}} + \underbrace{\beta^2 \|f^{\perp}\|^2}$$

$$= \beta \cdot \|f\|^2 + \alpha\left(\|f\|^2 - \|f^{\perp}\|^2\right) + \beta^2 \|f^{\perp}\|^2$$

$$\le \beta \cdot \|f\|^2 + \max(\alpha, \beta^2) \cdot \|f\|^2 \qquad \square$$

Constructing an infinite sequence of d-regular expander graphs

starting point: take $H = H_0$ to be $(d^4, d, \frac{1}{4})$-graph

take $G_1 = H^2$. $(d^4, d^2, \frac{1}{16})$-graph

$G_2 = (G_1)^2 ⓩ H$

G_1^2 $(d^4, d^4, \frac{1}{256})$-graph

$G_1^2 ⓩ H$ $(d^4 \cdot d^4, d^2, \gamma)$-graph

$\gamma \le \beta + \max(\alpha, \beta^2)$

$\beta = \frac{1}{4}$ $\alpha = \frac{1}{256}$

$\gamma \le \frac{1}{4} + \frac{1}{16} \le \frac{1}{2}$

$\vdots$

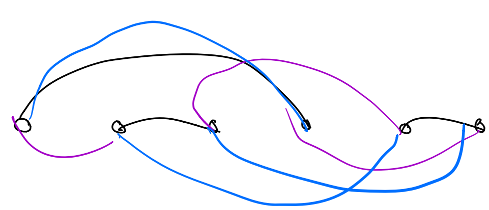
$G_{n+1} = (G_n)^2 ⓩ H$

Claim: G_n is a graph on d^{4n} vertices, degree d^2, $\forall n \geq 1$

$\max(|\lambda_2^n|, |\lambda_n^n|) \le \frac{1}{2}$

G_n is an $(d^{4n}, d^2, \frac{1}{2})$-graph

Altogether, algorithm enumerates graphs to find good H, then proceeds inductively.



N

extra

Ex: "modifications"

Let G be (n, d, λ) graph $\lambda \le 0.1$.

Suppose we remove edges from G so that

$G' = (V, E')$ $E' \subset E$ is $\frac{d}{2}$ regular.

is G' still an expander?