Lecture 1

Expanders: combinatorial definition

Slogan: An expander graph is a finite graph, which is

- highly-connected (in a robust way)
- sparse (not too many edges)

Remarkable facts:

(1) Such graphs exist

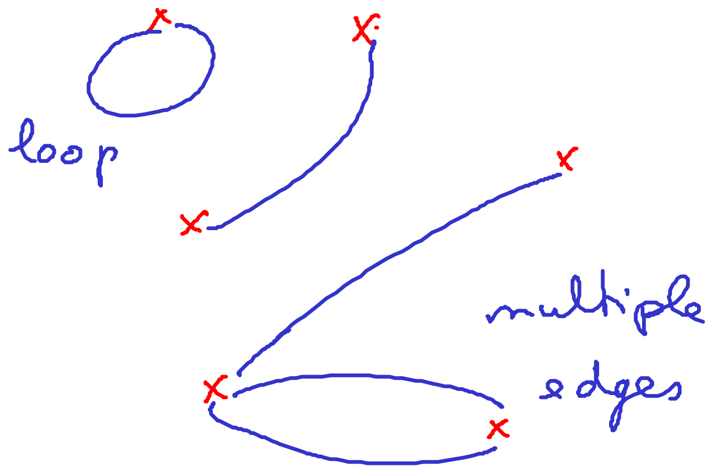
(2) They have amazing applications [in TCS, geometry, number theory, combinatorics, knot theory, arithmetic geometry ...]

1.1 – Graphs, examples

Informally:

- vertices
- edges

between some pairs of vertices [unoriented]



Often we work with finite graphs ($\iff$ V, E both finite);

vertices edges

we write $|\Gamma| = |V|$ for

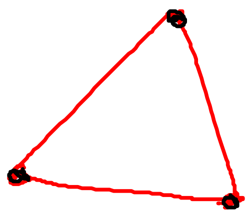
graph

The number of vertices.

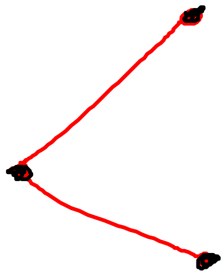
A graph is d-regular if

all vertices have d neighbors (with multiplicity)

[loop counts as 1]



2-regular

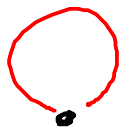


not regular

Examples:

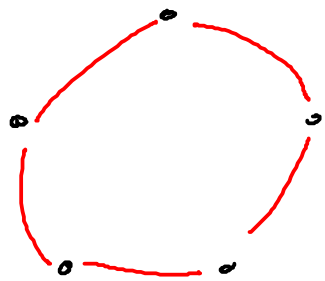
- $\underline{C_m}$ (cycles)

(1-regular) C_1

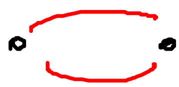


C_5

(2-regular)



C_2



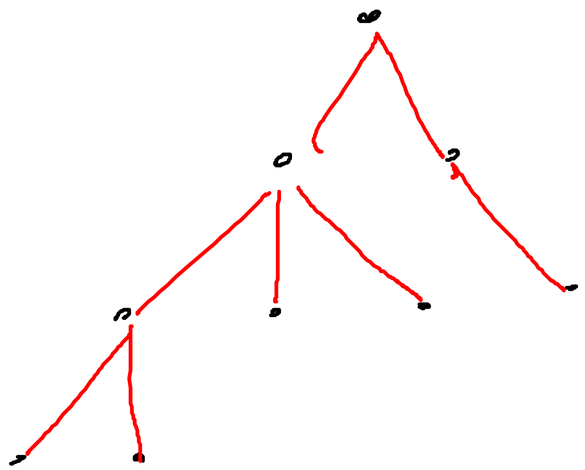
(2-regular)

Def. (Girth)

A graph Γ has *girth* equal to $m \geq 1$ if it contains an m-cycle, and no smaller one.

Def (Forest)

A graph is a forest if girth is <u>infinite</u>



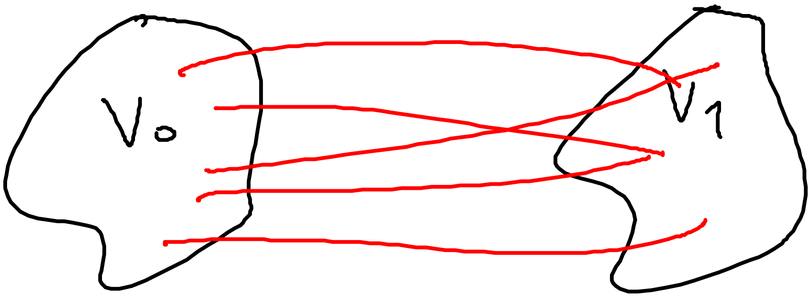
Def. (Bipartite)

Γ is bipartite if

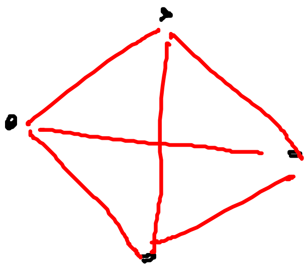
$$V = V_0 \sqcup V_1$$

disjoint

with all edges joining V_0 to V_1



Ex. K_m : complete graph on m vertices : one edge between any vertices $x \neq y$



$((m-1)$-regular$)$

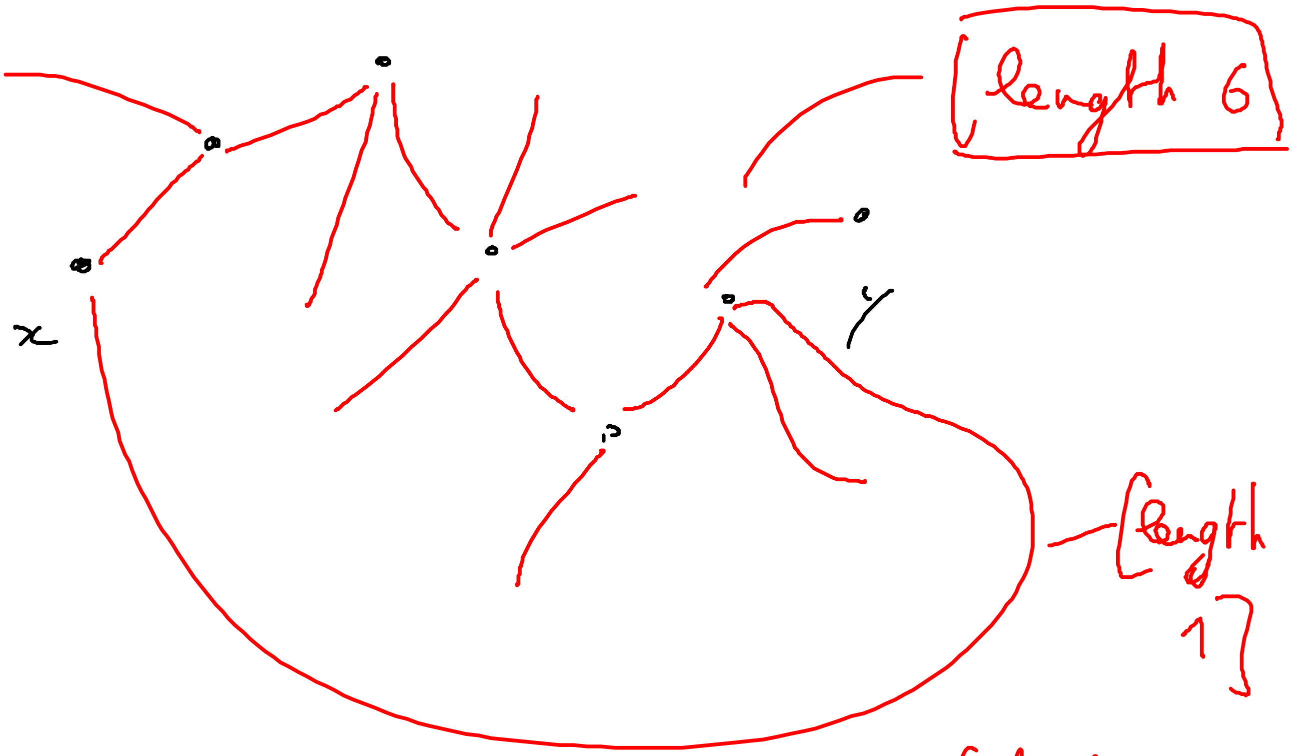
1.2. Distance

Def. $\Gamma = (V, E)$ graph

x, y in V

$d_\Gamma(x,y) = \min \{ \text{length}(\gamma) \,/\, \gamma$ is a path in Γ with extremities $x, y\}$

$$\in \{0, 1, \dots, \} \cup \underbrace{\{+\infty\}}_{x,\, y \text{ not connected}}$$



Prop. d is a distance on [if always finite]

$$\forall \left[\begin{array}{l} d(x,y)=0 \iff x=y \\ d(x,y)=d(y,x) \\ d(x,z)\le d(x,y)+d(y,z) \end{array}\right]$$

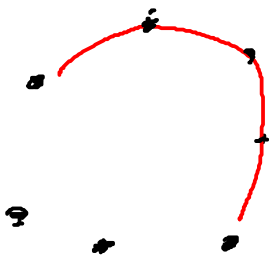
$\longrightarrow$ notions from geometry

- connectedness: if the distance is always finite
- diameter:

$$\mathrm{diam}(\Gamma) = \max_{(x,y)\in V^2} d(x,y)$$

For many applications, having a "small" diameter is a good thing.

Ex.: • $\mathrm{diam}(C_m) \simeq \frac{m}{2} \left(= \frac{|\Gamma|}{2} \right)$



- $\mathrm{diam}(K_m) = 1$

__Lemma.__ If $(\Gamma_n)_{n\geq 1}$ is a sequence of $\neq\emptyset$ finite d-regular graphs (d fixed), then $\exists\, c>0$, $\boxed{\mathrm{diam}(\Gamma_n) \geq c \log(|\Gamma_n|)}$

(nb of vertices)

$\boxed{\rightarrow}$ The best diameter bound, without increasing the number of neighbors ("valencies") is about $\log(|\Gamma|)$.

1.3. Cayley graphs

__Def.__ Let G be a group. Let $S \subset G$ be a subset [not nec. a subgroup], $\boxed{\text{with}}$

$\boxed{S = S^{-1}.}$ **The Cayley graph of G relative to S**, denoted

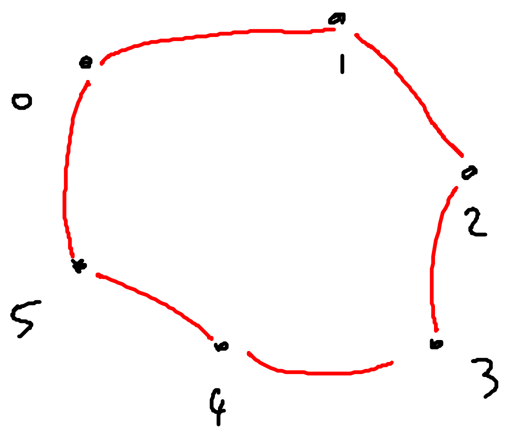
$$\mathcal{C}(G, S)$$

is the graph with

- vertices $= G$
- edges join g to gs for $g \in G$ and some $s \in S$

Ex. • $G = \mathbb{Z}/m\mathbb{Z}$

$$S = \{1, -1\}$$



$$\mathcal{C}(\mathbb{Z}/m\mathbb{Z}, \{1, -1\}) = C_m$$

[Changing S changes the graph!]

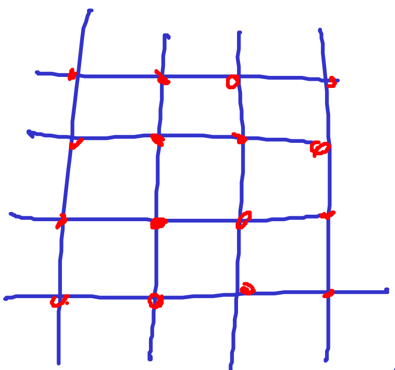
- $G = F_2$, free group on two generators

$$S = \{a, a^{-1}, b, b^{-1}\}$$

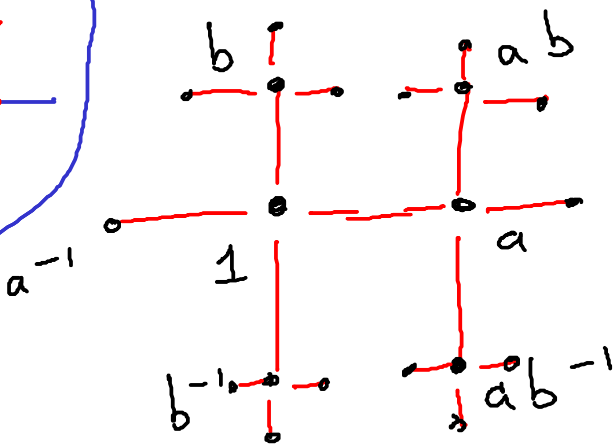
generators

$\mathbb{Z}^2$

$S = \{(\pm 1, 0), (0, \pm 1)\}$



grid



$\longrightarrow$ infinite

<u>4-regular</u>

<u>tree</u>

<u>Facts:</u>

(1) $\mathcal{C}(G, S)$ is $|S|$-regular

(2) $\mathcal{C}(G, S)$ is connected

$\Uparrow$

S generates G

Assume S generates G

(3) $\mathcal{C}(G,S)$ is bipartite [Exercise 1.9]

$\Updownarrow$

$\exists\ \varepsilon : G \longrightarrow \{\pm 1\}$ group morphism, surjective, with

$\varepsilon(s) = -1$ for all $s \in S$

[Counterex. $G = S_5$, $S = \{\tau_1, \tau_2\}$, $\tau_1, \tau_2 \in A_5$, ...]

1.4. Expansion in graphs

Want to define "sparse and robustly connected" sequences of graphs.

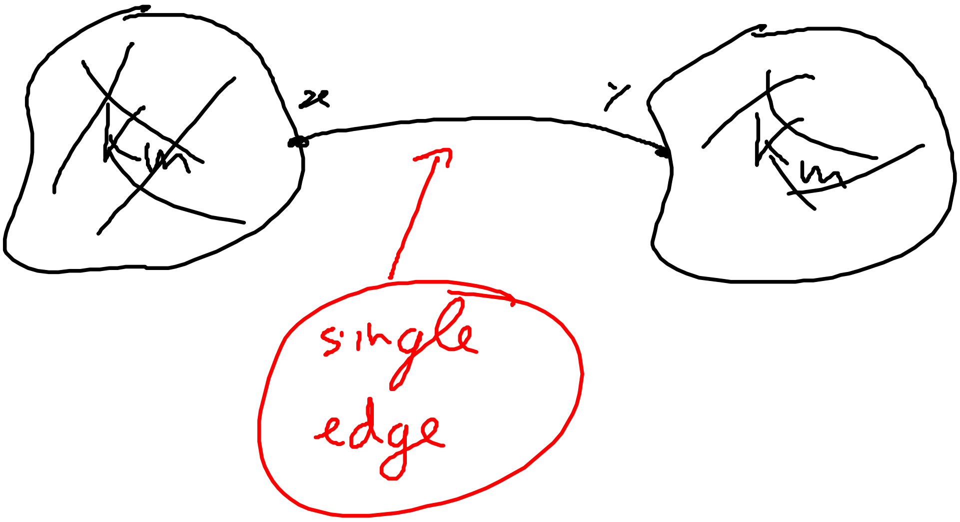
Sparse: every vertex has a $\underbrace{\text{bounded}}_{\text{uniformly}}$ number of neighbors

[ex. $(C_m)_{m \geq 1}$

not $(K_m)_{m \geq 1}$]

Connectedness: might think that we ask for small diameter

Not so good:



diameter: ≤ 3 but removing a single edge disconnects the graph!

To get a better definition, one defines an invariant that "measures" robustness.

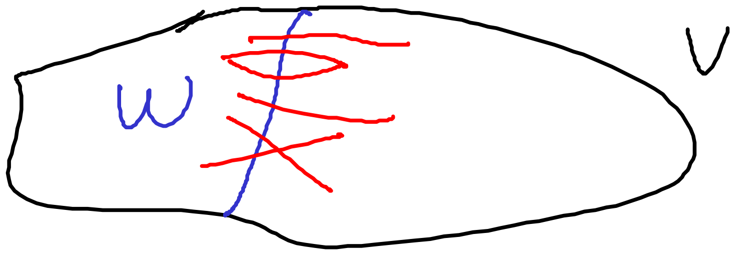
Def. (Cheeger constant)

Γ graph, finite

$$h(\Gamma) = \min_{\substack{W \subset V \\ \emptyset \neq W,\ |W| \leq \frac{|V|}{2}}} \frac{|\mathcal{E}(W)|}{|W|}$$

$\mathcal{E}(V-W)$

Cheeger constant



where $\mathcal{E}(W)$ is the set of edges with one extremity in W, the other outside W.

$h(\Gamma)$ "small" means that Γ can be disconnected "easily".

[Prop. 1.25]

Lemma

$h(\Gamma) > 0$

$\Updownarrow$

Γ is connected.

Def. (Expander graphs)

A sequence $(\Gamma_n)_{n\geq 1}$ of finite graphs is an <u>expander</u> (<u>family</u>) $\iff$

(i) $\lim_{n\to\infty} |\Gamma_n| = +\infty$

(ii) max. nb. of neighbors is <u>uniformly</u> bounded [e.g. all Γ_n is d-regular with d fixed].

sparsity

(iii) $\exists\ c > 0$ s.t. $\forall n \geq 1,\ h(\Gamma_n) \geq c.$ } robustness

Q. Do these exist ??