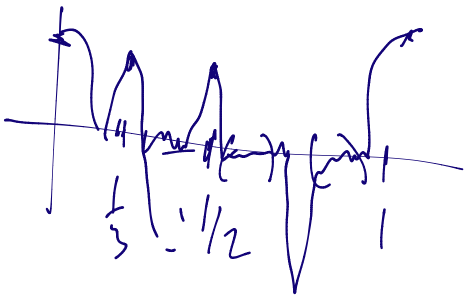
Orbital Circle Method II: Major Arcs + Expansion..

Last time: $$\hat{R}_N(\theta) = \sum_{\gamma\in\Gamma_N = \Gamma\cap B_N} e(\theta\langle\gamma v, w\rangle).$$

Goal: evaluate on major arcs



Let's try to evaluate at $\theta = \frac{r}{q} + \cancel{\beta}$:

$$\hat{R}_N\left(\frac{r}{q}\right) = \sum_{\gamma\in\Gamma_N} e_q(r\langle\gamma v, w\rangle)$$

$e_q(z) = e^{2\pi i z/q}$.

$$= \sum_{\gamma_0\in\Gamma\,(\mathrm{mod}\ q)} \ \sum'_{\substack{\gamma\in\Gamma_N \\ \gamma\equiv\gamma_0\,(q)}} e_q(r\langle\gamma_0 v, w\rangle)$$

$\mathfrak{M}$: $\frac{r}{q}$, $1\le q < Q_0 = N^{\varepsilon}$, $|\beta|$ small,

$$= \sum_{\gamma_0\in\Gamma\,(\mathrm{mod}\ q)} e_q(r\langle\gamma_0 v, w\rangle)\left[\sum'_{\substack{\gamma\in\Gamma_N \\ \gamma\equiv\gamma_0\,(q)}} 1\right].$$

"simple".

q grows (slowly-ish) with (a power of) N.

Need to count in archimedean balls in progressions, uniformly in q.

$$\sum'_{\substack{\gamma\in\Gamma_N \\ \gamma\equiv\gamma_0(q)}} 1 \overset{?}{=} \frac{1}{|\Gamma(\mathrm{mod}\ q)|}\sum_{\gamma\in\Gamma_N} 1 + O\left(|\Gamma_N|\cdot N^{-\Theta}\right)$$

spectral gap

$\Theta > 0$,

indep of $\underline{q}$! (expansion)

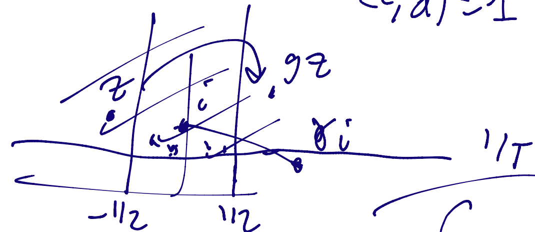
How to do this? Depends! In each context, done by different tools!

Apollonian – homogeneous dynamics, autom. reps, ...

Zaremba – Congruence transfer operators, thermodynamics ...

"Baby version": Count $\#\left\{(c,d) : \begin{array}{l} c^2+d^2 < N^2 \\ (c,d)=1 \end{array},\ \begin{array}{l} c \equiv 0\ (q) \\ d \equiv 1\ (q) \end{array}\right\}$, $N^2 = T$.

$\mathbb{H} = \{ z = x+iy \mid y>0 \} \subset \mathbb{C}$



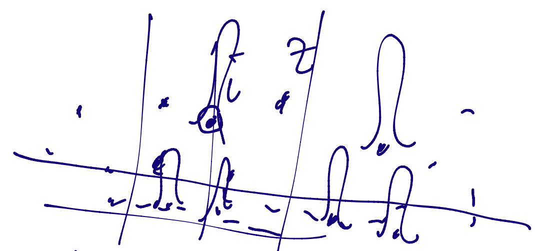
$\circlearrowright G = SL_2(\mathbb{R})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \dfrac{az+b}{cz+d}$. (Ex 1 $\in \mathbb{H}$).

Ex 2: $\operatorname{Im} gz = \dfrac{\operatorname{Im} z}{|cz+d|^2}$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL_2(\mathbb{Z})$, $\underline{a}\underline{d}-\underline{b}\underline{c}=1$, $\Gamma_\infty = \left\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\rangle$.

$\Rightarrow \operatorname{Im} \gamma i = \dfrac{1}{c^2+d^2} > \dfrac{1}{T}$.

So Count $= \displaystyle\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathbb{1}_{\{\operatorname{Im} \gamma i > \frac{1}{T}\}} = F_T(i)$, where

$$F_T(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathbb{1}_{\{\operatorname{Im} \gamma z > \frac{1}{T}\}}$$

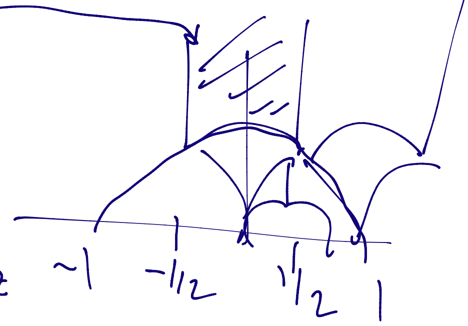


Let $\underline{\Phi}$ be a bump function of rad ε about i in $\Gamma\backslash\mathbb{H}$ (Approx delta spike at i)

Ex 3: $\boxed{\langle F_T, \Phi \rangle} = \text{Count} + O(\varepsilon \cdot T)$, $\displaystyle\int_{\Gamma\backslash\mathbb{H}} F_T(z)\,\Phi(z)\,\frac{dx\,dy}{y^2}$.

Let Δ = hyperbolic Laplacian/Beltrami $= -y^2(\partial_{xx} + \partial_{yy}) \circlearrowright L^2(\Gamma\backslash\mathbb{H})$

$f \in L^2(\Gamma\backslash\mathbb{H})$



$\displaystyle\int_{\Gamma\backslash\mathbb{H}} |f|^2 \underbrace{\frac{dx\,dy}{y^2}}_{G\text{-invariant}} < \infty$ Ex4:

$z \mapsto -\frac{1}{z} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} z$

Δ is self-adjoint, $\langle f, \Delta g\rangle = \langle \Delta f, g\rangle$, Ex5:

pos-def: $\langle \Delta f, f\rangle \geq 0$. Ex6: $\Rightarrow$ ~~eigenvalues~~ spectrum ≥ 0.

Analogy: $\mathcal{H}$ = vect space, Δ symm matrix $\Rightarrow$ diagonalizable & $\mathbb{R}$-eigen.

Pretend: $\mathcal{H} = L^2(\Gamma\backslash\mathbb{H})$ has orthonormal basis of eigenfuncts Maass.

$= \displaystyle\bigoplus_{j=1}^{\infty} \mathbb{C}\varphi_j$, $\Delta\varphi_j = \lambda_j \varphi_j = s_j(1-s_j)\varphi_j$,

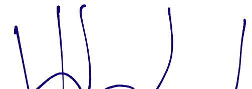
Then $\displaystyle\langle F_T, \Phi\rangle = \sum_j \langle F_T, \varphi_j\rangle \underbrace{\langle \varphi_j, \Phi\rangle}_{\approx\, \varphi_j(i)}$.

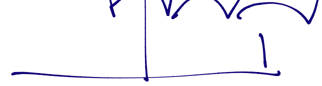
$\displaystyle\langle F_T, \varphi_j\rangle = \int_{\Gamma\backslash\mathbb{H} = \mathcal{F}} \left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathbb{1}_{\{\operatorname{Im}\gamma z > \frac{1}{T}\}}\right) \varphi_j(z)\frac{dx\,dy}{y^2}$

"Unfold"

$\displaystyle = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\mathcal{F}} \mathbb{1}_{\{\operatorname{Im}\gamma z > \frac{1}{T}\}} \varphi_j(z) \frac{dx\,dy}{y^2}$ change variables $z \mapsto \gamma^{-1} z$.

$= \sum \int \mathbb{1}$





$$\underbrace{\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma \mathcal{F}}}_{\text{add up tiles, up to translation}} \underbrace{1_{\{\mathrm{Im}\, z > \frac{1}{T}\}} \varphi_j(z) \frac{dx\,dy}{y^2}}_{\text{No } \gamma\text{'s!}}$$

$$= \int_{\Gamma_\infty \backslash \mathbb{H}} 1_{\{\mathrm{Im}\, z > \frac{1}{T}\}} \varphi_j(z) \frac{dx\,dy}{y^2} = \int_{\frac{1}{T}}^{\infty} \underbrace{\left[\int_{-1/2}^{1/2} \varphi_j(z)\,dx \right]}_{f(y)} \frac{dy}{y^2}$$

$$\Delta f(y) = -y^2 (\cancel{\partial_{xx}} + \partial_{yy}) f(y) = \int_{-1/2}^{1/2} \underline{\Delta \varphi_j}(z)\,dx = \lambda_j f(y).$$

$$-y^2 \partial_{yy} f(y) = \overbrace{s(1-s)}^{\lambda} f(y). \qquad 1 > s > \tfrac{1}{2}$$

Ex 7: $f(y) = y^s, \; y^{1-s}$

$f_y = s\, y^{s-1}$

$f_{yy} = s(s-1)\, y^{s-2}$.

Ex 8: $\lambda \geq 0 \Rightarrow \frac{1}{2} \leq s \leq \boxed{1}$ or $s = \frac{1}{2} + it$, $t \geq 0$.

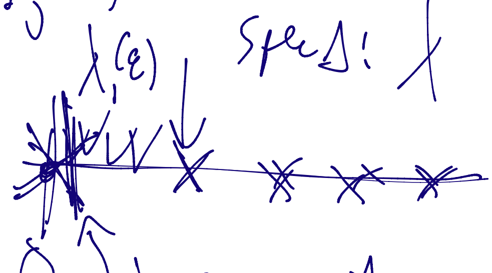
So ... $\Rightarrow f(y) = c_j\, y^{1-s_j}$. So: $\langle F_T, \varphi_j \rangle =$

$$\int_{\frac{1}{T}}^{\infty} c_j\, y^{1-s_j} \frac{dy}{y^2} = \boxed{c_j \frac{T^{s_j}}{s_j}} \qquad \text{(Ex 9:)}$$

$$\text{Count} \approx \langle F, \Phi \rangle = \sum_{j=0}^{\infty} \langle F_T, \varphi_j \rangle \varphi_j(i)$$

$$= \sum_{j=0}^{\infty} c_j'\, T^{s_j}$$

Spec:



$\eta_0 = 0,\ s_0 = 1.$

$= T^1 c + O\left(T^{s_1(q)}\right) \ll T^{1-\theta}$ $\boxed{\forall q}$

"Principal Congruence Subgroups" $\Gamma(q) = \{\gamma \in \Gamma \mid \gamma \equiv I\,(q)\}$

$\Gamma_1(q) = \left\{\gamma \in \Gamma \mid \gamma \equiv \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} (q)\right\}$, $\Gamma_0(q) = \left\{\gamma \in \Gamma \mid \gamma \equiv \begin{pmatrix}* & *\\ 0 & *\end{pmatrix} (q)\right\}$

Do all of the above with $\Gamma \rightsquigarrow \Gamma_1(q)$.

What happens to Spec $\Delta \circlearrowright L^2(\Gamma_1(q)\backslash \mathbb{H})$

Expansion ($\theta > 0$) $\Longleftrightarrow$ $\exists\, \theta > 0$ s.t. $s_1(q) < 1 - \theta$.

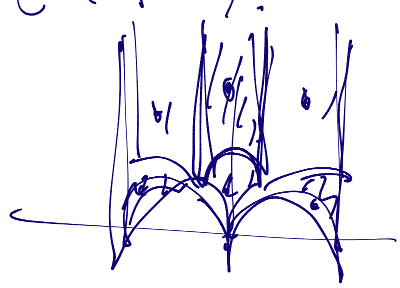
generalized $\longrightarrow$

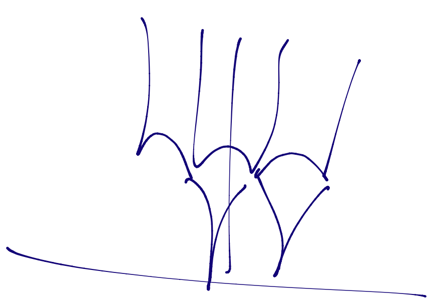
(Ramanujan Conj / Selberg $\frac{1}{4}$-Conj: $\mathrm{Re}(s_1(q)) = \frac{1}{2}$)

Selberg $3/16$ thm '60s.

$(\Gamma/\Gamma_1(q)) \circlearrowright$

Say φ is eigenfunction of Δ on $L^2(\Gamma_1(q)\backslash \mathbb{H})$ but not on $L^2(\Gamma\backslash \mathbb{H})$





Bourgain-Gamburd-Sarnak '10