Lecture 2

Spectral definition

Reminder: $(\Gamma_n)_{n\geq 1}$ **expander**

$\Updownarrow$

(i) $|\Gamma_n| \longrightarrow +\infty$

(ii) valencies uniformly bounded

(iii) $\exists\, c>0,\ \forall n$

$$h(\Gamma_n) = \min_{1\leq |W|\leq \frac{|V|}{2}} \frac{|\mathcal{E}(W)|}{|W|} \geqslant c$$

Cor. $\exists\, C \geqslant 0$

$$\operatorname{diam}(\Gamma_n) \leq C \log(|\Gamma_n|)$$

2.1. Markov operator

Convention: Γ finite graph

$V \neq \emptyset,\ E \neq \emptyset$

$v_+ =$ max. valency

$v_- =$ min. valency

no isolated vertex [e.g. Γ connected with $\geqslant 2$ vertices]

$$N = \sum_{x\in V} \operatorname{val}(x)$$

(Γ d-regular

$\Rightarrow\ v_+ = v_- = d\ ;\ N = d|V|$)

Def.

① Define $\mu(x) = \dfrac{\mathrm{val}(x)}{N}$
$\underset{V}{\cap}$

[so $\sum \mu(x) = 1$; μ is a probability on V]

② $L^2(\Gamma) = \{f : V \longrightarrow \mathbb{C}\}$

with inner product

$$\langle f_1, f_2 \rangle = \frac{1}{N} \sum_{x \in V} \mathrm{val}(x)\, f_1(x)\, \overline{f_2(x)}$$

[Γ d-regular: $\langle f_1, f_2 \rangle = \dfrac{1}{|V|} \sum_{x \in V} f_1(x)\, \overline{f_2(x)}$]
$\mathrm{val}(x) = d$, $N = d|V|$

③ M_Γ : $\begin{cases} L^2(\Gamma) \longrightarrow L^2(\Gamma) \\ f \longrightarrow M_\Gamma f \end{cases}$
linear

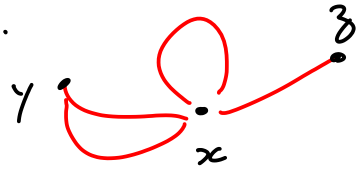
where

$$(M_\Gamma f)(x) = \frac{1}{\mathrm{val}(x)} \sum_{\substack{\text{edges} \\ \text{with } x \\ \text{as extremity}}} f(\text{"other extremity"})$$

nb. edges with x as one extremity

[Γ d-reg. : $\dfrac{1}{d} \sum_{\text{neighbors of } x} f(y)$]

Ex.



$$\frac{1}{4}\left(2f(y)+f(x)+f(z)\right) = Mf(x)$$

Prop. (spectral properties)

(1) M_Γ is self-adjoint

$$\left[\quad \langle Mf_1, f_2\rangle = \langle f_1, Mf_2\rangle \right]$$

(2) $\|M_\Gamma\| \leq 1$

$$\|M_\Gamma\| = \sup_{f\neq 0} \frac{\|Mf\|}{\|f\|}$$

norm as linear map

norms in L^2

$$\|f\| = \sqrt{\langle f,f\rangle}$$

(3) In fact,

$$-\langle f, f\rangle \leq \langle M_\Gamma f, f\rangle \leq \langle f, f\rangle$$

(4) The spectrum $Sp(M_\Gamma)$ of M_Γ is contained in $[-1, 1]$

(5) $1 \in Sp(M_\Gamma)$ has multiplicity equal to the number of connected components of Γ, with basis of the eigenspace given by characteristic functions of those components

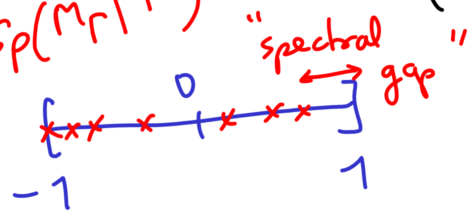
[so if Γ is connected, $\mathbb{1}$ has multiplicity 1, and the function $\mathbb{1}$ generates the eigenspace].

(6) If Γ is connected, then $-1 \in Sp(M_\Gamma) \iff \Gamma$ is bipartite; then for $V = V_0 \sqcup V_1$, a basis of the (-1)-eigenspace is

$$f(x) = \begin{cases} 1 & \text{if } x \in V_0 \\ -1 & \text{if } x \in V_1 \end{cases}$$

Note: Γ connected $\iff M_\Gamma$ has 1 as eigenv. with mult. 1 $\iff Sp(M_\Gamma \,|\, \mathbb{1}^\perp) \subset [-1, 1[$

$Sp(M_\Gamma|1^\perp)$



$\hookrightarrow \{f \in L^2 \,|\, \langle f, 1\rangle = 0\}$

$(M_\Gamma 1 = 1)$

This reminds us of

(Γ connected $\iff h(\Gamma) > 0$)

2.2 - M_Γ and expansion

There are quantitative relations between $h(\Gamma)$ and

The spectral gap:

(1) [Cheeger inequality]

$$\lambda_1(\Gamma) \leq \frac{2v_+}{v_-^2} h(\Gamma)$$

spectral gap

= 1 − (largest eigenvalue of $M_\Gamma | 1^\perp$)

$f \in 1^\perp \iff \langle f, 1\rangle = 0 \iff \frac{1}{N}\sum_x \mathrm{val}(x) f(x) = 0$

(Consequence: if a family (Γ_n) of finite graphs has $\lambda_1(\Gamma_n) \geq c > 0$ for all $n \geq 1$, then it satisfies

$$h(\Gamma_n) \geq \frac{v_-^2 c}{2v_+} > 0$$

for all n).

(2) [Buser inequality]

$$h(\Gamma) \leq v_+ \sqrt{2\lambda_1(\Gamma)}.$$

Cor. (Γ_n) is an expander

family $\iff$

easy $\begin{cases} \text{(i)} \ |\Gamma_n| \longrightarrow +\infty, \\ \text{(ii)} \ \exists k, \forall n, \forall x \in \Gamma_n, \ val(x) \leq k, \end{cases}$

(iii) $\exists \, c > 0, \ \lambda_1(\Gamma) \geq c$.

<u>Note:</u> (2) can be interpreted as a <u>consequence</u> of being an expander.

It's an example of "the expander philosophy": if we have a sequence (Γ_n) of graphs associated to some problem, then <u>if they are expanders</u>, the objects giving rise to the graphs are "<u>maximally complicated</u>".

<u>Ex.</u> One can show for C_m

$$h(C_m) \simeq \frac{1}{m}, \quad \lambda_1(C_m) \simeq \frac{1}{m^2}$$

Sketch of the Cheeger inequality

$$h(\Gamma) = \min_{1 \leq |W| \leq \frac{|V|}{2}} \frac{|\mathcal{E}(W)|}{|W|}$$

$$\lambda_1(\Gamma) = 1 - \text{largest eigenvalue}$$

$$= \text{smallest eigenvalue of } \mathrm{Id} - M_\Gamma \ (\text{on } 1^\perp)$$

lemma about self-adj. on Hilbert spaces

$$\boxed{=} \ \inf_{\substack{f \neq 0 \\ f \perp 1}} \frac{\langle (\mathrm{Id} - M_\Gamma) f, f \rangle}{\langle f, f \rangle}$$

$$= \inf_{\substack{f \neq 0 \\ f \perp 1}} \frac{1}{\langle f, f \rangle} \frac{1}{2N} \sum_{x,y \in V} a(x,y) |f(x) - f(y)|^2$$

nb. of edges joining x to y

Take
$$f = \mathbb{1}_W - \mu(W) \in 1^\perp$$
in the second formula.

Then
$$|f(x) - f(y)|^2 = |\mathbb{1}_W(x) - \mathbb{1}_W(y)|^2$$

so

$$a(x,y)\,|\mathbb{1}_W(x)-\mathbb{1}_W(y)|^2=\begin{cases} a(x,y)\geq 1, & x,y \text{ joined by an edge, one is in } W, \text{ the other in } V-W \\ 0 & \text{otherwise} \end{cases}$$

so

$$\boxed{\langle (Id-M)f, f\rangle = \frac{|\mathcal{E}(W)|}{N}}$$

and dividing by $\langle f, f\rangle$, we get

$$\frac{\langle (Id-M)f, f\rangle}{\langle f, f\rangle} = \frac{|\mathcal{E}(W)|}{N\,\mu(W)\mu(V-W)}.$$

From this, the Cheeger inequality will follow because

$$\inf_{\substack{f\neq 0\\ f\perp 1}} \frac{\langle (Id-M)f, f\rangle}{\langle f, f\rangle} \leq \inf_{\substack{f=\mathbb{1}_W-\mu(W)\\ W\neq\emptyset}} \frac{\langle (Id-M)f, f\rangle}{\langle f, f\rangle}$$

with some computation $\leq \frac{2v_+}{v_-^2}\,h(\Gamma)$