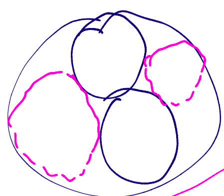
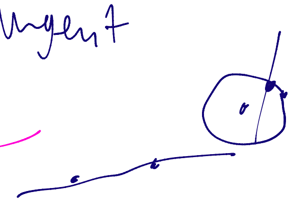
Applications of Expanders "in the Wild"

· "IMO" problem (Apollonius, ~250 BCE): Given 3 objects (points/lines/circles) construct tangent circle circles

Ex 1: Given:  Construct (straightedge+compass)

Ex 2: Given 3 tangent circles, construct further tangent circles



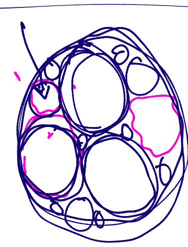


Ex3: Do this in "as few steps" as possible?

← uses of compass or straightedge.

Open: Prove (Baragar–K '20) construction is best possible!!!

Thm (Apollonius): $\exists$ exactly 2 (circle) solutions.

Leibniz ~1700:  ← "Apollonian Circle packing" 1950s...

Fix. P = packing

Natural Q: $\underbrace{N_{P}(x) = \#\{C \in P \mid \overset{\text{radius}}{r(C)} > \tfrac{1}{x}\}}_{\text{growth rate?}}$

Thm (KO '11): $N_{P}(x) = c_{P} \cdot x^{\boxed{\delta}}\left(1 + O(x^{-\varepsilon})\right)$, $\varepsilon > 0$.

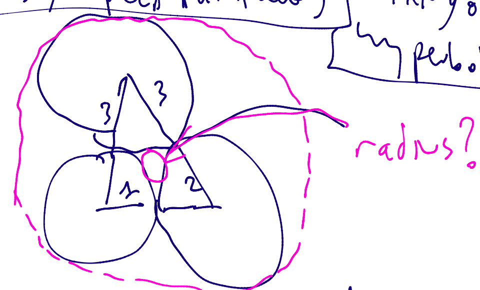
Vinogradov, Lee-Oh, ... (as $x \to \infty$).

$c_{P} > 0$

$1 < \delta = \mathrm{H.dim}(P) < 2$, $\approx \boxed{1.30\ldots}$

∞-volume automorphic forms/representations, homogeneous dynamics, spectral theory | mixing of flows on hyperbolic manifolds

(Soddy 1936): Ex 4:



$\exists$ configurations of P s.t. $\forall C \in P$, $r(C) = \frac{1}{n}$, $n \in \mathbb{Z}$.

Def: "bend" $\delta(C) = \frac{1}{r(C)}$

"curvature"

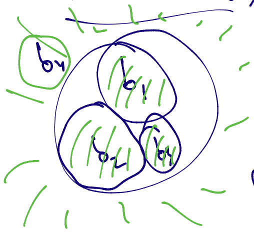
P (P is integral)

Thm (Soddy): $\exists\, P$ with all $\delta(C) \in \mathbb{Z}$.

Consequence Thm (Descartes 1630s) (Correspondence w/

Princess Elizabeth of Bohemia); redo geometry in coords:

Thm: Given 4 tangent circles



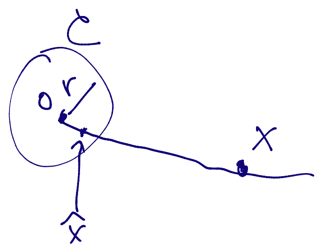
with bends $b_1, \dots, b_4$, $Q(\vec{b}) = 0$,

where $Q(b_1, \dots, b_4) = 2\left(\sum_{i=1}^{4} b_i^2\right) - \left(\sum b_i\right)^2$

"Descartes form"

(Caveat: $b(C) = 0 \iff$ line, $b(C) < 0 \iff$ concave)

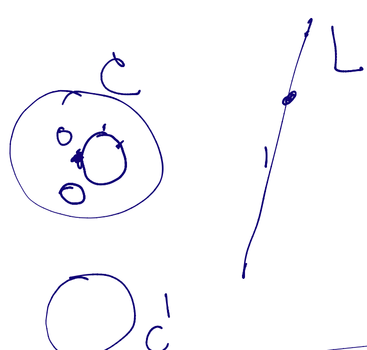
Def: Given C = "mirror", R_C = circle reflection/inversion in C.



$\hat{x} = R_C(x)$

$$\boxed{\frac{xO}{r} = \frac{r}{\hat{x}O}}$$

Fact: (or exercise)



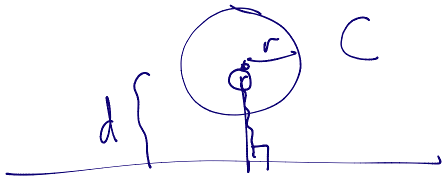
$R_C(L)$ = circle through O

L: line

$R_C(C')$ = circle

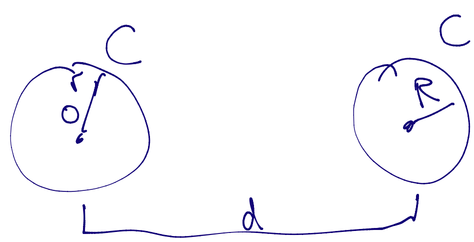
(Right way: think of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$).

Ex 5:



What is radius of $R_C(L)$?

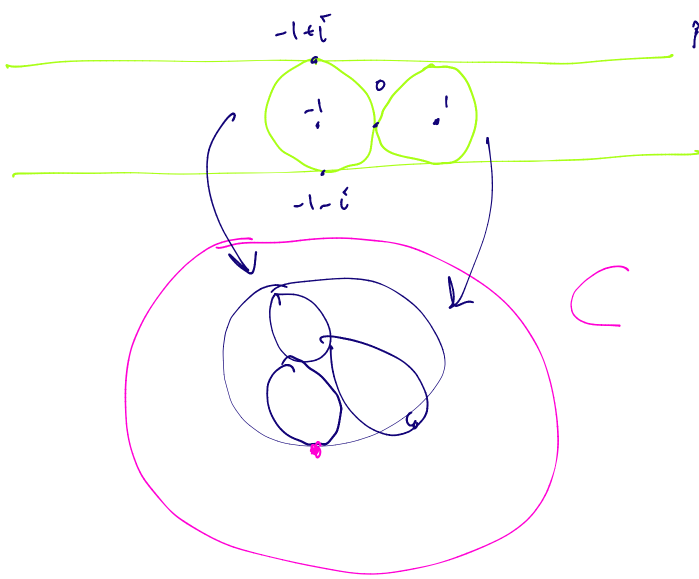
Ex 6:



What is radius of $R_C(C')$.

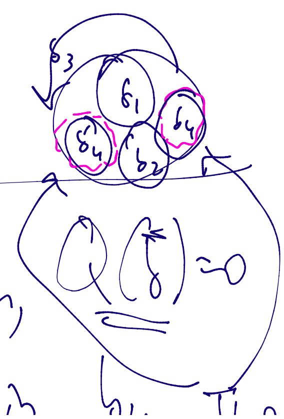
Ex 7: Prove Descartes' Thm

Hint: (Vrete) Given any 4 tangent circles



invert in aux mirror, center at pt of tangency

Do algebra



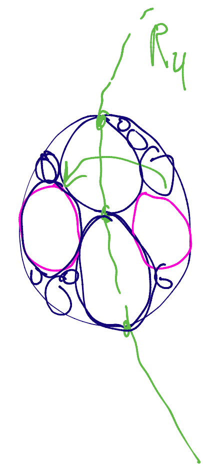
Consequence: If $\delta_1, \delta_2, \delta_3$ given,

becomes a quadratic eqn in δ_4. Two solns.

If solutions are $b_4 + b_4' = 2(b_1+b_2+b_3)$ (Ex 8)

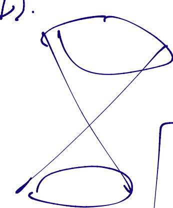
So: If know b_1, b_2, b_3, b_4

$$\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}}_{R_4} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4' \end{pmatrix}$$



$R_4 \in O_Q(\mathbb{Z})$

$$\langle R_1, R_2, R_3, R_4 \rangle = \Gamma$$



Ex 9: $\mathrm{sgn}\, Q = (3,1)$

Pf (so odd b's): If $(b_1,\dots,b_4) \in V(\mathbb{Z})$: $Q = 0$

then all $\gamma \vec{b} \in V(\mathbb{Z})$, $\forall \gamma \in \Gamma$.

where $O_Q = \{ g \in GL_4 \mid Q(\vec{x}) = Q(g\vec{x})\ \forall \vec{x} \}$

$g^t Q g = Q$

Ex 10: Check $R_j \in O_Q$

Now we know $\exists$ integral $\mathcal{P}$ (all $b(C) \in \mathbb{Z}$).

Q2: (Graham–Lagarias–Mallows–Wilks–Yan '03)

Which integers arise?

Let $\mathcal{B}_P = \{ b(C) \mid C \in P \} = \{2, 3, 6, \dots\}$

What is this? **Obs:** $\mathcal{B}_P \pmod{\overset{2^3\cdot 3}{24}} \neq \mathbb{Z}/24$.

There are "local obstructions".

$\boxed{\text{Ex: 11 Prove this from } R_1,\dots,R_4}$

Fix P. $\boxed{\text{primitive}}$ $\boxed{\text{if } d \mid b(C)\ \forall C \in P \Rightarrow d \mid 1.}$

Def: n is admissible (for P)

if $n \in \mathcal{B}_P \pmod q$, $\forall q \geq 1$.

$\boxed{\text{Fuchs: } n \text{ admissible} \iff n \in \mathcal{B}(24)}$

Local-Global Conjecture for Integral Ap Packings:

Fix P. Every suff large admissible arises.

Eg: $n = 24{,}000{,}002$ is admissible. Is $n \in \mathcal{B}$?

Thm (Bourgain–K '14): Almost every ...

almost every admissible

$\mathcal{B}$ represented,

circle method,

expanders

$$\frac{\#\mathcal{B}\cap[1,x]}{\#\{n<x : \text{admissible}\}} \to 1.$$