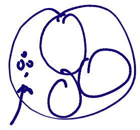
Orbital Circle Method I: Minor Arcs

Recall: Apollonian: $\Gamma = \langle R_1, \dots, R_4 \rangle \curvearrowright$ acting on $\begin{pmatrix} b_1 \\ \vdots \\ b_4 \end{pmatrix}$ of bends

$\mathcal{O} = \Gamma \cdot b_0$, $\quad \mathcal{B} \supset \bigcup_{\gamma \in \Gamma} \underbrace{\langle \gamma \cdot b_0, \vec{e}_1 \rangle}$ in $\mathcal{B}$.



Zaremba: $\Gamma = \langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} : a \le A \rangle^+$, $\quad \mathcal{O} = \Gamma \cdot \vec{e}_1 \subset \mathbb{Z}^2$,

$\{d's\} \supset \bigcup_{\gamma \in \Gamma} \langle \gamma \cdot e_1, e_1 \rangle$, $\quad \boxed{\Gamma_N = \Gamma \cap B_N}$

In either setting write $R_N(n) = \sum_{\gamma \in \Gamma_N} \mathbb{1}_{\{n = \langle \gamma v, w \rangle\}}$

Fix v, w $\quad (n \asymp N)$

If $R_N(n) \neq 0 \Rightarrow n$ is represented!

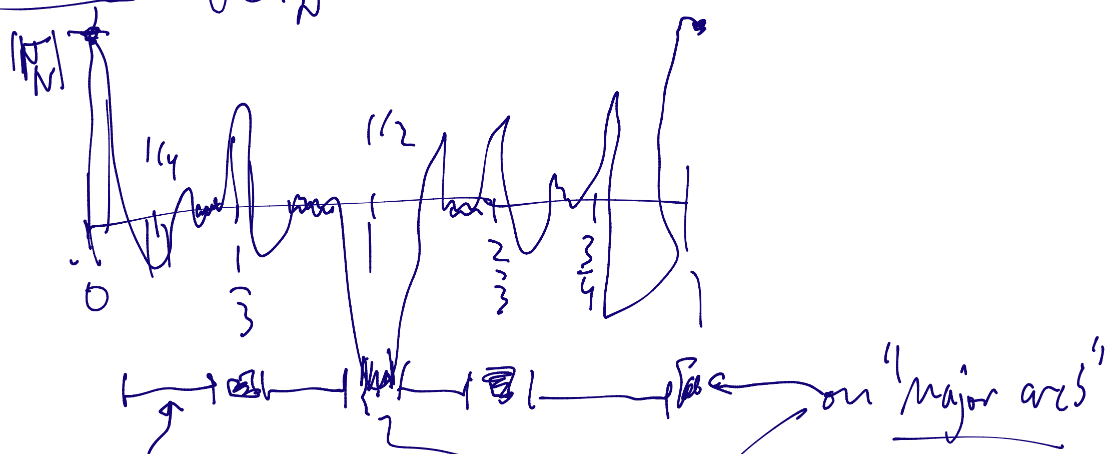
(Not circle method!): $R_N(n) = \int_0^1 \hat{R}_N(\theta)\, e(-n\theta)\, d\theta$,

where $\hat{R}_N(\theta) = \sum_{\gamma \in \Gamma_N} e(\theta \langle \gamma v, w \rangle)$

Ex 1: $\quad$ Ex 2:

$|\hat{R}_N(\theta)|$

$\hat{R}_N(0) = |\Gamma_N|$



Circle "Method": ("catch" main terms & get cancellation elsewhere (minor arcs) most of $\mathbb{R}/\mathbb{Z}$.

Before Major arcs, need "Dirichlet approx":

Ex3: $\forall \theta \in \mathbb{R}$, $\forall M \geq 1$ "depth of approx",

$\exists\, q \leq M$ $\exists\, (r,q)=1$ s.t. $\left|\underbrace{\theta - \frac{r}{q}}_{\beta}\right| \leq \frac{1}{qM}$. $r(\theta,M)$, $q(\theta,M)$.

{Hint/pf: Look at $n\theta \bmod 1$ $\underset{0}{|}$ ~~xxxxxxx~~ $\underset{1}{|}$

"pigeonhole" $n=1,\dots,M$, get M pts, so at least 2 of them are within $\frac{1}{M}$ of each other.}

$|n_1\theta - n_2\theta - k| \leq \frac{1}{M}$. $\boxed{M = N^{1/2},\ Q_0, K_0.}$

Let $\mathcal{M}$ be "major arcs", that is, $\theta = \frac{r}{q} + \beta$, with $q < Q_0 (= N^{\varepsilon})$ & $|\beta| < \frac{K_0}{N}$, $(K_0 = N^{\varepsilon})$.

$$R_N(n) = \int_{\mathbb{R}/\mathbb{Z}} \hat{R}_N(\theta)\, e(-n\theta)\, d\theta = \mathcal{M}_N(n) + \mathcal{E}_N(n),$$

where $\mathcal{M}_N(n) = \int_{\mathcal{M}} \hat{R}_N(\theta)\, e(-n\theta)\, d\theta$

Next time (Expanders!!): for $n \sim N$, admissible: $\mathcal{M}_N(n) \gg \frac{|T_N|}{N} \mathfrak{S}(n)$ (##)

singular series ~1

If we could prove: $|\mathcal{E}_N(n)| = o\left(\frac{|\Gamma_N|}{N}\right)$ $\Rightarrow$ Full

If true: L^2 bound: $\sum_{n<N} |\mathcal{E}_N(n)|^2 = o\left(\frac{|\Gamma_N|^2}{N^2} \cdot N\right)$ (✡)

Apoll/Zaremba (Gorj)

Can prove $\longrightarrow$ ↑

Ex 4: (✡) $\Rightarrow$ Almost all local global.

Hint/pf: $\underbrace{\sum_{\substack{n<N \\ \text{admissible}}} \mathbb{1}_{\{R_N(n)=0\}}}_{} \leq \sum_{\substack{n<N \\ \text{admissible}}} \mathbb{1}_{|M_N(n)| \leq |\mathcal{E}_N(n)|}$

tomorrow

$\rightarrow$ (✡✡) $\leq \sum_{n<N} \mathbb{1}_{\frac{|\Gamma_N|}{N} \ll \cancel{|M_N(n)|} \leq |\mathcal{E}_N(n)|} \leq \frac{\sum_{n<N} |\mathcal{E}_N(n)|^2}{\left(\frac{|\Gamma_N|}{N}\right)^2}$.

$\boxed{\rightarrow}$ (✡) $= o\left(\frac{\cancel{\frac{|\Gamma_N|^2}{N^2}} \cdot N}{\cancel{\left(\frac{|\Gamma_N|^2}{N^2}\right)}}\right) = o(N)$.

Ex 5: $\sum_n |\mathcal{E}_N(n)|^2 = \int_{\mathbb{R}/\mathbb{Z} \setminus \mathcal{M}} |\widehat{R}_N(\theta)|^2 \, d\theta$.

Parseval identity: $\|f\|_{\ell^2} = \|\hat{f}\|_{L^2(\mathbb{R}/\mathbb{Z})}$

What do we need? $|\widehat{R}_N(\theta)| \leq |\Gamma_N| = \widehat{R}_N(0)$

Effectively, need "$|\hat{R}_N(\theta)| \le \frac{|\Gamma_N|}{N^{1/2+}}$" on avg. (minor arc)

In L^∞? i.e. Can we see cancellation individually?

$\theta = \frac{r}{q} + \beta$

modular archimedean

$$\hat{R}_N\left(\frac{r}{q}+\beta\right) = \sum_{\gamma \in \Gamma_N} e\left(\left(\frac{r}{q}+\beta\right)\langle \gamma v, w\rangle\right).$$

don't have "good" Poisson sum formula in groups.

Key Idea: (van der Corput / Vinogradov)

Create bilinear/multilinear forms.

Replace R_N by:

$$R_N(n) = \sum_{\gamma \in \Gamma_{\boxed{N^{1/2}}}} \sum_{\alpha \in \Gamma_{\boxed{N^{1/2}}}} \mathbb{1}\{n = \langle \underbrace{\gamma \cdot \alpha}_{\Gamma} \cdot v, w\rangle\};$$

Claim: $R_N(n) \neq 0 \Rightarrow n$ rep'd.

Rmk: In $\mathbb{Z}$, $B_{\frac{N}{2}} + B_{\frac{N}{2}} \supseteq B_N = [-N, N]$

$$\hat{R}_N(\theta) = \sum_{\gamma \in \Gamma_{N^{1/2}}} \sum_{\alpha \in \Gamma_{N^{1/2}}} e(\theta \langle \cancel{\gamma} \alpha v, \overbrace{\gamma^t w}^{=y}\rangle)$$

(Ex6) $= \sum \sum e(\theta\langle \alpha v, \dots$

$$\sum_{\substack{y\in\mathbb{Z}^2\\ |y|<N^{1/2}}} \underbrace{\sum_{\alpha\in\Gamma_{N^{1/2}}} e(\theta\langle\alpha v, y\rangle)}\ \underbrace{\sum_{\gamma\in\Gamma_{N^{1/2}}} \mathbb{1}_{\{\gamma^t w = y\}}}$$

(Zaremba)

Cauchy-Schwarz: $|R_N(\theta)|^2 \le \sum_{y\in\mathbb{Z}^2}\left(\underbrace{\sum_{\gamma\in\Gamma_{N^{1/2}}}} \mathbb{1}_{y=\gamma^t w}\right)^2 \cdot$

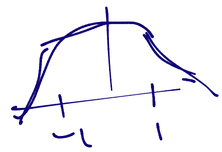
$$\boxed{\left(\sum a_n b_n\right)^2 \le \sum a_n^2 \cdot \sum b_n^2}$$

$$\sum_{\substack{y\in\mathbb{Z}^2\\ |y|<N^{1/2}}} \left|\sum_{\alpha\in\Gamma_{N^{1/2}}} e(\theta\langle\alpha v, y\rangle)\right|^2$$

How often does $\gamma_1^t w = \gamma_2^t w$?

Never. So $\left(\sum\right)^2 = \sum$

$$\frac{1}{7+\frac{1}{5}} = \frac{1}{7+\frac{1}{4+\frac{1}{1}}}.$$



$$\sum_{\alpha_1,\alpha_2\in\Gamma_{N^{1/2}}} \sum_{y\in\mathbb{Z}^2} \Upsilon\left(\frac{y}{N^{1/2}}\right) e\left(\underset{=}{\theta}\langle\underline{(\alpha_1-\alpha_2)v}, y\rangle\right)$$

$\frac{r}{q}+\beta$

Apply Poisson Summation

& work to get cancellation...

HUGE loss from CS replacing $\gamma^t w$, $\gamma\in\Gamma$

By y. (price to pay for good Poisson Summation).

Argument only works (not too expensive) if

$|\Gamma_N| \asymp N^{2\delta_A}$, where $\delta_A = \mathrm{H.dim}\, \mathcal{C}_A$.

$\mathcal{C}_A = \{ [0, a_1, \ldots, a_\ell, \ldots] : a_j \le A \}$.

"uniformly badly approx" numbers. $\boxed{\text{Should be true } \delta_A > 1/2.}$

as $A \to \infty$, $\delta_A \to 1$. $\boxed{\text{If } \delta_A < 1/2, \text{ no chance of local-global}}$

Montgomery-Vaughn: all but $N^{1-\varepsilon}$ evens up to N are $p+q$

Why is $\gamma \mapsto \gamma^t w$ injective?

Ex 6: If $\gamma_1 e_1 = \gamma_2 e_2 \Rightarrow \gamma_1 \gamma_2^{-1} =$ parabolic $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$,

~~Ex 7:~~ Fact: $(\Gamma \cap SL_2)$ has no parabolics.

$$\gamma = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & x \\ \delta & x \end{pmatrix}.$$