A Slice of Pi

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Mathematics

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Abstract

π is one of the longest-studied irrational constants, with one of the first rigorous studies due to Archimedes in the third century BCE when he was able to approximate π with arbitrary accuracy using regular polygons and the method of exhaustion. In more recent history, and in particular thanks to the development of Calculus in the seventeenth century AD, π has been calculated to more than 1.24 trillion digits; however it is still unknown if π is normal in any base. We look at a new approach to the problem of normality which involves chaotic dynamics, as well as possible implications if it were normal. We also discuss known properties of π and their proofs, in particular the properties of transcendency and irrationality. We introduce and examine some of the varied methods for calculating π, including the recent BBP formula which may have a connection to the question of normality. Furthermore, we examine some of these formulas using the Maple software package in order to determine the rates of convergence and, more importantly, the efficiency of these formulas for long approximations of π on modern computers. In conclusion, we present an answer to the question of why we need to compute long approximations of π.

Introduction

This article is the outcome of an independent research project suggested by my professor, Dr. Jan Rychtář. Originally, the goal was to do general research on what is known about π, but the project expanded to include original research on a few of the methods for computing π in order to become familiar with computer algebra systems in preparation for solving mathematics-related biological problems in a math-biology research fellowship. Specifically, this original research was conducted using the Maple software package to compare the convergence and efficiency of various formulas for π.

During the general research, I also found some amusing trivia such as the field of piphilology, which is the study of mnemonic techniques for remembering approximations of π. If one counts the number of letters in each word, the following example will give π to 14 decimals:

"How I need a drink, alcoholic of course, after the heavy lectures involving quantum mechanics."

History of Computing π

Perhaps the first person to be able to compute π with arbitrary precision was Archimedes in the 3rd century BCE. By inscribing a regular polygon inside a circle and by circumscribing a regular polygon with the same number of sides outside the circle, one can set an upper and lower bound for the circumference of the circle, and therefore also an upper and lower bound for π (see Figure 1). By increasing the number of sides, one is able to more closely approximate π. With this method Archimedes was also able to place an upper and lower bound on π, stating that it is between 223/71 and 22/7 (see Eymard and Lafon, 2004).

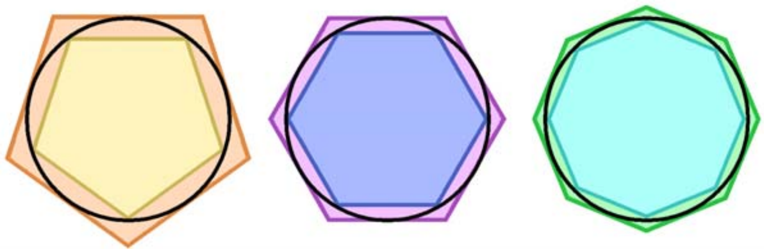

Figure 1

The next leap in computing π came after the development of the Maclaurin series for the $\arctan(x)$ function during the 17th century AD (see Table 1). Although one can use the identity

$$\frac{\pi}{4} = \arctan(1)$$

with the Maclaurin series to compute π with arbitrary precision, the series converges so slowly that it is useless on its own. However, in 1706 John Machin developed a variation on the formula by using the tangent double angle and tangent difference identities (see Table 1 and Baker, 1975, Chapter 1, Theorem 1.4). By taking only five terms of the series for this new formula, one is able to get eight correct digits for π (it converges on average at a rate of roughly 1.4 correct digits per iteration).

In 1996, David Bailey, Peter Borwein, and Simon Plouffe published a new formula for computing digits of π, which they dubbed the BBP formula (see Table 1). This formula can be used to compute any digit of π in hexadecimal base directly, without computing any of the previous digits. There is no known equivalent formula for base 10. For more information on the method of calculation, see Bailey *et. al, 1997*.

Other interesting formulas were developed by Madhava of Sangamagrama, who discovered the formula

$$\pi = 6\arctan\left(\frac{1}{\sqrt{3}}\right)$$

in the 15th century AD. Other Machin-like formulae were developed by L.K. Schulz von Strassnitzky in 1844 and Carl Friedrich Gauss in 1863. Strassnitzky's formula converges to π somewhat slower than Machin's original formula, and Gauss's formula converges remarkably faster.

Properties of π

The number π is known to be transcendental (transcendental numbers are ones which are not a root of a non-zero polynomial with rational coefficients). Here is a short proof that π is transcendental:

The Lindemann-Weierstrass theorem states (see Baker, 1975, http://en.wikipedia.org/wiki/Lindemann-Weierstrass_theorem/): If $\alpha_1,....,\alpha_n$ are distinct algebraic numbers, then their exponentials $\exp(\alpha_1),....,\exp(\alpha_n)$ are linearly independent over the algebraic numbers.

First, recall Euler's formula:

$$\exp(ix) = \cos(x) + i\sin(x).$$

If π were algebraic, i.e. the root of a nontrivial polynomial with rational coefficients, then 2π i would be algebraic too (since 2i is algebraic and the product of two algebraic numbers is an algebraic number, see Herstein, p. 213). Then by the Lindemann-Weierstrass theorem

$$\{\exp(0), \exp(2i\pi)\} = \{1,1\}$$

would be linearly independent over the algebraic numbers. This is a contradiction, so π is transcendental. Furthermore, because all transcendental numbers are also irrational, it follows that π is irrational.

Maclaurin series for arctan(x)	$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$
Madhava	$\pi = 6\arctan\left(\frac{1}{\sqrt{3}}\right) = \sqrt{12}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)3^k}$
Machin	$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$
Gauss	$\frac{\pi}{4} = 12\arctan\left(\frac{1}{18}\right) + 8\arctan\left(\frac{1}{57}\right) - 5\arctan\left(\frac{1}{239}\right)$
Strassnitzky	$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$
BBP	$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$

Table 1: Formulas for π

It is currently unknown if π is a normal number (i.e. if a string of length m occurs with frequency b^{-m} in base b (http://mathworld.wolfram.com/NormalNumber.html/). However, the BBP algorithm may provide an answer to the question of π's normality. In the step immediately before the hexadecimal result, the algorithm yields a result between 0 and 1. The result appears to oscillate erratically, suggesting a link to chaotic dynamics. Bailey and Crandall have hypothesized that if this result is evenly distributed, then π can be said to be normal to base 2, thus transforming the problem of π's normality into one involving chaos theory (see Peterson, 2001 for more information).

Computing π in Maple

One goal of this research was to write a program implementing certain formulas for computing π in the Maple software package in order to compare their convergence and efficiency. The program works on formulas which are based on infinite series (such as the formulas in Table 1), and then it determines the number of correct digits which were computed. The results are shown in the charts and graphs below. The tests were run on a modern-day 3Ghz Pentium 4 computer using Maple v10.

One can see immediately that the Gauss formula clearly converges to π at the fastest rate— in fact, it is almost twice as fast as any other formula shown here. Based on the results in the graph Precision vs. Iterations, we can derive the table shown below. The Gauss formula clearly wins for convergence to π: one only needs to sum from 0 to 398 in order to get 1000 correct digits of π.

Interestingly, even though the BBP formula converges at less than half the rate of the Gauss formula, it is by far the most efficient algorithm considered here in terms of correct digits calculated per second. Note how after 60 seconds it has correctly computed around 68,600 digits of π, whereas the Gauss formula has only computed about 37,600 correct digits in the same time.

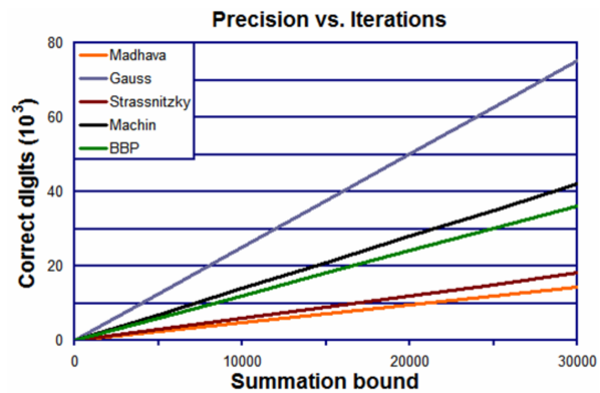

Figure 2

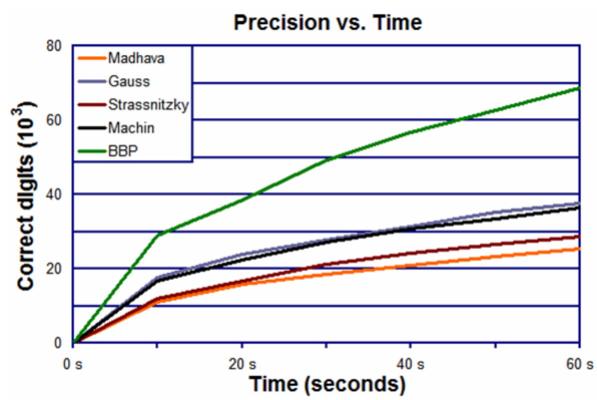

Figure 3

Correct Digits	Madhava sum to…	Gauss sum to…	Strassnitzky sum to…	Machin sum to…	BBP sum to…
1000	2093	398	1659	715	829
2000	4186	796	3318	1430	1659

Table 2: The sum required to reach 1000 and 2000 correct digits

Why Should We Care?

The current world record for computed digits of π is 1.24 trillion digits, set by Yasumasa Kanada of the University of Tokyo in 2002 using several Machin-like formulae (http://www.super-computing.org/pi_current.html). David Bailey pointed out that one only needs about 40 decimals of π to calculate the circumference of something the size of the Milky Way galaxy with an error less than the size of a proton (Bailey *et. al*, 1997). So why should we care to compute so many digits of π?

One possible answer is that these extensive calculations can be used in disclosing computer hardware errors. If two independent systems compute the same digits of π, it is likely that both systems performed the operations flawlessly (Bailey *et. al*, 1997).

Furthermore, if it can be shown that π is a normal number, it may be possible to use π as a pseudo-random number generator for scientific purposes or computer simulations (Bailey *et. al*, 1997).

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Mathematics

Robert Gove is an undergraduate student from the University of North Carolina at Greensboro and will be graduating in May 2009 with Bachelor of Science degrees in Computer Science and Applied Math. He is currently working on a math-biology undergraduate research fellowship funded by the NSF. In April 2007 he was awarded a travel grant for the summer of 2007. His abstract was accepted for NCUR 21 and he gave his poster presentation there in April 2007 on his research. Additionally, in November 2006 he gave an oral presentation at UNCG Regional Undergraduate Mathematics Conference and a poster presentation at the State of North Carolina Undergraduate Research and Creativity Symposium. He is planning to pursue a Masters degree in Computer Science.

Robert's faculty mentor, **Jan Rychtář**, received his Ph.D. in 2004 from University of Alberta, Edmonton, AB, Canada. The topic of the dissertation was Banach spaces and the interplay between analysis and topology. After completing his dissertation, he accepted position at University of North Carolina Greensboro. In addition to his work on Banach spaces, he started several projects on game theory and its applications to biology. He has mentored 6 undergraduate students whose work was presented on various regional, national and international conferences as well as published in peer reviewed journals. He is currently PI of NSF funded grant for training undergraduate students in mathematical and biological sciences and leads a group of 9 undergraduate students working on projects in mathematical biology.