Function Fields in Magma

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Overview

Magma has two extensive packages for number fields and algebraic function fields of transcendence degree one which are to a large part analogous.

In the following an overview is given over

- the representation of function fields,
- the available functionality.

There is also a curve data type. The functionality of curves in Magma is based partly on the function field package and adds also own functionality. This is not discussed in the following (but nevertheless relevant).

Functionality for function fields

Function fields (over perfect base field):

- Creation of function fields and field extensions in various forms, elements, orders, ideals.
- Places, valuations, residue class fields, completions.
- Divisors, differentials, Riemann-Roch spaces, genus, exact constant field, holomorphic differentials, canonical divisor, ...
- Weierstrass places, gap numbers, differentiations.
- Automorphisms and isomorphisms.

Extensions of function fields:

- Characteristic and minimal polynomials, norm and trace, discriminants, differents, decomposition of primes, ramification behavior.

Functionality for function fields

- Modules over Dedekind domains (maximal orders).
- Galois groups of polynomials over $\mathbb{F}_q[t]$ and $\mathbb{Q}[t]$.
- Automorphism groups, subfields and isomorphisms over $\mathbb{Q}(t)$.

Global function fields:

- Places of prescribed degree.
- Unit and S-unit groups, ideal class groups and ray class groups.
- Cartier operator, p-rank of class group, Hasse-Witt invariant.

Class field theory:

- Creation of abelian extensions from divisor groups.
- Conductor, discriminant, genus, degree of exact constant field, decomposition types of places, number of places of degree one.
- Witt ring, cyclic extensions from Witt vectors.

Representation of function fields

Method 1):

This is basically the same as for number fields. Function fields may be created as recursive finite, simple or distributive, extensions of a rational function field.
They play a double role of field and finite field extension.

Method 2):

Since the rational function field is not a canonically determined subfield of a function field and for better formulation of mathematical context (e.g. a better fit with the curve type), there are also function fields viewed as infinite extensions over the constant field.

Their creation is possible via curves (or the defining multivariate polynomials).

Representation of function fields

For algorithmic purposes, a representation as finite extension of a rational function field is required. Conversion functions between the two versions of function fields are available.

For finite extensions, orders are available like in the number field case. There are no extra visible fields of fractions of orders.

Embeddings and field operations like compositum are not yet as comprehensive as in the number field case.

Ex 1

Places and divisors

In Magma, places of a function field may be created

- by decomposing a place of a base function field,
- by computing (common) zeros and poles of functions, or by computing the support of principal divisors,
- by computing all places of a prescribed degree in the global case.

A place defines a valuation, a residue class field and a completion.

Divisors can be obtained

- by adding multiples of places.
- by computing divisors of functions,
- via special commands (e.g. canonical divisors).

Ex 2.1

Riemann-Roch spaces

Given a divisor D of the function field F/k, Magma can compute the k-vector space $L(D) = \{a \in F^\times \mid (a) + D \geq 0\} \cup \{0\}$, by providing a basis.

This yields

- a test whether a divisor is principal, or whether two divisors are linearly equivalent.
- the exact constant field via $L(0)$.
- the genus.

Ex 2.2

Global function fields

Global function fields are specially supported.

Magma can compute

- all places of a prescribed degree, divisors of degree one,
- divisors class groups by giving an isomorphism to an abstract abelian group, S-units similarly,
- Hasse-Witt invariants,
- ...

The product representation of elements is a particular useful feature here.

Ex 3

Class field theory

Magma enables a number of class field computations for global function fields,
such as:

- Creation of abelian extensions from divisor groups.
- Conductor, discriminant, genus, degree of exact constant field, decomposition types of places, number of places of degree one.
- ...

The computations for the last line only use information coded in the defining divisor class group.

The creation of explicit equations is (much) more involved.

Ex 4

Differentials, Weierstrass places

Differentials are simply expressed in the form $f dx$ for $f, x \in F$ and x separating.

Magma can compute

- Divisors, valuations, residues,
- differential spaces $\{\omega \,|\, (\omega) \geq D\}$.

Weierstrass places can also be determined. (A Weierstrass place of degree > 1 consists of a Galois orbit of Weierstrass points in the usual sense defined over the normal closure of its residue class field.)

Ex 5

Isomorphisms and Automorphisms

This is about isomorphisms of function fields, not finite extensions of function fields (which Magma can also handle) !

Isomorphisms and automorphisms are represented by their images on field generators.

Magma can compute

- List of all isomorphisms and automorphisms
- Group of automorphisms
- Representation on Riemann-Roch spaces or spaces of differentials

Ex 6