Lecture 1

Function Fields, Curves and Global sections

Summer School UNCG 2016

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Function Fields vs. Curves

Function fields vs. regular complete curves:

▶ Essentially boil down to the same thing - there is an equivalence of categories.

▶ If base field is $\mathbb{C}$ then there is another equivalence of categories, to compact Riemann surfaces and covering maps.

▶ So using one term over the other is more a sociological question about one's mathematical genesis or point of view ...

▶ Best to know all three ...

Curves can also be singular, this gives some added ways of expressing matters.

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Function Fields

Let K be a field. An algebraic function field of one variable is a field extension F/K of transcendence degree one.

This means that there is $x \in F$ such that x is transcendental over K and $F/K(x)$ is finite.

The exact constant field of F/K is the algebraic closure K' of K in F.

The extension F/K' is also an algebraic function field of one variable, the x from above is still transcendental over K' and $F/K'(x)$ is finite.

In theory one can always assume w.l.o.g. that $K' = K$. In practice one can not or should not.

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Separating Elements

The element x is called separating for F/K if $F/K(x)$ is
separable. It is a theorem that if K is perfect then there is
always a separating element for F/K.

Fields of characteristic zero, finite fields and algebraically closed
fields are perfect. Any algebraic extension field of a perfect
field is perfect.

Example. The polynomial $y^2 + x^2 + t \in \mathbb{F}_2(t,x)[y]$ is
irreducible and purely inseparable. Thus

$$F = \mathbb{F}_2(t,x)[y]/\langle y^2 + x^2 + t \rangle$$

is a purely inseparable field extension of degree two of $\mathbb{F}_2(t,x)$.
Then $F/\mathbb{F}_2(t)$ is an algebraic function field without a
separating element.

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Local rings and Points

We give a "function field" based approach to curves in the spirit of Hartshorne I.6, including singular curves.

Let F/K be an algebraic function field. A subring of F/K is a proper subring $\mathcal{O}$ of F with $K^\times \subseteq \mathcal{O}^\times$ and $\mathrm{Quot}(\mathcal{O}) = F$.

If $\mathcal{O}$ is subring of F/K and a local ring with maximal ideal $\mathfrak{m}$ we call it a point P of F/K with local ring $\mathcal{O}_P = \mathcal{O}$ and maximal ideal $\mathfrak{m}_P = \mathfrak{m}$.

A place of F/K is regarded as point of F/K.

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Domination

Let P und Q be points of F/K. We say that P is dominated by Q if $\mathcal{O}_P \subseteq \mathcal{O}_Q$ and $\mathfrak{m}_P \subseteq \mathfrak{m}_Q$ holds.

We define supp(P) as the set of places Q of F/K such that P is dominated by Q.

Theorem. The sets supp(P) are non-empty and finite. The residue class fields $\mathcal{O}_P/\mathfrak{m}_P$ are finite over K.

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Sets of Points and Curves

We will only consider sets U of points of F/K that are

- admissible, i.e. almost all points of U are places.
- separated, i.e. for every place Q of F/K there is at most one $P \in U$ such that P is dominated by Q.

Let U^c denote the set of places of F/K that are not contained in $\cup_{P \in U} \mathrm{supp}(P)$. Then U is called cofinite, complete, and affine if U^c is finite, empty and non-empty respectively.

A curve C over K is an admissible separated cofinite set of points of F/K.

The function field of C is $K(C) = F$.

A point $P \in C$ is regular if P is a place, otherwise singular. The curve is regular if all points of C are regular.

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Subrings

Let $P \in C$ and $U \subseteq C$. We define $\mathcal{O}_{C,P} = \mathcal{O}_P$ and

$$\mathcal{O}_C(U) = \cap_{P \in U} \mathcal{O}_{C,P},$$

where the empty intersection is defined as F.

Theorem. Suppose U is affine.

1. The rings $\mathcal{O}_C(U)$ are subrings of F/K and the maps

   $$P \mapsto \mathcal{O}_C(U) \cap \mathfrak{m}_P \text{ and } \mathfrak{m} \mapsto \mathcal{O}_C(U)_{\mathfrak{m}}$$

   give mutually inverse bijections from U to the set of non-zero maximal ideals of $\mathcal{O}_C(U)$.

2. Every point in U is regular if and only if $\mathcal{O}_C(U)$ is a Dedekind domain.

3. With $D_U(f) = \{P \in U \mid f \notin \mathfrak{m}_P\}$ for $f \in \mathcal{O}_C(U)$,

   $$\mathcal{O}_C(D_U(f)) = \mathcal{O}_C(U)[f^{-1}].$$

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Affine Curves

If R is a subring of F/K we define $\mathrm{Specm}(R)$ to be the set of points of F/K defined by $R_{\mathfrak{m}}$ where $\mathfrak{m}$ ranges over the maximal ideals of R.

Theorem. The map $C \mapsto \mathcal{O}_C(C)$ gives an inclusion-reversing bijection of the set of *affine curves* C over K with $K(C) = F$ to the set of subrings R of F/K that are *finitely generated* K-algebras. Its inverse is given by $R \mapsto \mathrm{Specm}(R)$.

This provides the link to the usual definition of affine curves.

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Curves as Topological Spaces*

Let C be a curve over K. A subset U of C is called open if U is empty or $C \setminus U$ is finite.

Theorem. Let C be a curve over K.

1. Then C with its open sets is a topological space.

2. Moreover, it is an irreducible, one-dimensional T_1-space and any open subset of C is quasicompact.

3. If C is affine the sets $D_C(f)$ form a basis of the open sets of C.

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Morphisms

Let X and Y be curves over K. A morphism $\phi : X \to Y$ is defined by a K-algebra monomorphism $\phi^\# : K(Y) \to K(X)$ such that $\phi^\#$ restricts for each $P \in X$ to

$$\phi_P^\# : \mathcal{O}_{Y, \phi(P)} \to \mathcal{O}_{X, P}.$$

Then $\phi(P) = (\phi_P^\#)^{-1}(P)$, and if $U \subseteq Y$ we obtain by further restriction

$$\phi^\#(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U)).$$

The degree of ϕ is $\deg(\phi) = [K(X) : \phi^\#(K(Y))]$.

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Properties

Theorem.

1. ϕ has finite fibres and is continous.

2. If X is complete then Y is complete and $\mathcal{O}_X(\phi^{-1}(U))$ is finite over $\mathcal{O}_Y(U)$.

3. If $P \in X$ is regular and Y is complete, then any morphism $X \setminus \{P\} \to Y$ can be uniquely extended to a morphism $X \to Y$.

4. The map $\phi \mapsto \phi^{\#}$ gives a bijection of the sets of morphismus $X \to Y$ of regular complete curves and of K-algebra monomorphisms $K(Y) \to K(X)$.

If $\phi : X \to Y$ is a morphism, one says that ϕ is separable or that ϕ is ramified over $Q \in Y$ etc., if the corresponding properties hold for the extension $K(X)/\phi^{\#}(K(Y))$ and involved places.

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Example

Let F/K be the rational function field over K.

We define $\mathbb{A}^1$ as the set of places of F/K corresponding to the maximal ideals of $K[x]$, where x is a generator of F/K. This is a regular affine curve.

We define $\mathbb{P}^1$ as the set of places of F/K. This is a regular complete curve.

There is a bijection between the set of generators of F/K and the set of morphisms $\mathbb{A}^1 \to \mathbb{P}^1$ of degree one.

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Normalisation

Let C be curve over K.

The normalisation $\tilde{C}$ of C is the set of places of $K(C)$ that dominate points of C.

There is a morphism $\phi : \tilde{C} \to C$ of degree one, mapping each place to the point of C that it dominates.

The normalisation $\tilde{C}$ of C is a regular curve. If C is complete then $\tilde{C}$ is also complete.

$\mathcal{O}_{\tilde{C}}(\phi^{-1}(U))$ is the integral closure of $\mathcal{O}_C(U)$ in $K(C)$.

Normalisation is thus also desingularisation!

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Representation and Definition of Function Fields and Curves

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General Idea

Task: Represent

- irreducible complete regular curve C over a field K, with
- morphism $\phi : C \to \mathbb{P}^1$ of degree n.

This can be done using $K[x]$-algebras that are finitely generated, free modules over $K[x]$ of rank n, called $K[x]$-orders.

Advantages and disadvantages:

- Linear algebra over $K[x]$ vs. Gröbner basis computations.
- Many existing algorithms from algebraic number theory, e.g. normalisation, ideal arithmetic, valuations, residue class fields, different etc.

There are of course other approaches and points of view (projective, geometric, Khuri-Makdisi).

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Representation using orders

We embed $K(\mathbb{P}^1)$ via ϕ^* into $K(C)$ and choose $x \in K(\mathbb{P}^1)$ to correspond to ϕ. The pole of x in $\mathbb{P}^1$ is denoted by ∞.

Thus have function field $K(C)/K$ and field extension $K(C)/K(x)$ of degree n.

Cover $\mathbb{P}^1$ by two affine open subsets U_0, U_∞ isomorphic to $\mathbb{A}^1$ with $\mathcal{O}_{\mathbb{P}^1}(U_0) = K[x]$ and $\mathcal{O}_{\mathbb{P}^1}(U_\infty) = K[1/x]$.

Then $V_0 = \phi^{-1}(U_0)$ and $V_\infty = \phi^{-1}(U_\infty)$ are open affines that cover C.

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Representation using orders

Write $R_0 = \mathcal{O}_C(V_0)$ and $R_\infty = \mathcal{O}_C(V_\infty)$.

We know that R_0 is finite over $K[x] = \mathcal{O}_{\mathbb{P}^1}(U_0)$ and R_∞ is finite over $K[1/x] = \mathcal{O}_{\mathbb{P}^1}(U_\infty)$.

Thus R_0 and R_∞ are $K[x]$- and $K[1/x]$-orders of rank n.

We can fix bases of R_0 and R_∞ of length n whose relation ideals are generated by quadratic polynomials (and form a Gröbner basis).

These bases are related by a transformation matrix in $K(x)^{n \times n}$, which describes the overlap (glueing) of V_0 and V_∞.

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Definition Via Affine Curve

How do we explicitly define such a C as above?

Start with

- irreducible affine algebraic curve C_0 over a field K,
- a finite map $\alpha_0 : C_0 \to \mathbb{A}^1$.

Then complete and normalise!

Representation of C_0:

- Coordinate ring R_0 of C_0 as quotient of polynomial ring by suitable ideal such that R_0 is $K[x]$-order.
- Often $\alpha_i = y^i$ with $f(x, y) = 0$ and f irreducible, monic and of degree n in y.

Example. $f(x, y) = y^2 - x^7 + 1$.

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Completion Step

Complete as follows:

- ▶ Divide generators of R_0 by suitable powers of x such that they become integral over $K[1/x]$ and hence resulting relations are also defined over $K[1/x]$.
- ▶ Results in $K[1/x]$-order R_∞.
- ▶ Then have $C_0 = \text{Specm}(R_0)$, $C_\infty = \text{Specm}(R_\infty)$ and $\alpha_0 : C_0 \to \mathbb{A}^1$, $\alpha_\infty : C_\infty \to \mathbb{A}^1$.
- ▶ Since R_0 is integral over $K[x]$, every zero of x dominates a maximal ideal of R_0.
- ▶ Since R_∞ is integral over $K[1/x]$, every pole of x dominates a maximal ideal of R_∞.
- ▶ This combines (glues) to a complete curve $C_{0,\infty} = C_0 \cup C_\infty$ and morphism $\alpha : C_{0,\infty} \to \mathbb{P}^1$.

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Completion Step

Example.

- $C_0 : y^2 = x^7 - 1$.
- y/x^4 is integral over $\mathbb{Q}[1/x]$: $(y/x^4)^2 = 1/x - (1/x)^8$.
- Thus $R_0 = K[x, y]$, $R_\infty = K[1/x, y/x^4]$, and
- $C_{0,\infty} = \mathrm{Specm}(R_0) \cup \mathrm{Specm}(R_\infty)$.
- Is regular in characteristic $\neq 2, 7$.

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Normalisation Step

Normalise and hence desingularise $C_{0,\infty}$ as follows:

- Compute $\tilde{R}_0 = \mathrm{Cl}(R_0, K(C_0))$, $\tilde{R}_\infty = \mathrm{Cl}(R_\infty, K(C_0))$.
- The normalisations of C_0 and C_∞ are $\tilde{C}_0 = \mathrm{Specm}(\tilde{R}_0)$ and $\tilde{C}_\infty = \mathrm{Specm}(\tilde{R}_\infty)$.
- Define $C = \tilde{C}_0 \cup \tilde{C}_\infty$. This gives the regular complete curve C and the normalisation morphism $\beta : C \to C_{0,\infty}$.
- Composing yields the morphism $\phi = \alpha \circ \beta : C \to \mathbb{P}^1$.

Data to be stored: Defining relations for R_0, transformation matrices between bases of R_0 and R_∞, between bases of $\tilde{R}_0$ and R_0, and between bases of $\tilde{R}_\infty$ and R_∞. These matrices are in $K(x)^{n \times n}$ or even $K[x]^{n \times n}$.

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Normalisation Algorithms

There are various normalisation and desingularisation algorithms. Some require α to be separable, K to be perfect or even char$(K) = 0$.

Some references:

- Zassenhaus (Round2, Round4)
- Grauert-Remmert (Decker, ...)
- van Hoeij
- Montes-Nart
- Chistov: Polynomial time equivalent to factoring discriminant of f.

Recent activity:

- J. Bauch: Computation of Integral Bases, 2015.
- Singular Group at Kaiserslautern, 2015.
- What is when the fastest method?

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Magma

Let ∞ denote the pole of x in $\mathbb{P}^1$ and $\mathcal{O}_\infty$ the local ring of ∞.

In Magma and its function field package,

- R_0 and $R_\infty\mathcal{O}_\infty$ are called finite and infinite (equation) orders, $\tilde{R}_0$ and $\tilde{R}_\infty\mathcal{O}_\infty$ are called finite and infinite maximal orders.
- Places are uniquely represented as maximal ideals in the maximal orders, by explicit generators.
- The poles of x are called places at infinity.
- A host of algorithms from algebraic number theory is quasi readily available, e.g. integral closures, valuations, residue class fields.

These objects are more considered of internal type. One can work with places rather like in Stichtenoth, without knowing those background details.

There is a curve data type in Magma, but it is different from (although equivalent to) that presented here.

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Global Sections,
Riemann-Roch and an Application

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Outline

Start with function field F/K and divisor D of F/K.

Compute the K-vector space

$$L(D) = \{f \in F^{\times} \,|\, \text{div}(f) \geq -D\} \cup \{0\}$$

of global sections of D!

Approaches are based on:

▶ Curves and Brill-Noether method of adjoints

▶ Integral closures and series expansions

▶ Sheaves and Grothendiecks theorem

Recent activity:

▶ J. Bauch: Lattices over Polynomial Rings and Applications to Function Fields, 2014.

▶ I. Stenger: Computing Riemann-Roch Spaces - a geometric approach, 2014.

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Sheaves

Let C be a curve over K with function field F.

Let M an F-vector space and $\mathcal{F}_P$ submodules of the $\mathcal{O}_{C,P}$-modules M such that $F\mathcal{F}_p = M$ for all $P \in C$ and each $f \in M$ is contained in almost all $\mathcal{F}_P$. Define

$$\mathcal{F}(U) = \cap_{P \in U}\mathcal{F}_P,$$

where the empty intersection is defined as M.

Each $\mathcal{F}(U)$ is a torsion-free $\mathcal{O}_C(U)$-module and $\mathcal{F}$ is called a sheaf of locally torsion-free $\mathcal{O}_C$-modules.

The elements of $\mathcal{F}(U)$ are called sections over U, and global sections when $U = C$.

Example. $\mathcal{O}_C$ is such a sheaf, or better a sheaf of rings, and is called structure sheaf of C.

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Sheaves

Theorem. Let $\mathcal{F}$ be a sheaf of locally torsion-free $\mathcal{O}_C$-modules.

1. We have

$$\mathcal{F}(U) \subseteq \mathcal{F}(V) \quad \text{and} \quad \mathcal{F}(U) = \cap_{i \in I} \mathcal{F}(U_i)$$

   for $V \subseteq U$ and for any family $(U_i)_{i \in I}$ with $U = \cup_{i \in I} U_i$.

2. For all $U \subseteq C$ affine, $P \in U$ and $\mathfrak{m}$ the corresponding maximal ideal of $\mathcal{O}_C(U)$,

$$\mathcal{F}(U)_{\mathfrak{m}} = \mathcal{F}_P.$$

3. For all $U \subseteq C$ affine and $f \in \mathcal{O}_C(U)$,

$$\mathcal{F}(D_U(f)) = \mathcal{F}(U)[f^{-1}].$$

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Sheaves

A sheaf $\mathcal{F}$ of locally torsion-free $\mathcal{O}_C$-modules is said to be locally finitely generated if all $\mathcal{F}_P$ are finitely generated and if each basis of M is also a basis of $\mathcal{F}_P$ for almost all $P \in C$.

Theorem. Let $\mathcal{F}$ be a sheaf of locally torsion-free and finitely generated $\mathcal{O}_C$-modules. Then each $\mathcal{F}(U)$ for U affine is finitely generated.

Example. The structure sheaf $\mathcal{O}_C$ is locally torsion-free and finitely generated.

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Sheaf of a divisor

Let C denote a regular complete curve with function field F and D a divisor of C resp. F/K.

The sheaf $\mathcal{O}_C(D)$ associated to D is defined by

$$\mathcal{O}_C(D)(U) = \{f \in F^\times \mid v_P(f) \geq v_P(-D) \text{ for all } P \in U\} \cup \{0\}.$$

It is a locally torsion-free and finitely generated sheaf of $\mathcal{O}_C$-modules with

$$L(D) = \mathcal{O}_C(D)(C).$$

In other words, the $\mathcal{O}_C(D)(U)$ are non-zero fractional ideals of the Dedekind domains $\mathcal{O}_C(U)$.

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Representation using two free modules

Since V_0 and V_∞ are an open affine cover of C, the sheaf $\mathcal{F}$ can be represented by the torsion-free finitely generated modules $\mathcal{F}(V_0)$ of R_0 and $\mathcal{F}(V_\infty)$ of R_∞ and

$$\mathcal{F}(C) = \mathcal{F}(V_0) \cap \mathcal{F}(V_\infty).$$

The modules $\mathcal{F}(V_0)$ and $\mathcal{F}(V_\infty)$ are also torsion-free and finitely generated $K[x]$- and $K[1/x]$-modules and thus are free of rank $n \dim_F(M)$ inside the $K(x)$-vector space M of dimension $n \dim_F(M)$. They can thus be explicity described by their bases.

To compute the intersection we need to find all $f \in M$ which can be written as a $K[x]$-linear combination of the basis of $\mathcal{F}(V_0)$ and as a $K[1/x]$-linear combination of the basis of $\mathcal{F}(V_\infty)$ *simultaneously*.

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Diagonalisation

The key proposition is as follows:

Proposition. Let $A \in \mathrm{GL}(n, K[x, 1/x])$. Then there are $S \in \mathrm{GL}(n, K[x])$ and $T \in \mathrm{GL}(n, K[1/x])$ such that

$$TAS = (x^{d_i}\delta_{i,j})_{i,j}$$

with $d_1 \geq \cdots \geq d_n$ uniquely determined.

The proof essentially uses

- ▶ matrix reduction (Dedekind-Weber, weak Popov form, lattice reduction in function fields),
- ▶ or Birkhoff's matrix decomposition.

Thus need to find $\lambda \in K[x]$ such that $x^{-d}\lambda \in K[1/x]$. These are precisely the $\lambda \in K[x]$ with $\deg(\lambda) \leq d$.

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Global Sections

Denote by $\mathcal{F}(r)$ the sheaf defined by

$$\mathcal{F}(r)(V_0) = \mathcal{F}(V_0) \ \text{ and } \ \mathcal{F}(r)(V_\infty \setminus V_0) = x^r \cdot \mathcal{F}(V_\infty \setminus V_0).$$

Theorem. Recall $n = [K(C) : K(x)]$. There exist $K(x)$-linearly independent $f_1, \ldots, f_n \in M$ and uniquely determined $d_1 \geq \cdots \geq d_n$ such that for all r:

$$\mathcal{F}(r)(C) = \left\{ \sum_{i=1}^{n} \lambda_i f_i \,\middle|\, \lambda_i \in K[x] \text{ and } \deg(\lambda_i) \leq d_i + r \right\}.$$

Moreover,

- the $f_1, \ldots, f_n$ are a $K[x]$-basis of $\mathcal{F}(V_0)$ and
- the $x^{d_1} f_1, \ldots, x^{d_n} f_n$ are a $K[1/x]$-basis of $\mathcal{F}(V_\infty)$.

These bases are called reduced.

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Push Forward of a Sheaf[*]

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Let $\phi : X \to Y$ be a morphism of the curves X and Y, and $\mathcal{F}$ a locally torsion-free sheaf of $\mathcal{O}_X$-modules.

We define the push forward $\phi_*(\mathcal{F})$ of $\mathcal{F}$ along ϕ via

$$\phi_*(\mathcal{F})(U) = \mathcal{F}(\phi^{-1}(U))$$

for any $U \subseteq Y$.

Theorem. Then $\phi_*(\mathcal{F})$ is a locally torsion-free sheaf of $\mathcal{O}_Y$-modules. If X is complete and $\mathcal{F}$ is finitely generated, then $\phi_*(\mathcal{F})$ is also finitely generated.

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Isomorphisms of Sheaves[*]

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_C$-modules inside the F-vector spaces M and N respectively.

A morphism $f : \mathcal{F} \to \mathcal{G}$ is given by an F-linear map $M \to N$ that restricts to $\mathcal{O}_{X,P}$-module homomorphisms

$$f_P : \mathcal{F}_P \to \mathcal{G}_P.$$

It then also restricts to $\mathcal{O}_X(U)$-module homomorphisms

$$f(U) : \mathcal{F}(U) \to \mathcal{G}(U).$$

We say f is an isomorphism if all f_P are isomorphisms. Then all $f(U)$ are also isomorphisms.

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Direct Sum of Sheaves[*]

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_C$-modules inside the F-vector spaces M and N respectively.

We define $\mathcal{F} \oplus \mathcal{G}$ as the sheaf of $\mathcal{O}_C$-modules inside $M \oplus N$ defined by

$$(\mathcal{F} \oplus \mathcal{G})_P = \mathcal{F}_P \oplus \mathcal{G}_P$$

for all $P \in C$. Then also

$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

for all $U \subseteq C$.

If $\mathcal{F}$ and $\mathcal{G}$ are locally torsion-free then $\mathcal{F} \oplus \mathcal{G}$ is locally torsion-free. If in addition $\mathcal{F}$ and $\mathcal{G}$ are locally finitely generated then $\mathcal{F} \oplus \mathcal{G}$ is locally finitely generated.

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Grothendiecks Theorem[*]

Let C be complete and $\phi : C \to \mathbb{P}^1$ a morphism of degree n.
Let $\mathcal{F}$ be a locally torsion-free and finitely generated sheaf of
$\mathcal{O}_C$-modules.

Grothendieck's Theorem:

$$\phi_*(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^1}(d_1\infty) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n\infty)$$

with $d_1 \geq \cdots \geq d_n$ uniquely determined.

We have indeed computed $\mathcal{F}(C)$ via

$$\begin{aligned}
\mathcal{F}(C) &= \phi_*(\mathcal{F})(\mathbb{P}^1) \\
&\cong \mathcal{O}_{\mathbb{P}^1}(d_1\infty)(\mathbb{P}^1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n\infty)(\mathbb{P}^1) \ !
\end{aligned}$$

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Relation to Riemann-Roch

$\mathcal{O}_C(C)$ is the algebraic closure of K in C. Suppose $K = \mathcal{O}_C(C)$ and let g denote the genus of C.

Let $\mathcal{F} = \mathcal{O}_C(D)$. The numbers d_i satisfy

- $d_1 \geq \cdots \geq d_n$.
- $\sum_{i=1}^{n} d_i = \deg(D) + 1 - g - n$
- $L(D) \neq 0$ iff $d_1 \geq 0$.
- $\deg(D) \geq d_1 \gtrsim (\deg(D) - g)/n$,
- D non-special implies $d_n \geq 0$.
- $d_n \gtrsim (\deg(D) - 2g)/n$.
- $d_1 - d_n \lesssim 2g/n$.
- $\mathcal{O}_C(C) = L(0)$.

The $d_1, \ldots, d_n$ are thus balanced.
If $D = 0$ then g can be computed from $d_1, \ldots, d_n$.

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Application: Special Models

When applied to $\mathcal{I} = \mathcal{O}_C$ the theorem yields

- a specific representation of C and
- also gives an embedding of C in a weighted n-dimensional projective space, depending on ϕ.
- The weights are given by the $-d_i$.

Example. $C : y^2 = zx^7 - z^8$ over $\mathbb{Q}$ where $w(x) = w(z) = 1$ and $w(y) = 4$, is regular.

The affine ring R_0 of C is generated by x and n additional variables. Relations are at most quadratic in these variables and of degree $O(g/n)$ in x.

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Gonality

In practice rather sensitive to n.

Thus

- minimize n, find ϕ of lowest degree. But in general $n = \Theta(g)$.
- substitute variables by powers of others, if possible.

Recent activity:

J. Schicho and D. Sevilla: Effective radical parametrization of trigonal curves, 2011.

M. C. Harrison: Explicit solution by radicals, gonal maps and plane models of algebraic curves of genus 5 or 6, 2013.

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Magma and other Implementations

Probably not exhaustive ...

Global sections:

- ▶ via Grothendiecks theorem: Magma
- ▶ via saturation of homogenous ideals: Magma,
  MacCaulay2, Singular.

Maps of minimal degree:

- ▶ via Schicho and Sevilla: Magma
- ▶ via Harrison: Magma

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Excercises

1. Compute a complete regular curve C in the sense of these slides with function field $\mathbb{Q}(x, y)$, where $y^7 - y^2 = x^2$, and show by the approach presented here that the genus of C is 2.

2. Suppose C is a regular curve and let $U \subseteq C$ be finite. Show that $\mathcal{O}_C(U)$ is a principal ideal domain.

3. Suppose C is a regular curve and let $U \subseteq C$ be affine. Show that every fractional ideal of $\mathcal{O}_C(U)$ can be generated by two elements of $K(C)$.

4. Find a complete curve C over K where $\mathcal{O}_C(C) \neq K$. Verify the latter using Magma.

5. Find a curve C over some K such that there is no separable morphism $C \to \mathbb{P}^1$.

6. Provide examples that in the relation of domination the cases $\mathcal{O}_P^\times \subsetneq \mathcal{O}_Q^\times$ and $\mathfrak{m}_P = \mathfrak{m}_Q$ as well as $\mathcal{O}_P^\times = \mathcal{O}_Q^\times$ and $\mathfrak{m}_P \subsetneq \mathfrak{m}_Q$ can indeed occur.

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Excercises[*]

For the following excercises let C be a complete curve over K.

7. Show that there is a morphism $C \to \mathbb{P}^1$ and a non-zero $K(\mathbb{P}^1)$-linear map $K(C) \to K(\mathbb{P}^1)$.

8. Show that for every sheaf $\mathcal{F}$ of locally torsion-free and finitely generated $\mathcal{O}_C$-modules there is a sheaf $\mathcal{F}^{\#}$ of locally torsion-free and finitely generated $\mathcal{O}_C$-modules such that if $\phi_*(\mathcal{F}) \cong \oplus_i \mathcal{O}_{\mathbb{P}^1}(d_i)$ then $\phi_*(\mathcal{F}^{\#}) \cong \oplus_i \mathcal{O}_{\mathbb{P}^1}(-d_i)$.

9. In the situation of excerise 8 show there is a sheaf $\mathcal{F}^*$ of locally torsion-free and finitely generated $\mathcal{O}_C$-modules such that $\phi_*(\mathcal{F}^*) \cong \oplus_i \mathcal{O}_{\mathbb{P}^1}(-d_i - 2)$.

10. Adapt matters if necessary and define a degree $\deg(\mathcal{F})$ of locally torsion-free and finitely generated $\mathcal{O}_C$-modules such that

$$\dim_K(\mathcal{F}(C)) - \dim_K(\mathcal{F}^*(C)) = \deg(\mathcal{F}) + c,$$

where c depends only on C and $\dim_{K(C)} \mathcal{F}(\emptyset)$.