LATTICES AND MODULAR FORMS – EXERCISES

1. Lattices and elliptic modular forms

A *lattice in* $\mathbb{C}$ is a subgroup $L \subset \mathbb{C}$ of the form $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with ω_1 and ω_2 linearly independent over $\mathbb{R}$.

Lattices L and M in $\mathbb{C}$ are *homothetic* if there is a complex number λ such that $M = \lambda L$.

Let $\mathbb{L}$ be the set of lattices in $\mathbb{C}$. A function $F : \mathbb{L} \to \mathbb{C}$ is *homogeneous of degree* $-k$ if $F(\lambda L) = \lambda^{-k} F(L)$ for all $\lambda \in \mathbb{C}^\times$.

(1) Show that a subgroup L of $\mathbb{C}$ is a lattice in $\mathbb{C}$ if and only if L is finitely generated and contains an $\mathbb{R}$-basis of $\mathbb{C}$.

(2) Let L be a lattice in $\mathbb{C}$. Show that there is a complex number

$$\tau \in \mathfrak{h} = \{x + iy \in \mathbb{C} : y > 0\}.$$

such that L is homothetic to $\mathbb{Z} + \mathbb{Z}\tau$. Show that $\mathbb{Z} + \mathbb{Z}\tau$ is homothetic to $\mathbb{Z} + \mathbb{Z}\tau'$ if and only if there is a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that

$$\tau' = \gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

(3) Let $F : \mathbb{L} \to \mathbb{C}$ be homogeneous of degree L. Define $f = f_F : \mathfrak{h} \to \mathbb{C}$ by

$$f(\tau) = F(\mathbb{Z} + \mathbb{Z}\tau).$$

Show that f is a modular function[1] of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Further, show that every modular function of weight k arises in this way.

(4) Define $F_k : \mathbb{L} \to \mathbb{C}$ by

$$F_k(L) = \sum_{\ell \in L} \ell^{-k}.$$

For which L does F_k converge absolutely? Show that $F_k = 0$ if k is odd. Identify the modular form f_{F_k} for even k.

(5) How do you define a slash operator $\cdot|_k$ on functions $F : \mathbb{L} \to \mathbb{C}$ such that $f_F|_k\gamma = f_{F|_k\gamma}$ for all

$$\gamma \in \mathrm{GL}_2^+(\mathbb{R}) = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) : \det\gamma > 0\}.$$

[1] a set-theoretic function $f : \mathfrak{h} \to \mathbb{C}$ such that $f(\gamma\tau) = (cz + d)^k f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

(6) Verify that

$$(f_F|T_p)(\tau) = \sum_L F(L),$$

where L varies over the set of index p sublattices of $\mathbb{Z} + \mathbb{Z}\tau$.

2. Bilinear and quadratic forms

Let B be an F-bilinear form on V. For $g \in \mathrm{GL}(V)$, define a bilinear form ${}^gB$ by

$${}^gB(x, y) = B(xg, yg).$$

(1) Let $[B]_e$ be the matrix of B with respect to a basis e of V. Show that the matrix of ${}^gB$ is $g[B]_e g^t$. Conclude that the class of $\det[B]_e$ in $F^\times/F^{\times 2}$ does not depend on e and, thus, can be denoted $\det B$.