GUÀRDIA EXERCISES I

(1) Let $K = \mathbb{Q}(\theta)$, where θ is a root of $f(x) = x^3 + 5x^2 + 6x + 5$. Determine the decomposition of $p = 5$. In other words, compute the factorization of the principal ideal $5\mathbb{Z}_K$.

(2) Let $K = \mathbb{Q}(\theta)$, where θ is a root of $f(x) = x(x+2)(x+4) + 2^7$.

   (a) Show that $2\mathbb{Z}_K$ factors as a product of three distinct primes,
   $$2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3.$$

   (b) Use the previous part to show that K has no intgeral power basis, i.e., $\mathbb{Z}[\alpha]$ is a proper subset of $\mathbb{Z}_K$ for all $\alpha \in K$. (We say $\mathbb{Z}_K$ is not *monogenic*.)

(3) Fix a prime $p \in \mathbb{Z}$. Let $\nu_p \colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be the p-adic valuation. Define $\nu \colon \mathbb{Z}[x] \to \mathbb{N} \cup \{\infty\}$ by
$$\nu(\sum_i a_i x^i) = \min_i \{\nu(a_i)\}.$$
Prove that ν is a valuation.