Low dimensional Euclidean buildings

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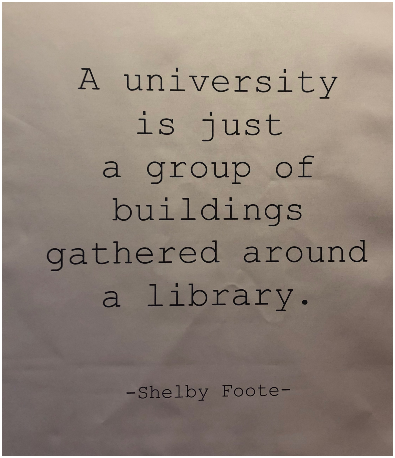
Seen at the library of the University of Jyväskyä



Motivation for low rank/dimension building

- Buildings were introduced by Belgian mathematician **Jacques Tits** to unify the classification of semi-simple Lie groups.
- Existence of particular subgroups B and N in an ambient group G.
- Tits recognized that B and N and their conjugates were living in G in an organized fashioned which could be encoded by a simplicial complex satisfying some properties.
- He extracted the axioms of building which are more general than the classical/algebraic setting of $B, N < G$.
- He later realized that only the chambers (maximal simplices) matter and the **chamber system** contains all the information.

Motivation for low rank/dimension building

Tits' classification: all spherical buildings ($|W|$ finite) of rank ≥ 3 and all Euclidean buildings of rank ≥ 4:

- "There is always a big group of symmetries G with subgroups B, N."

However in low rank (≤ 3), things are more flexible and allow for **exotic** behavior. So much so that there is no hope for classifying Euclidean buildings of rank 3.

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However in low rank (≤ 3), things are more flexible and allow for **exotic** behavior. So much so that there is no hope for classifying Euclidean buildings of rank 3.

- We will come back to the **classification** later.

Goal 1: Radu's lattice

In a paper of 2016, **Nicolas Radu** gave the first example of

- *a cocompact lattice in a $\widetilde{A}_2$-building with non-Desarguesian residues*

Question asked by Kantor in 1986.

All the credit for the code and illustrations goes to him.

Goal 1: Radu's lattice

In a paper of 2016, **Nicolas Radu** gave the first example of *a cocompact lattice in a $\widetilde{A}_2$-building with non-Desarguesian residues* (answering a question of Kantor from 1986).

- Rank 2 residues in an $\widetilde{A}_2$-building are subbuildings of type A_2 called **projective planes**.
- Projective planes of the form $A_2(k)$ satisfy **Desargues' Theorem**.
- A (cocompact) **lattice** is a discrete group acting on the building with finitely many orbits.
- A theorem of Cartwright-Mantero-Stegger-Zappa (CMSZ) shows that to find such building and lattice we can look for two combinatorial objects in a finite projective plane:
  - A **point-line correspondence** $\lambda : P \to L$.
  - A **triangular presentation** $\mathcal{T}$ compatible with λ.

Goal 1: Radu's lattice

- CMSZ found all those triangular presentations in the case of $A_2(\mathbb{F}_2)$ and $A_2(\mathbb{F}_3)$ (up to equivalence).
- Radu took the smallest non-Desarguesian projective plane, **Hughes plane**, and made a search.
- His C++ search is not perfect and actually introduces inaccuracies to speed up the process and find the one example.

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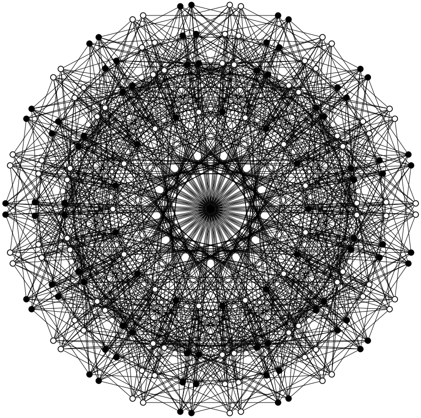
- His C++ search is not perfect and actually introduces inaccuracies to speed up the process and find the one example.

Goal: get familiar with the construction, the algorithm, C++, and possibly improve to find new examples.

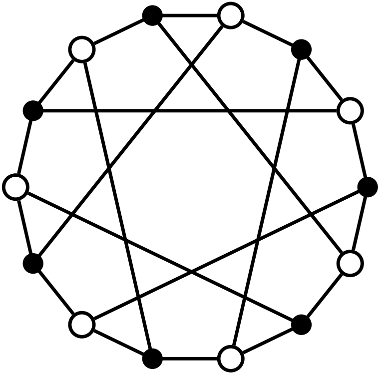
Goal 1: Radu's lattice

- Hughes plane of order $q = 9$.



Goal 1: Radu's lattice

- Finite projective plane $A_2(\mathbb{F}_2)$.



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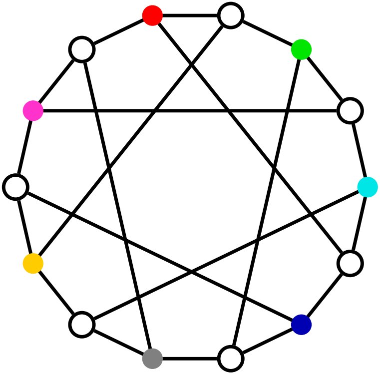
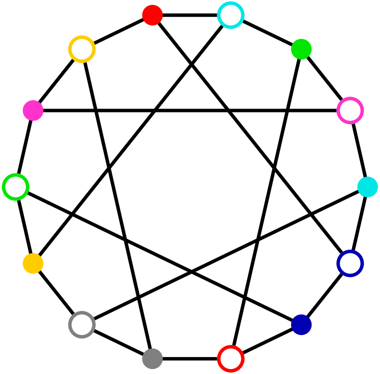


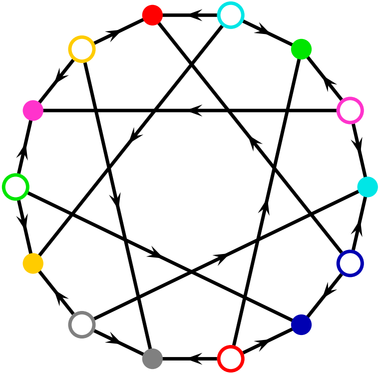
Goal 1: Radu's lattice

- A point-line correspondence λ forming pairs.



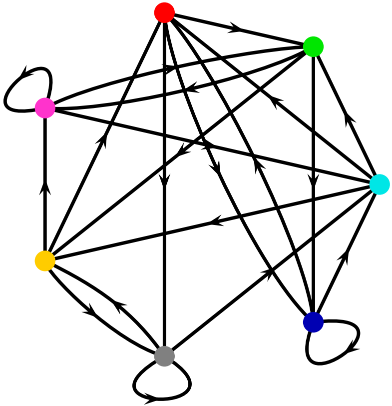
Goal 1: Radu's lattice

- The incidence relation: $point \subset line$



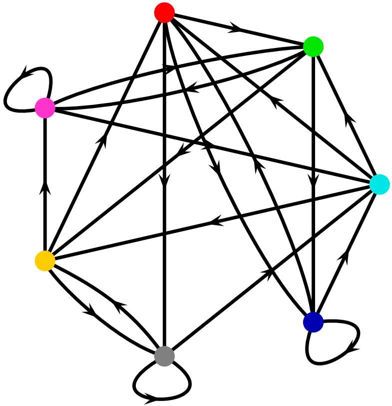
Goal 1: Radu's lattice

- A graph G_λ associated to the point line correspondence λ.



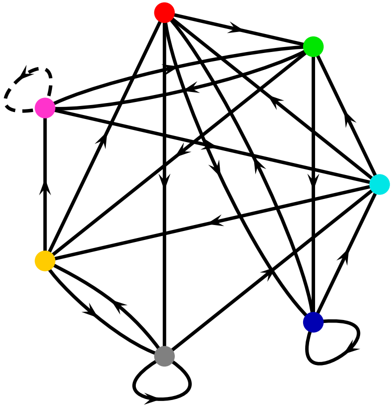
Goal 1: Radu's lattice

- The triangle presentation $\mathcal{T}$ is a cover of G_λ by disjoint of triangles.



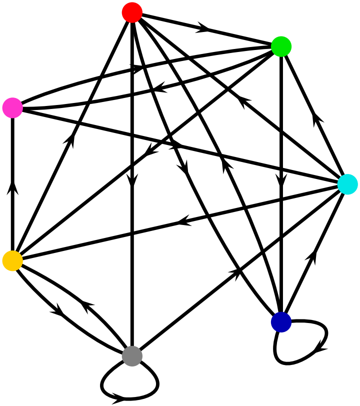
Goal 1: Radu's lattice

- Triangle can also mean loop.



Goal 1: Radu's lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$.



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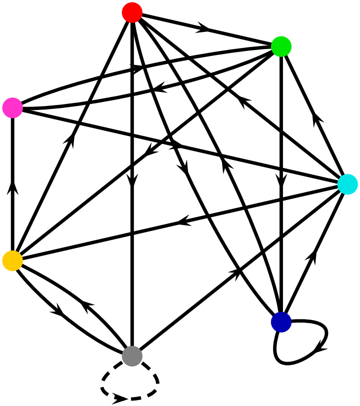

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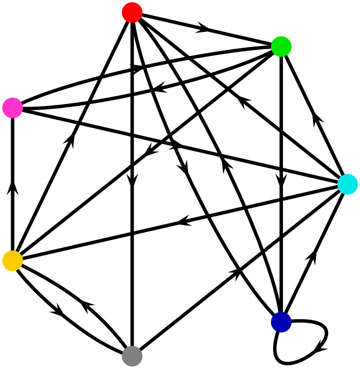

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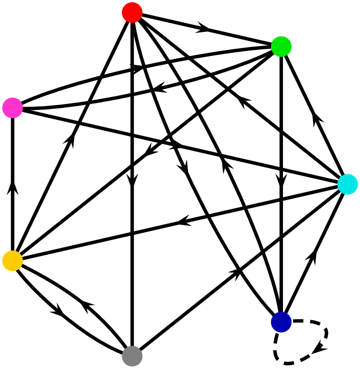

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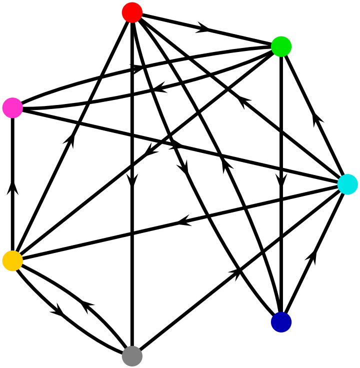

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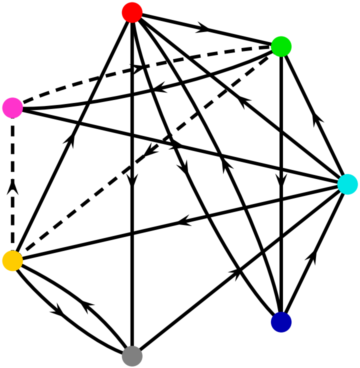

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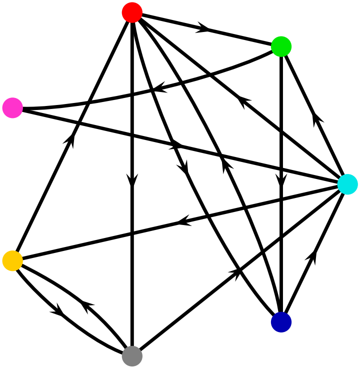

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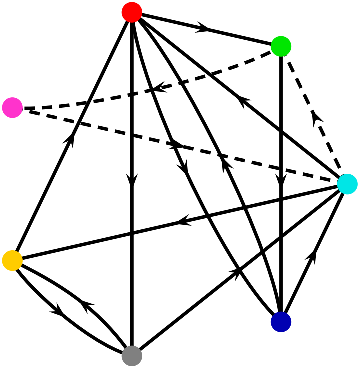

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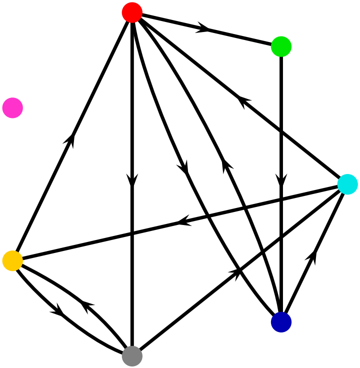

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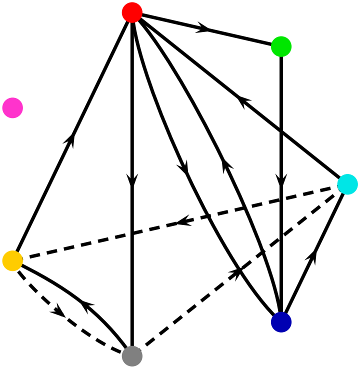

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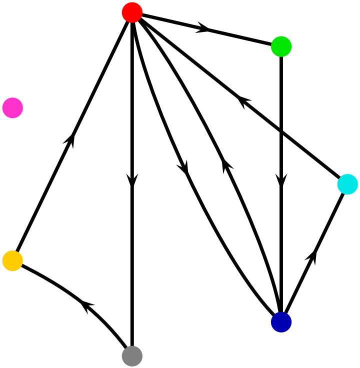

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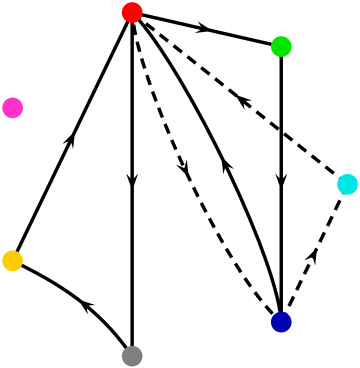

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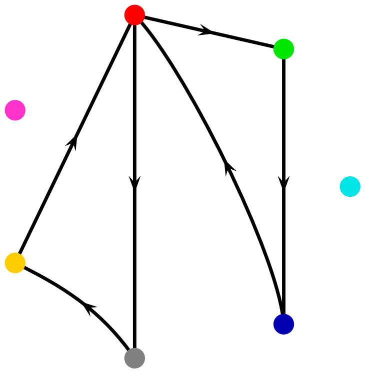

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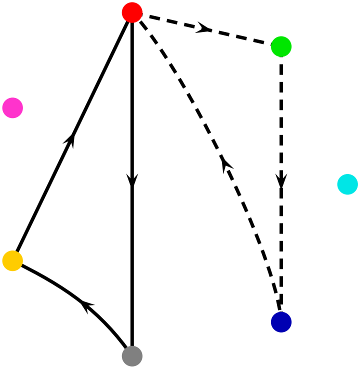

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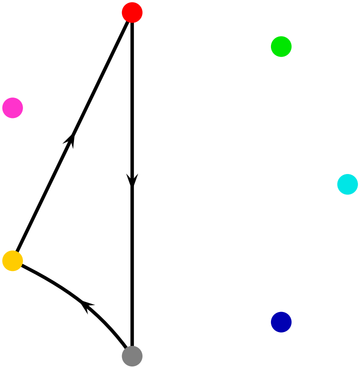

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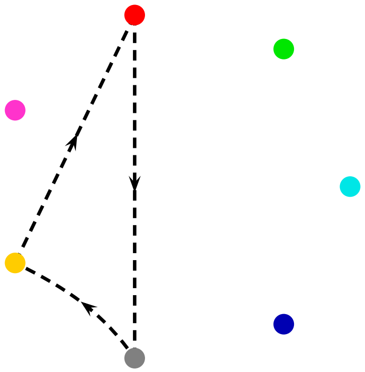
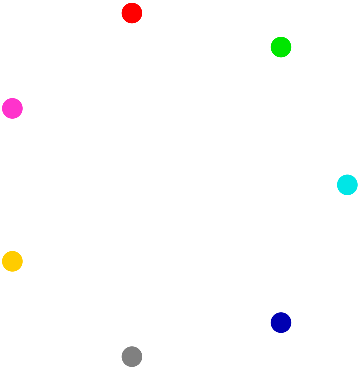


Goal 1: Radu's lattice

- No triangle left, so we the triangle we removed form a cover of G_λ. Pretty lucky!



Goal 2: An estimate motivated by buildings

Let q, n be positive integers and $q \geq 2$.

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Let q, n be positive integers and $q \geq 2$. Here are some functions $\mathbf{R} \to \mathbf{R}$:

- f_n piecewise linear and $h(x) = q^{-|x|}$.

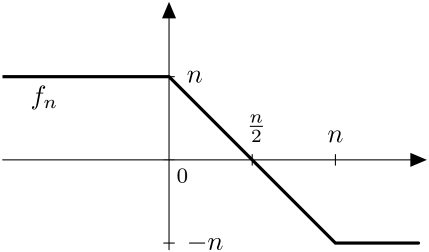

Figure: Graph of f_n.

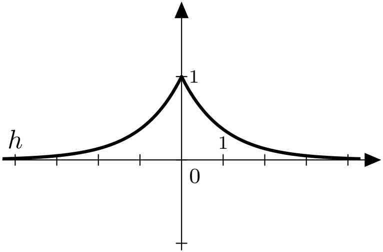

Figure: Graph of h.

Goal 2: An estimate motivated by buildings

- g represents a signed measure (on $\mathbf{Z}$):

$$g(x) = \begin{cases} h(x) & \text{if } x \leq 0, \\ 1 - 2x & \text{if } 0 \leq x \leq 1, \\ -h(x-1) & \text{if } 1 \leq x, \end{cases}$$

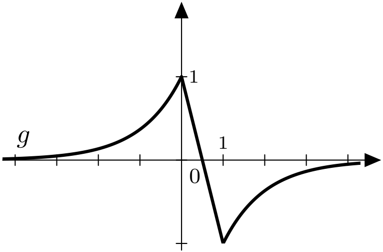

Figure: Graph of g.

Goal 2: An estimate motivated by buildings

Finally:

- μ_n a positive weight function (on $\mathbf{Z}$):

$$\mu_n(x) = \begin{cases} q^{|x|} & \text{if } x \leq 0, \\ 1 & \text{if } 0 \leq x \leq n, \\ q^{x-n} & \text{if } n \leq x, \end{cases}$$

Goal 2: An estimate motivated by buildings

Let f_n, g, μ_n be as above and let P_n be defined as follows:

$$P_n(i) = \sum_{k \in \mathbf{Z}} f_n(k) g(k - i)$$

Theorem

This is a constant $C = C(q)$ such that for all $n \in \mathbf{N}$:

$$\|P_n\|^2_{\ell^2(\mathbf{Z}, \mu_n)} = \sum_{i \in \mathbf{Z}} P_n(i)^2 \mu_n(i) \leq C \cdot n.$$

Groups acting on buildings

White board