Low dimensional Euclidean buildings: III

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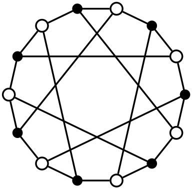
Long time ago, in a building far away

Finite projective plane

A building of type A_2 is called a **projective planes**. It's a graph of diameter 3 and girth 6 with two type of vertices called **points** or **lines**.

- If it is finite, every vertex has the same number of neighbor, $q+1$ (with $q \geq 2$ if thick).
- A projective plane has q^2+q+1 vertices of each types (points or lines).
- A projective plane has $(q+1)(q^2+q+1)$ edges (chambers).



The game: Dobble

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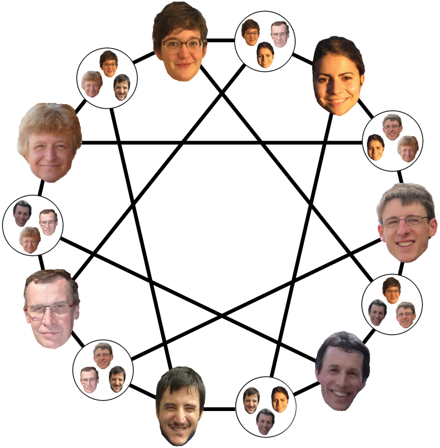

Triangle Lattices

Let Δ be a thick locally finite building of type $\tilde{A}_2$, shortly **a triangle building**.

- Let $q \geq 2$ denote the regularity parameter of Δ.
- Let $I = \{0, 1, 2\}$ denote the types and V_i the set of residues of type $\{j, k\}$ where $\{i, j, k\} = \{0, 1, 2\}$.
- In other words, V_i is the set of vertices of type i.
- The residues are finite projective plane of order q (equivalently a finite thick A_2-building).

Triangle Lattices

Let Γ be a group acting on Δ. Assume the action is:

- **type-rotating**: either $g \in G$ fixes all types or permutes them cyclically.
- **simply-transitive** on the set of vertices on $V = V_0 \cup V_1 \cup V_2$: for every $v, w \in V$ there is a unique g mapping v to w.
  - The elements of G are in bijection with the vertices. (Think of $\mathbb{Z}^n$ acting on itself by translation).

Triangle Lattices

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Theorem (CMSZ)

Any such action gives a point-line correspondence and a compatible a triangular presentation. Conversely, any point-line correspondence in a projective plane admitting a triangular presentation yields a triangle building and a lattice as above.

Triangle Lattices

Generators and Relations

Let Γ act simply transitively on the set $\mathscr{V}$ of vertices of a building Δ. Fix some $v_0 \in \mathscr{V}$, and let $\mathscr{N}$ denote the set $\{v \in \mathscr{V} : d_{\mathscr{V}}(v_0, v) = 1\}$ of nearest neighbors of v_0, i.e. the vertex set of the *residue* of v_0. For each $v \in \mathscr{N}$, there must be a unique $g_v \in \Gamma$ such that $g_v v_0 = v$. If $v \in \mathscr{N}$, then

$$d_{\mathscr{V}}(g_v^{-1} v_0, v_0) = d_{\mathscr{V}}(v_0, g_v v_0) = 1,$$

and so $g_v^{-1} v_0 \in \mathscr{N}$. Write $g_v^{-1} v_0 = \lambda(v)$. Then $g_v^{-1} v_0 = g_{\lambda(v)} v_0$, so that

$$g_{\lambda(v)} = g_v^{-1} \quad \text{for each } v \in \mathscr{N}.$$

Note that $g_{\lambda(\lambda(v))} = g_{\lambda(v)}^{-1} = g_v$, so that $\lambda: \mathscr{N} \to \mathscr{N}$ is an involution. Suppose that $u, v \in \mathscr{N}$ and that $d_{\mathscr{V}}(\lambda(u), v) = 1$. Then

$$d_{\mathscr{V}}(v_0, g_u g_v v_0) = d_{\mathscr{V}}(g_u^{-1} v_0, g_v v_0) = d_{\mathscr{V}}(\lambda(u), v) = 1.$$

Thus $g_u g_v v_0 \in \mathscr{N}$. Write $g_u g_v v_0 = \lambda(w) = g_w^{-1} v_0$. Then $g_u g_v = g_w^{-1}$, so that

$g_u g_v g_w = 1$. Conversely, if $g_u g_v g_w = 1$ for some $w \in \mathscr{N}$, then reversing the above steps, we see that $d_{\mathscr{V}}(\lambda(u), v) = 1$ must hold. Let $\mathscr{T} = \{(u, v, w) \in \mathscr{N}^3 : g_u g_v g_w = 1\}$. Then

given $u, v \in \mathscr{N}, (u, v, w) \in \mathscr{T}$ for some $w \in \mathscr{N}$ if and only if $d_{\mathscr{V}}(\lambda(u), v) = 1$.

Triangle Presentation

Definition 2.1. Let P and L be the sets of points and lines respectively in a projective plane Π. A bijection $\lambda\colon P \to L$ is called a **point-line correspondence** in Π. A subset $\mathcal{T} \subseteq P^3$ is then called a **triangle presentation** compatible with λ if the two following conditions hold:

1. For all $x, y \in P$, there exists $z \in P$ such that $(x, y, z) \in \mathcal{T}$ if and only if $y \in \lambda(x)$ in Π. In this case, z is unique.
2. If $(x, y, z) \in \mathcal{T}$, then $(y, z, x) \in \mathcal{T}$.

Example 2.2. The projective plane PG(2, 2) can be defined by $P = L = \mathbf{Z}/7\mathbf{Z}$ with line $x \in L$ being adjacent to the points $x + 1$, $x + 2$ and $x + 4$ in P. Consider the point-line correspondence $\lambda\colon P \to L\colon x \in P \mapsto x \in L$ in Π. Then

$$\mathcal{T} := \{(x, x + 1, x + 3), (x + 1, x + 3, x), (x + 3, x, x + 1) \mid x \in P\}$$

is a triangle presentation compatible with λ. Indeed, (ii) is obviously satisfied and, for $x, y \in P$, it is apparent that there exists (a unique) $z \in P$ such that $(x, y, z) \in \mathcal{T}$ if and only if $y \in \{x + 1, x + 2, x + 4\}$, which is exactly the set of points on the line $\lambda(x)$.

Score

Definition 3.5. Let $\lambda\colon P \to L$ be a point-line correspondence in a projective plane Π. A subset $\mathcal{T} \subseteq P^3$ is called a **triangle partial presentation** compatible with λ if the two following conditions hold:

(1) For all $x, y \in P$, if there exists $z \in P$ such that $(x, y, z) \in \mathcal{T}$ then $y \in \lambda(x)$ and z is unique.

(2) If $(x, y, z) \in \mathcal{T}$, then $(y, z, x) \in \mathcal{T}$.

We directly have the following.

Lemma 3.6. *Let $\lambda\colon P \to L$ be a point-line correspondence in a projective plane Π of order q. A subset $\mathcal{T} \subseteq P^3$ is a triangle presentation compatible with λ if and only if it is a triangle partial presentation compatible with λ and $|\mathcal{T}| = (q+1)(q^2+q+1)$.*

Proof. This is clear from the definitions, since there are exactly $(q+1)(q^2+q+1)$ pairs $(x, y) \in P^2$ with $y \in \lambda(x)$. □

We now define the *score* of a point-line correspondence as follows.

Definition 3.7. Let $\lambda\colon P \to L$ be a point-line correspondence in a projective plane Π of order q. The **score** $S(\lambda)$ of λ is the greatest possible size of a triangle partial presentation compatible with λ.

Graph G_λ

3.1 The graph associated to a point-line correspondence

In the context of triangle presentations, it is natural to associate a particular graph to each point-line correspondence $\lambda\colon P \to L$ of a projective plane Π.

Definition 3.1. Let $\lambda\colon P \to L$ be a point-line correspondence in a projective plane Π. The **graph** G_λ **associated to** λ is the directed graph with vertex set $V(G_\lambda) := P$ and edge set $E(G_\lambda) := \{(x, y) \in P^2 \mid y \in \lambda(x)\}$.

For λ, admitting a triangle presentation can now be rephrased as a condition on its associated graph G_λ. In order to state this reformulation, we first define what we will call a *triangle* in a directed graph.

Definition 3.2. Let G be a directed graph. A set $\{e_1, e_2, e_3\}$ of edges in G such that the destination vertex of e_1 (resp. e_2 and e_3) is the origin vertex of e_2 (resp. e_3 and e_1) is called a **triangle**. If two of the three edges e_1, e_2 and e_3 are equal, then they are all equal. In this case, the triangle contains only one edge and is also called a **loop**.

Lemma 3.4. *Let $\lambda\colon P \to L$ be a point-line correspondence in a projective plane Π. There exists a triangle presentation compatible with λ if and only if there exists a partition of the set of edges $E(G_\lambda)$ of G_λ into triangles.*

Proof. Via the above bijection, a partition of $E(G_\lambda)$ into triangles exactly corresponds to a triangle presentation compatible with λ. □

Score of a Correlation

3.3 Scores of correlations

When $\lambda\colon P \to L,\ L \to P$ is a **correlation** of a (self-dual) projective plane Π of order q, i.e. a map such that $\lambda(p) \ni \lambda(\ell)$ if and only if $p \in \ell$, there is an explicit formula for the score of the point-line correspondence $\lambda\colon P \to L$.

Proposition 3.10. *Let $\lambda\colon P \to L,\ L \to P$ be a correlation in a projective plane Π of order q. Let $a(\lambda)$ be the number of points $p \in P$ such that $\lambda^3(p) \ni p$ and let $b(\lambda)$ be the number of points $p \in P$ such that $\lambda^3(p) \ni p$ and $\lambda^6(p) = p$. Then*

$$S(\lambda) = (q+1)(q^2+q+1) - (2q-3)\cdot a(\lambda) - b(\lambda).$$

Theorem 3.11 (Devillers–Parkinson–Van Maldeghem)**.** *Let $\lambda\colon P \to L,\ L \to P$ be a correlation in a finite projective plane Π. Then there exists $p \in P$ such that $p \in \lambda(p)$.*

Score of a Correlation

# of concerned λ	$a(\lambda)$	$b(\lambda)$	$S(\lambda)$	$s(\lambda)$ (mean)
6318	4	4	846	846.00
4212	10	2	758	757.97
6318	10	10	750	750.00
4212	16	0	670	669.92
6318	16	16	654	654.00
6318	22	22	558	558.00

Table 3.1: Scores of the correlations of the Hughes plane of order 9.

Score: Algorithm 1

While there exists $e \in E(G_\lambda)$ such that there is a unique triangle t in G_λ containing e, choose this triangle t, remove the edge(s) of t from G_λ and start again this procedure. If, at the end, there is no more triangles in G_λ, then we say that the score-algorithm **succeeds** and that the **estimated score** $s(\lambda)$ of λ is the number of edges that are covered by the chosen triangles. Otherwise, there still are triangles in G_λ but all edges are contained in 0 or at least 2 triangles. In this case, we say that the score-algorithm **fails**. For a pseudo-code, see Algorithm 1.

Score: Algorithm 1

Algorithm 1: Computing the estimated score $s(\lambda)$ of λ

```
score ← 0;
edgesInOneTriangle ← true;
while edgesInOneTriangle = true do
    edgesInOneTriangle ← false;
    for e in E(G_λ) do
        if e is contained in exactly one triangle t of G_λ then
            edgesInOneTriangle ← true;
            remove the edge(s) of t from E(G_λ);
            if t is a loop then
                score ← score + 1;
            else
                score ← score + 3;
if there still are triangles in G_λ then
    return FAIL
else
    return score
```

Improving the Score: Algorithm 2

Lemma 3.15. *Let* $\lambda\colon P \to L$ *be a point-line correspondence in a projective plane* Π *of order* q *and let* $a, b \in P$. *Define* $\lambda_{a,b}\colon P \to L$ *by* $\lambda_{a,b}(x) := \lambda(x)$ *for all* $x \in P \setminus \{a, b\}$, $\lambda_{a,b}(a) := \lambda(b)$ *and* $\lambda_{a,b}(b) := \lambda(a)$. *Then* $|S(\lambda_{a,b}) - S(\lambda)| \leq 6(q+1)$.

Algorithm 2: Finding a point-line correspondence λ with $s(\lambda) = 910$

```
λ ← some correlation of the Hughes plane;
while s(λ) < 910 do
    visited[λ] ← true;
    bestA ← −1; bestB ← −1;
    bestScore ← −1;
    for a in P and b in P do
        if visited[λ_{a,b}] = false and s(λ_{a,b}) > bestScore then
            bestScore ← s(λ_{a,b});
            bestA ← a;
            bestB ← b;
    λ ← λ_{bestA,bestB};
return λ;
```

Results

λ	_0	_1	_2	_3	_4	_5	_6	_7	_8	_9
0_	20	0	44	75	78	77	50	76	37	3
1_	54	39	30	8	88	68	18	34	65	57
2_	70	82	42	23	38	90	81	13	61	69
3_	73	4	83	22	58	28	59	55	64	60
4_	56	2	87	84	26	45	53	11	80	41
5_	25	14	63	72	7	32	62	86	51	46
6_	36	27	31	29	79	33	16	71	85	24
7_	89	35	17	19	5	47	67	10	66	43
8_	6	21	1	52	74	40	12	48	9	15
9_	49									

(0,3,41)	(0,10,82)	(0,29,54)	(0,34,9)	(0,56,88)	(0,61,31)	(0,66,1)	(0,67,13)	(0,68,74)
(0,69,80)	(1,1,1)	(1,2,16)	(1,3,47)	(1,4,72)	(1,5,89)	(1,6,86)	(1,7,51)	(1,8,77)
(1,9,27)	(2,3,62)	(2,19,12)	(2,43,61)	(2,54,73)	(2,60,65)	(2,65,55)	(2,73,35)	(2,75,17)
(2,81,63)	(3,3,3)	(3,14,8)	(3,23,6)	(3,27,56)	(3,49,7)	(3,76,4)	(3,84,5)	(4,15,28)
(4,20,37)	(4,28,15)	(4,39,81)	(4,48,29)	(4,53,20)	(4,74,79)	(4,85,71)	(5,17,90)	(5,22,67)
(5,31,33)	(5,40,60)	(5,45,53)	(5,50,30)	(5,55,40)	(5,71,22)	(6,13,25)	(6,24,78)	(6,30,26)
(6,35,21)	(6,44,10)	(6,59,19)	(6,78,24)	(6,87,59)	(7,18,39)	(7,21,75)	(7,32,57)	(7,37,83)
(7,46,69)	(7,58,32)	(7,63,64)	(7,82,18)	(8,11,87)	(8,26,52)	(8,36,58)	(8,52,68)	(8,57,36)
(8,64,50)	(8,80,70)	(8,90,85)	(9,20,43)	(9,42,38)	(9,55,44)	(9,56,42)	(9,57,48)	(9,58,45)
(9,59,46)	(9,60,11)	(10,17,23)	(10,26,33)	(10,27,82)	(10,35,73)	(10,48,77)	(10,67,81)	(10,73,50)
(10,79,69)	(11,11,11)	(11,23,71)	(11,37,82)	(11,54,24)	(11,55,85)	(11,67,61)	(11,72,49)	(11,83,47)
(12,12,12)	(12,24,68)	(12,39,86)	(12,47,45)	(12,58,56)	(12,67,53)	(12,80,82)	(12,81,57)	(12,89,76)
(13,31,48)	(13,46,35)	(13,52,26)	(13,62,79)	(13,67,27)	(13,75,52)	(13,85,82)	(13,86,64)	(14,15,21)
(14,22,63)	(14,33,82)	(14,41,37)	(14,44,32)	(14,51,58)	(14,57,51)	(14,65,43)	(14,67,22)	(15,16,30)
(15,36,34)	(15,45,82)	(15,49,59)	(15,59,89)	(15,67,83)	(15,88,65)	(16,18,82)	(16,40,25)	(16,53,66)
(16,60,84)	(16,67,46)	(16,70,36)	(16,84,55)	(16,90,18)	(17,17,38)	(17,46,20)	(17,59,70)	(17,65,82)
(17,68,40)	(17,74,72)	(18,22,42)	(18,29,39)	(18,42,78)	(18,62,87)	(18,87,62)	(18,90,29)	(19,22,77)
(19,32,79)	(19,34,75)	(19,54,45)	(19,59,87)	(19,77,22)	(19,84,41)	(19,85,61)	(20,24,62)	(20,33,64)
(20,53,37)	(20,56,70)	(20,62,77)	(20,77,24)	(21,21,38)	(21,29,68)	(21,44,60)	(21,53,31)	(21,55,83)
(21,80,77)	(22,71,74)	(22,74,63)	(22,78,42)	(23,28,80)	(23,41,36)	(23,43,26)	(23,58,90)	(23,83,77)
(23,86,28)	(23,90,57)	(24,40,34)	(24,51,85)	(24,81,84)	(24,88,32)	(25,25,25)	(25,30,58)	(25,37,65)
(25,47,48)	(25,50,77)	(25,57,56)	(25,61,29)	(25,74,49)	(26,39,55)	(26,45,76)	(26,60,39)	(26,63,88)
(26,66,77)	(27,31,61)	(27,41,34)	(27,54,29)	(27,74,68)	(27,80,69)	(27,88,66)	(28,32,40)	(28,40,80)
(28,42,52)	(28,61,32)	(28,73,42)	(28,86,61)	(29,44,89)	(29,64,76)	(29,71,70)	(30,30,38)	(30,45,57)
(30,54,74)	(30,62,61)	(30,69,37)	(31,54,83)	(31,76,78)	(31,78,36)	(31,89,81)	(31,90,43)	(32,46,84)
(32,54,64)	(32,66,72)	(33,33,33)	(33,40,51)	(33,47,46)	(33,59,56)	(33,72,39)	(33,75,61)	(34,34,34)
(34,50,39)	(34,63,37)	(34,88,47)	(34,90,35)	(35,57,71)	(35,62,41)	(35,70,86)	(35,71,58)	(35,72,83)
(36,53,44)	(36,64,67)	(36,78,65)	(36,79,51)	(37,52,89)	(37,89,52)	(38,41,41)	(38,42,56)	(38,64,64)
(38,66,66)	(38,81,81)	(38,85,85)	(39,49,41)	(39,74,43)	(40,76,41)	(40,87,48)	(42,47,47)	(42,73,52)
(43,43,43)	(43,51,68)	(43,59,64)	(43,70,47)	(44,44,44)	(44,62,75)	(44,74,80)	(44,79,47)	(45,45,45)
(45,68,69)	(45,86,62)	(46,46,46)	(46,71,76)	(46,80,50)	(48,48,48)	(48,49,63)	(48,58,81)	(48,65,84)
(49,53,73)	(49,69,85)	(49,73,53)	(49,89,59)	(50,55,76)	(50,73,51)	(50,76,60)	(50,88,87)	(51,73,54)
(51,87,66)	(52,75,84)	(52,84,68)	(53,90,88)	(54,84,70)	(55,56,79)	(55,63,86)	(56,83,60)	(57,75,72)
(58,72,75)	(60,79,85)	(60,86,63)	(63,69,70)	(65,66,86)	(65,78,69)	(66,89,71)	(68,87,83)	(69,78,72)
(70,70,70)	(71,75,88)	(72,78,76)	(79,79,79)	(79,90,89)	(80,81,87)	(83,83,83)	(88,88,88)	

Table B.2: Triangle presentation $\mathcal{T}$ compatible with λ.

Results

A The Hughes plane of order 9

0	0	1	2	3	4	5	6	7	8	9
1	0	10	11	12	13	14	15	16	17	18
2	0	19	34	35	36	37	38	39	40	41
3	0	20	27	42	55	56	57	58	59	60
4	0	21	33	48	54	61	76	78	89	90
5	0	22	30	43	49	63	68	72	79	80
6	0	23	28	44	50	69	70	77	81	82
7	0	24	29	45	51	64	73	74	83	84
8	0	25	31	46	52	62	67	75	85	86
9	0	26	32	47	53	65	66	71	87	88
10	1	10	19	20	21	22	23	24	25	26
11	1	11	34	42	43	44	45	46	47	48
12	1	12	28	35	55	61	62	63	64	65
13	1	13	31	41	54	56	74	80	82	88
14	1	14	33	36	50	58	68	73	85	87
15	1	15	29	37	52	59	71	76	79	81
16	1	16	27	38	51	66	72	77	86	89
17	1	17	32	39	49	57	69	75	78	83
18	1	18	30	40	53	60	67	70	84	90
19	2	10	35	42	49	50	51	52	53	54
20	3	10	29	34	56	61	66	67	68	69
21	4	10	31	38	48	57	63	81	84	87
22	5	10	33	40	47	59	64	72	75	82
23	6	10	28	41	43	58	71	83	86	90
24	7	10	27	37	45	65	70	78	80	85
25	8	10	30	39	46	55	73	76	77	88
26	9	10	32	36	44	60	62	74	79	89
27	2	11	19	27	28	29	30	31	32	33
28	2	13	21	34	57	62	70	71	72	73
29	2	14	22	37	48	60	64	69	86	88
30	2	12	24	39	47	58	67	80	81	89
31	2	18	20	41	45	61	75	77	79	87
32	2	16	26	40	44	56	63	76	83	85
33	2	15	25	36	43	55	66	78	82	84
34	2	17	23	38	46	59	65	68	74	90
35	4	11	22	35	58	66	70	74	75	76
36	5	11	21	39	50	56	65	79	84	86
37	3	11	26	36	52	57	64	77	80	90
38	6	11	20	40	51	62	68	78	81	88
39	9	11	23	37	54	55	67	72	83	87
40	8	11	24	38	49	60	61	71	82	85
41	7	11	25	41	53	59	63	69	73	89
42	5	14	19	42	63	67	71	74	77	78
43	4	13	19	47	51	55	69	79	85	90
44	3	16	19	43	54	60	65	73	75	81
45	9	12	19	45	53	57	68	76	82	86
46	6	15	19	46	49	56	64	70	87	89
47	7	17	19	44	52	58	61	72	84	88
48	8	18	19	48	50	59	62	66	80	83
49	5	18	23	29	35	43	57	85	88	89
50	3	13	24	30	35	44	59	78	86	87
51	8	14	25	32	35	45	56	72	81	90
52	7	15	21	31	35	47	60	68	77	83
53	9	16	20	33	35	46	69	71	80	84
54	6	17	26	27	35	48	67	73	79	82
55	4	14	20	30	34	52	65	82	83	89
56	5	17	25	28	34	51	60	76	80	87
57	6	12	22	32	34	54	59	77	84	85
58	9	15	24	27	34	50	63	75	88	90
59	8	16	23	31	34	53	58	64	78	79
60	7	18	26	33	34	49	55	74	81	86
61	4	15	23	32	40	42	61	73	80	86
62	3	12	25	33	38	42	70	79	83	88
63	8	13	26	28	37	42	68	75	84	89
64	9	17	21	30	41	42	64	66	81	85
65	7	16	22	29	39	42	62	82	87	90
66	6	18	24	31	36	42	65	69	72	76
67	4	12	26	29	41	46	50	60	72	78
68	4	16	21	28	36	45	49	59	67	88
69	4	18	25	27	39	44	54	64	68	71
70	4	17	24	33	37	43	53	56	62	77
71	5	13	22	27	36	46	53	61	81	83
72	5	12	20	31	37	44	49	66	73	90
73	5	15	26	30	38	45	54	58	62	69
74	5	16	24	32	41	48	52	55	68	70
75	3	14	23	27	41	47	49	62	76	84
76	3	18	21	32	37	46	51	58	63	82
77	3	17	22	31	40	45	50	55	71	89
78	3	15	20	28	39	48	53	72	74	85
79	7	13	20	32	38	43	50	64	67	76
80	9	13	25	29	40	48	49	58	65	77
81	6	13	23	33	39	45	52	60	63	66
82	6	14	21	29	38	44	53	55	75	80
83	7	14	24	28	40	46	54	57	66	79
84	9	14	26	31	39	43	51	59	61	70
85	8	12	21	27	40	43	52	69	74	87
86	7	12	23	30	36	48	51	56	71	75
87	9	18	22	28	38	47	52	56	73	78
88	8	15	22	33	41	44	51	57	65	67
89	8	17	20	29	36	47	54	63	70	86
90	6	16	25	30	37	47	50	57	61	74

Radu's C++ program

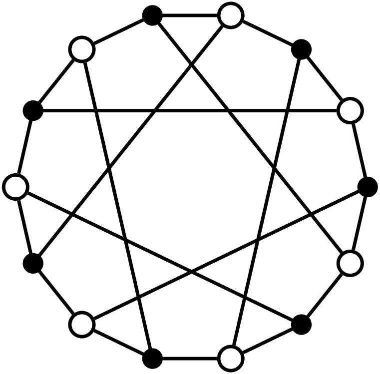
A few things to know about the C++ code:

- Radu uses the fact that the lines 0, 1, 10 and 30 generate the Hughes plane.
- It was too slow to check all pairs a, b, so he tests and selects only a few pairs. Especially the vertices for which few triangles have been used. He calls them **bad** vertices.

What the code does:

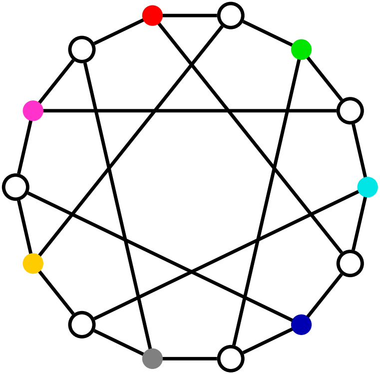
- Generates correlations until it finds one with a good score ≥ 750.
- Apply the improving algorithm, which permutes some a and b to see if it gets to the score max of 910. (Keeps track of the permutations to not fall in a local maximum).
- If after 150 steps the score is still low, it moves on to the next correlation.

- Finite projective plane $A_2(\mathbb{F}_2)$.



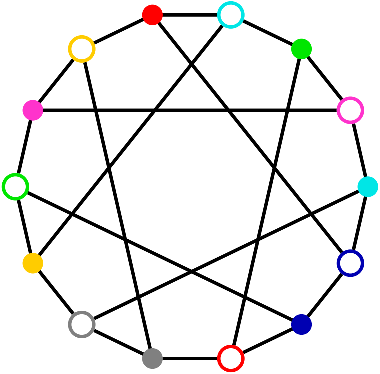
Goal 1: Radu's lattice

- Finite projective plane $A_2(\mathbb{F}_2)$.



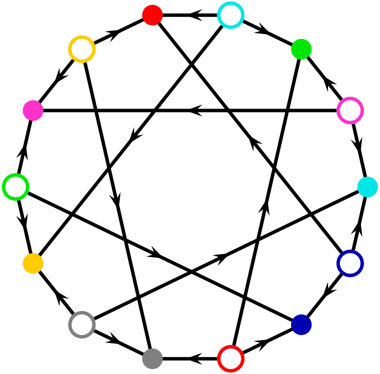
Goal 1: Radu's lattice

- A point-line correspondence λ forming pairs.



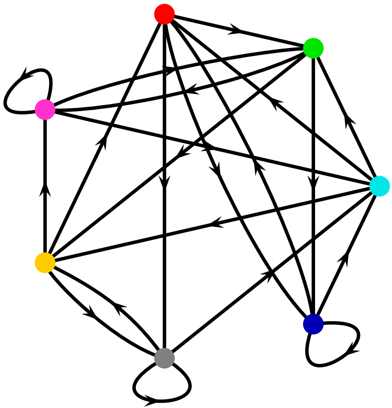
Goal 1: Radu's lattice

- The incidence relation: $point \subset line$



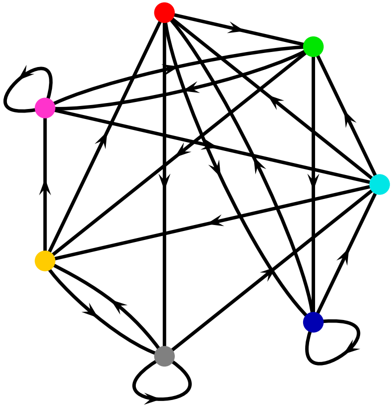
Goal 1: Radu's lattice

- A graph G_λ associated to the point line correspondence λ.



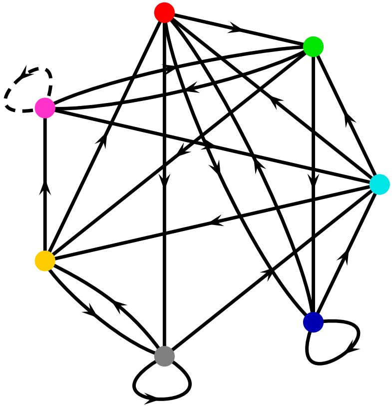
Goal 1: Radu's lattice

- The triangle presentation $\mathcal{T}$ is a cover of G_λ by disjoint of triangles.



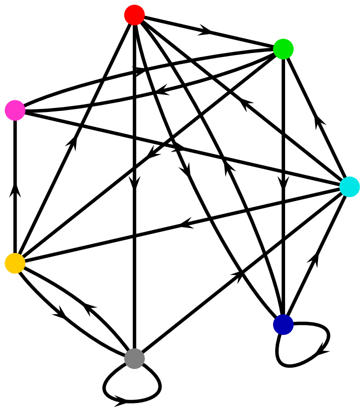
Goal 1: Radu's lattice

- Triangle can also mean loop.



Goal 1: Radu's lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$.



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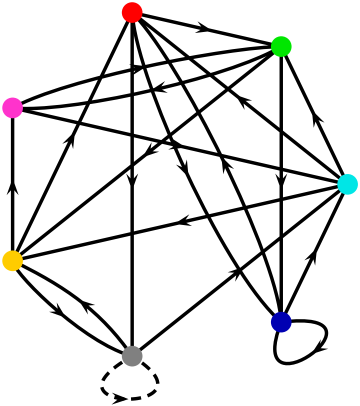

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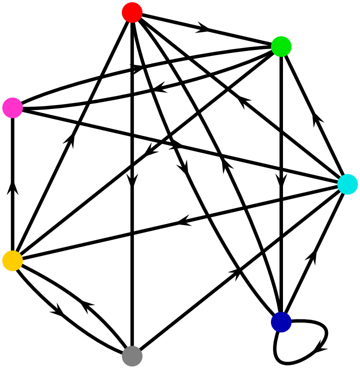

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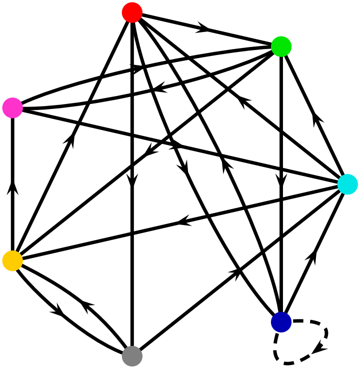

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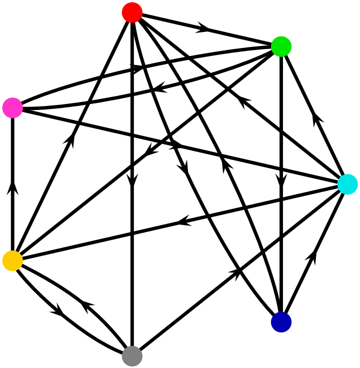

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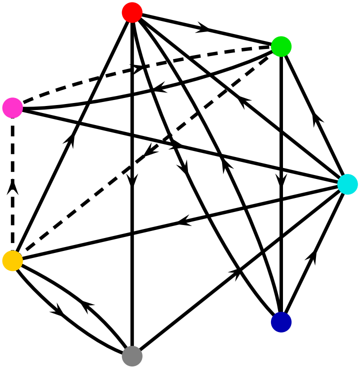

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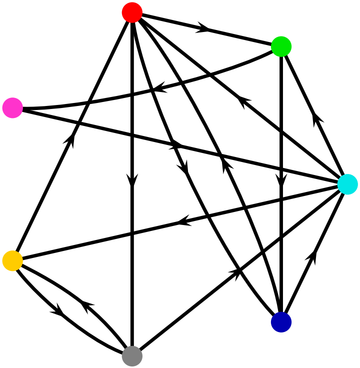

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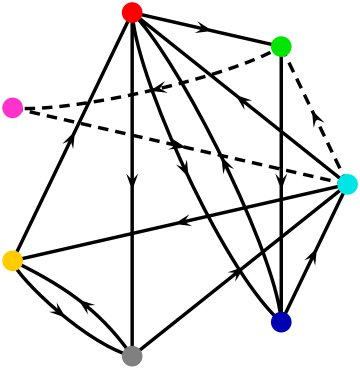

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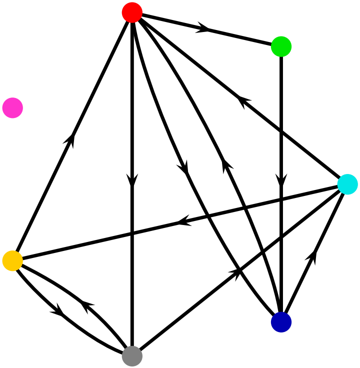

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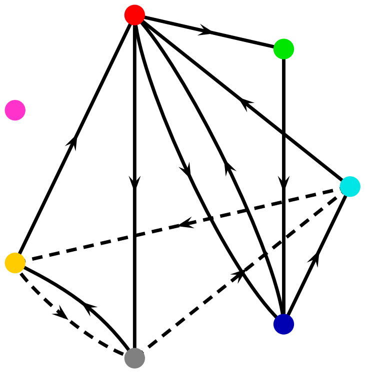

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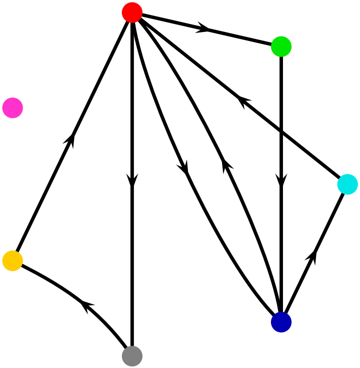

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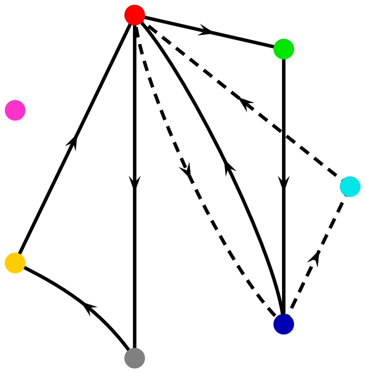

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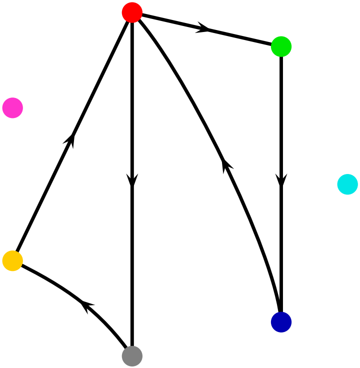

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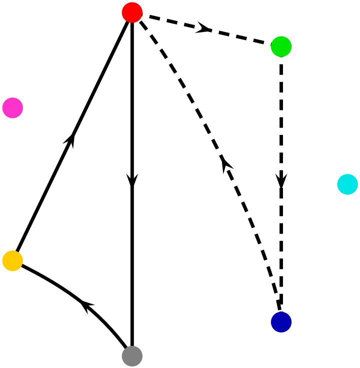

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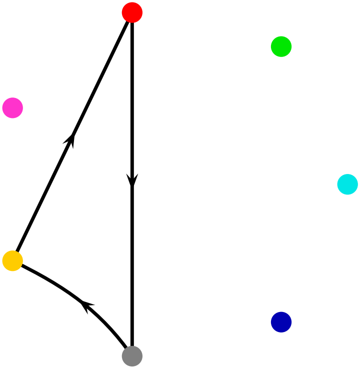

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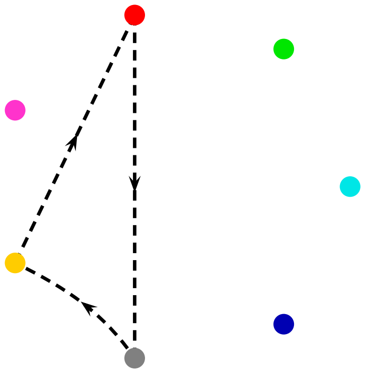

Goal 1: Radu's lattice

- No triangle left, so we the triangle we removed form a cover of G_λ. Pretty lucky!