Some basics on Coxeter groups and cross ratio on the boundary

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June 27, 2019
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The plan

1. Cross ratio on the boundary;
2. Tits representation;
3. Deletion condition and exchange condition;
4. Longest element in a finite Coxeter group

Cross ratio on metric spaces

Classical cross ratio: for $a, b, c, d \in \mathbb{C} \cup \{\infty\}$,

$$cr(a, b, c, d) = \frac{(a - d)(b - c)}{(a - c)(b - d)}.$$

Let X be a metric space, and $a, b, c, d \in X$, define

$$cr(a, b, c, d) = \frac{d(a, d)d(b, c)}{d(a, c)d(b, d)}.$$

Metric on the boundary of a tree

Let T be a tree, and $p \in T$. Define a metric on ∂T as follows: for $a \neq b \in \partial T$,

$$d_p(a,b) = e^{-d(p,ab)},$$

where $d(p,ab)$ is the distance from p to the geodesic ab.

It is easy to see that d_p satisfies the triangle inequality. In fact, d_p is an ultra metric:

$$d_p(a,c) \leq \max\{d_p(a,b), d_p(b,c)\}.$$

For $a, b, c, d \in \partial T$ we have

$$cr(a,b,c,d) = \frac{e^{-d(p,ad)}e^{-d(p,bc)}}{e^{-d(p,ac)}e^{-d(p,bd)}} = e^{d(p,ac)+d(p,bd)-d(p,ad)-d(p,bc)}.$$

Cross ratio on the boundary of a tree

The cross ratio (cross difference) on ∂T:

$$(a, b, c, d) = d(p, ac) + d(p, bd) - d(p, ad) - d(p, bc).$$

Exercise: (a, b, c, d) is independent of p, and is the signed distance from $m(a, b, c)$ to $m(a, b, d)$, with + sign if the direction from $m(a, b, c)$ to $m(a, b, d)$ is the same direction as from a to b, and $-$ sign otherwise.

Theorem. A metric tree T with no vertex of valence one is determined up to isometry by the cross ratio on ∂T.

Trees associated with Euclidean buildings

Let Δ be a locally finite thick Euclidean building. Let Y be a wall in Δ. A subset Y' is said to be parallel to Y if there is some $c \geq 0$ such that $d(y, Y') = c = d(y', Y), \forall y \in Y, \forall y' \in Y'$.

Let P_Y be the union of all the sets parallel to Y. A basic fact in $CAT(0)$ space is that P_Y splits isometrically as a product $P_Y = Y \times Z$ for some convex subset of Δ. Due to the dimension consideration it is easy to see that Z is a tree.

By the above discussion, the tree Z can be recovered from the cross ratio. This idea can be used to classify some Euclidean buildings.

Tits representation

Let (W, S) be a Coxeter system with Coxeter matrix $M = (m_{ij})$. An injective homomorphism $W \to GL_n(\mathbb{R})$ (where $n = |S|$) due to Tits is constructed as follows.

Write $S = \{s_1, \cdots, s_n\}$. Fix a basis (e_i), $1 \leq i \leq n$, for $\mathbb{R}^n$. Let B be the symmetric bilinear form determined by

$$B(e_i, e_j) = -\cos\frac{\pi}{m_{ij}}.$$

Note $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ for $i \neq j$.

Let H_i be the hyperplane $H_i = \{v \in \mathbb{R}^n : B(v, e_i) = 0\}$. For each i, define $\sigma_i : \mathbb{R}^n \to \mathbb{R}^n$ by:

$$\sigma_i(v) = v - 2B(e_i, v)e_i.$$

Tits representation II

It is easy to check that $\sigma_i(e_i) = -e_i$ and fixes all points in H_i. So $\sigma_i^2 = id$. It can also be checked that $\sigma_i\sigma_j$ has order m_{ij} for $i \neq j$. Hence the map $s_i \to \sigma_i$ defines a group homomorphism

$$\rho : W \to GL_n(\mathbb{R}).$$

The map ρ is in fact injective. Hence all Coxeter groups are linear.

Selberg's lemma Finitely generated linear groups are virtually torsion free (have torsion free subgroups of finite index).
Malcev's Theorem Finitely generated linear groups are residually finite.

Deletion and Exchange conditions

Let W be a group generated by a finite set of order 2 elements $S \subset W$. Then the following are equivalent:
1. (W, S) is Coxeter system;
2. The deletion condition holds:
if $s_1 s_2 \cdots s_k$ is NOT a reduced word in S, then there are $i < j$ such that

$$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k,$$

where $\hat{s}_i$ means s_i is removed.
3. The exchange condition holds:
if $s_1 \cdots s_k$ is a reduced word in S and $s \in S$, then either $l(sw) = k + 1$ or there is some i such that $w = ss_1 \cdots \hat{s}_i \cdots s_k$.

Longest element in a finite Coxeter group

Let (W, S) be a finite Coxeter group. Then:
1. There is unique element w_0 with the maximal length;
2. Every reduced word in S arises as the initial word for w_0; that is, for any $w \in W$, there is some $w' \in W$ satisfying:
$l(w) + l(w') = l(w_0)$ and $ww' = w_0$;
3. $w_0^2 = 1$ and $w_0 S w_0 = S$.