Quasi-isometric rigidity of Fuchsian buildings

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Quasi-isometries

Let X, Y be metric spaces, and $f : X \to Y$ a map.

• f is a (L, C)– quasi-isometry (QI) for some $L \geq 1$, $C \geq 0$ if:

(1) $d(y, f(X)) \leq C \quad \forall y \in Y$;

(2) $\forall x_1, x_2 \in X$,

$$\frac{1}{L} \cdot d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq L \cdot d(x_1, x_2) + C.$$

Remark. QIs are not assumed to be continuous.

Quasi-isometries II

QIs arise naturally in group actions

If a group G acts geometrically (that is, properly discontinuously and cocompactly by isometries) on two proper geodesic metric spaces X_1 and X_2, then X_1 and X_2 are quasi-isometric: pick $x_1 \in X_1$, $x_2 \in X_2$, then

$$g(x_1) \mapsto g(x_2), \quad g \in G$$

defines a QI from X_1 to X_2.

Some QI rigidity theorems

Theorem (Pansu) Every QI of quarternionic hyperbolic spaces is at a finite distance of an isometry.

Theorem (Kleiner-Leeb) Every QI of an irreducible higher rank symmetric space or Euclidean building is at a finite distance of an isometry.

Fuchsian buildings

Let X be a convex polygon in the hyperbolic plane with angles of the form π/m where $m \geq 2$ is an integer. Let s_i, $1 \leq i \leq n$, be the isometric reflections about the sides of X. Let W be the group generated by $S = \{s_1, \cdots, s_n\}$. Then (W, S) is a Coxeter system.

A Fuchsian building is a thick regular building of type (W, S).

Theorem(Right angled case by Bourdon-Pajot, general case by Xie) Let Δ_1, Δ_2 be two Fuchsian buildings. If Δ_1, Δ_2 admit cocompact lattices, then every QI $f : \Delta_1 \to \Delta_2$ is at a finite distance of an isometry.

Some ideas in the proof

Δ_i is $CAT(-1)$ and so has a boundary at infinity $\partial\Delta_i$, which supports the so called visual metrics (similar to the case of trees). Furthermore, any QI between $CAT(-1)$ spaces induces a homeomorphism $F : \partial\Delta_1 \to \partial\Delta_2$.

A key observation is that for $\xi, \eta \in \partial\Delta_i$, whether $\xi\eta$ lies in the 1-skeleton of Δ_i can be detected by the cross ratio.

The first step is to show that F preserves the cross ratio. Since F preseves cross ratio, if $\xi\eta$ lies in the 1-skeleton of Δ_1, then $F(\xi)F(\eta)$ lies in the 1-skeleton of Δ_2. We call $F(\xi)F(\eta)$ the image of $\xi\eta$. The next step is to show that for any vertex $v \in \Delta_1$, the images of all the geodesics in the 1-skeleton of Δ_1 that contains v intersect in a single vertex w in Δ_2. Then it is not hard to see that the map $v \to w$ extends to an isomorphism $\Delta_1 \to \Delta_2$.

Davis complex: Right angled case

A Coxeter system (W, S) is right angled if $m_{st} \in \{2, \infty\}$ for any $s \neq t \in S$.

Given a finite simplicial graph Γ, there is an associated RACG W_Γ given by

$$W_\Gamma =< v \in V(\Gamma) | uv = vu, \forall (u, v) \in E(\Gamma) > .$$

When (W, S) is right angled, the Davis complex Σ admits a structure of $CAT(0)$ cube complex. The 1-skeleton of Σ is simply the Cayley graph of (W, S). For any $w \in W$ and any $s \neq t \in S$ with $m_{st} = 2$, attach a square to the 4-cycle $w, ws, wst, wsts = wt, w$ in the Cayley graph. In general, for $w \in W$ and any subset $T \subset S$ with W_T finite, attach a $|T|$-cube to wW_T. The resulting Σ is a $CAT(0)$ cube complex.

QI classification of a class of RACGs

Theorem (Bounds-Xie). For $i = 1, 2$, let Γ_i be a finite thick generalized m_i-polygon, with $m_i \in \{3, 4, 6, 8\}$. Then W_{Γ_1} and W_{Γ_2} are QI iff Γ_1, Γ_2 are isomorphic.

Proof: Let Σ_i be the Davis complex for W_{Γ_i}. Then Σ_i is a $CAT(0)$ square complex. Σ_i becomes a Fuchsian building after replacing each square in Σ_i with a regular 4-gon in the hyperbolic plane with angles π/m_i. So every QI between Σ_1, Σ_2 lies at a finite distance from an isometry. In particular, Σ_1, Σ_2 are isometric, and so have isomorphic vertex links, which are Γ_1, Γ_2 respectively.