Geometric realizations of Coxeter groups and buildings

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Overview

A building is a union of apartments, and an apartment is a copy of the Coxeter group. We first talk about geometric realizations of Coxeter groups.

Main topics:

1. The basic construction
2. Coxeter complex
3. Geometric reflection groups
4. Davis complex

Some examples

1. Dihedral groups;
2. Euclidean reflection groups;
3. Hyperbolic reflection groups

The basic construction I: Mirror structure

Def. Let (W, S) be a Coxeter system, X a connected, Hausdorff top. space. A mirror structure on X over S is a collection $(X_s)_{s \in S}$, where each X_s is a non-empty, closed subset of X.

The X_s are the mirrors. We always assume $X \neq \cup_{s \in S} X_s$.

Examples:

The idea of the basic construction is to glue $|W|$-many copies of X along mirrors.

The basic construction II

For $x \in X$, let

$$S(x) = \{s \in S | x \in X_s\}.$$

Note that $S(x)$ is empty for some $x \in X$. Define an equivalence relation on $W \times X$:

$$(w, x) \sim (w', x') \iff x = x' \text{ and } w^{-1}w' \in W_{S(x)}.$$

So if $x \in X_s$, then $s \in S(x)$ and $(w, x) \sim (ws, x)$. So if two chambers are s-adjacent, then the corresponding copies of X are glued together via the identity map on X_s.

Equip W with the discrete top. and $W \times X$ with the product top., the basic construction is the quotient

$$\mathcal{U}(W, X) = W \times X / \sim$$

with the quotient top.

Examples:

Coxeter complex

Let (W, S) be a Coxeter system, and X a simplex with codimension-1 faces $\{\Delta_s | s \in S\}$ and mirrors $X_s = \Delta_s$. The corresponding basic construction $\mathcal{U}(W, X)$ is the Coxeter complex.

Example

Coxeter complex in general is not locally finite, for example, for

$$W =< s_1, s_2, s_3 | s_i^2 = 1, (s_1 s_2)^3 = (s_2 s_3)^3 = 1 > .$$

Geometric reflection groups

Let $\mathbb{X}^n$ be $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. A convex polytope $X \subset \mathbb{X}^n$ is a compact intersection of a finite number of closed half spaces in $\mathbb{X}^n$, with nonempty interior. The link of a vertex v is the $(n-1)$-dimensional spherical polytope obtained by intersecting X with a small sphere centered at v. Say X is simple if all its vertex links are simplices.

Theorem. Let X be a simple convex polytope in $\mathbb{X}^n$, $n \geq 2$. Let $\{X_i\}_{i\in I}$ be the collection of codimension-1 faces of X, with each face X_i supported by the hyperplane $\mathcal{H}_i$. Suppose that for all $i \neq j$, if $X_i \cap X_j \neq \emptyset$ then the dihedral angle between X_i and X_j is $\frac{\pi}{m_{ij}}$ for some integer $m_{ij} \geq 2$. Put $m_{ii} = 1$ for every $i \in I$ and $m_{ij} = \infty$ if $X_i \cap X_j = \emptyset$. For each $i \in I$, let s_i be the isometric refelction of $\mathbb{X}^n$ across the hyperplane $\mathcal{H}_i$. Let W be the group generated by $\{s_i\}_{i\in I}$. Then W has the presentation

$$W =< s_i | (s_i s_j)^{m_{ij}} = 1, \forall i, j \in I > .$$

Basic construction and geometric refelction groups

A group W is called a geometric reflection group if W is either a dihedral group or as in the above Theorem. Say W is spherical, Euclidean or hyperbolic if $\mathbb{X}^n$ is $\mathbb{S}^n$, $\mathbb{R}^n$, or $\mathbb{H}^n$.

A building Δ of type (W, S) is called a spherical building, Euclidean building or hyperbolic building if W is a spherical, Euclidean or hyperbolic geometric reflection group. By replacing each chamber of the building with a copy of X, and then gluing two s-adjacent chambers via the identity map on the s-mirrors, we get a geometric realization of Δ. Now each apartment is a copy of $\mathbb{X}^n$.

Davis complex I

Let (W, S) be a Coxeter system. For any subset $T \subset S$, let W_T be the subgroup generated by T.

The nerve L of (W, S) is the simplicial complex with vertex set S, where a subset $T \subset S$ spans a simplex iff W_T is finite. Let L' be the barycentric subdivision of L, and X be the cone over L'. For each $s \in S$, let X_s be the union of closed simplices in L' that contain s. The basic construction corresponding to this mirror structure is the Davis complex.

Σ is locally finite.

Examples

Davis complex as a CW complex

A CW complex structure can be put on Σ inductively as follows. The vertex set is W. Two vertices w_1, w_2 are joined by an edge iff $w_2 = w_1 s$ for some $s \in S$. Hence the 1-skeleton is just the Cayley graph of (W, S). For any $s_i \neq s_j \in S$ satisfying $m_{ij} < \infty$ and any $w \in W$, we attach a 2-cell to the cycle $w, ws_i, ws_is_j, \cdots, ws_is_j \cdots s_i = ws_j, w$. In general, if $w \in W$ and $T \subset S$ is such that W_T is finite, we attach a $(|T| - 1)$ cell to wW_T.

With a suitable metric on this CW-complex, Σ becomes a $CAT(0)$ space. In particular, Σ is contractible.

Davis complex: Right angled case

A Coxeter group (W, S) is right angled if $m_{st} \in \{2, \infty\}$ for any $s \neq t \in S$.

Examples

In this case Σ admits a structure of $CAT(0)$ cube complex. As above, the 1-skeleton of Σ is simply the Cayley graph of (W, S). For any $w \in W$ and any $s \neq t \in S$ with $m_{st} = 2$, attach a square to the 4-cycle $w, ws, wst, wsts = wt, w$ in the Cayley graph. In general, for $w \in W$ and any subset $T \subset S$ with W_T finite, attach a $|T|$-cube to wW_T. The resulting Σ is a $CAT(0)$ cube complex.