Retraction, curvatue aspects of buildings

Xiangdong Xie
Department of Mathematics and Statistics
Bowling Green State University

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University of North Carolina, Greensboro

The plan

1. retractions;
2. curvature aspects of buildings;
3. Spherical building at infinity of Euclidean buildings

Another definition of building

This is historically the first definition. It is equivalent to the one in terms of W-distance metric.

Def. A simplicial complex Δ is a building if it contains a collection of subcomplexes (called apartments) isomorphic to the Coxeter complex of a fixed Coxeter system, that satisfies the following two conditions:
1. Given any two simplices B_1, B_2, there is an apartment that contains both B_1, B_2;
2. Given two apartments A, A' that contain a common chamber there is an isomorphism from A to A' that fixes $A \cap A'$ pointwise.

Example: simplicial trees where each vertex is incident to at least two edges.

Retraction

Let Δ be a building, A an apartment of Δ and c a chamber in A. The retraction $r_{A,c} : \Delta \to A$ is defined as follows. For any chamber c', let A' be an apartment containing both c and c'. Then there is an isomorphism $f : A' \to A$ fixing c pointwise. Define $r_{A,c}|_{c'} = f|_{c'}$.

If c_1, c_2 are adjacent chambers, then either $r_{A,c}(c_1)$ and $r_{A,c}(c_2)$ are adjacent or $r_{A,c}(c_1) = r_{A,c}(c_2)$. So retraction sends galleries to galleries (with possibly repeated chambers).

Equality above is possible, example: trees.

Applications of retraction

Convexity of Apartment: Let A be an apartment, and c, c' two chambers in A. Then every minimal gallery from c to c' lies in A.

Gate property: Let c be a chamber and R a residue. Then there exists a chamber $\tilde{c}$ in R such that $d(c, \tilde{c}) < d(c, D)$ for every chamber D in R different from $\tilde{c}$.

Curvature bounds in metric spaces

Let X be a geodesic metric space. X is called a $CAT(0)$ space if every geodesic triangle in X is at least as thin as in the Euclidean space. One similarly defines $CAT(1)$ and $CAT(-1)$ spaces by comparing triangles with those in the round sphere and real hyperbolic planes.

Fact: Spherical buildings are $CAT(1)$, Euclidean building are $CAT(0)$, hyperbolic buildings are $CAT(-1)$.

Davis complex admits a metric making it a $CAT(0)$ space. Every building also admit a geometric realization (Davis realization) with a $CAT(0)$ metric.

Boundary at infinity of a $CAT(0)$ space

Let X be a $CAT(0)$ space. Two rays in X are equivalent if the distance between them is finite. ∂X is the set of equivalence classes of rays in X.

When X is locally compact, given any $\xi \in \partial X$ and any p, there is a ray starting from p and belonging to ξ.

Examples: Euclidean spaces, other examples(product of trees with Euclidean spaces).

Boundary at infinity of Euclidean buildings

Let Δ be a locally finite Euclidean building. The boundary of each apartment A is a sphere with a triangulation cut out by the finite number family of parallel hyperplanes in A. Each maximal simplex in ∂A will be called an ideal chamber.

Each ray is contained in an apartment. Hence every point in $\partial\Delta$ is contained in a sphere. By considering a ray starting at a chamber c and ending in an ideal chamber S, we see that given any chamber c and any ideal chamber S, there is an apartment A containing c and such that ∂A contains S.

Retraction based on an ideal chamber

Let S be an ideal chamber and A an apartment such that ∂A contains S. We now define a retraction $r_{A,S} : \Delta \to A$ as follows.

For any chamber c, let A' be an apartment containing c and s.t. $\partial A'$ contains S. Then there is an isomorphism $f : A' \to A$ fixing $A \cap A'$ pointwise. Define $r_{A,S}|_c = f|_c$.

Spherical building at infinity of an Euclidean building

As observed above, the ideal boundary of each apartment is a sphere which is a union of ideal chambers, and $\partial\Delta$ is a union of spheres. One can check $\partial\Delta$ satisfies the two conditions of a building, with apartments being ∂A for apartments A of Δ. This is the spherical building at infinity of an Euclidean building.

Condition 2:

Let A_1, A_2 be two apartments so that $\partial A_1 \cap \partial A_2$ contains an ideal chamber S. The retraction $r_{A_1,S} : \Delta \to A_1$ restricted to A_2 is an isomorphism, so induces and isomorphism $\partial A_2 \to \partial A_1$ that fixes $\partial A_1 \cap \partial A_2$ pointwise.

Condition 1 can also be verified.

Note the building $\partial\Delta$ is NOT locally finite when Δ is a thick building.