Buildings Summer School
Exercise Set 1

1. Prove that every J-residue of Δ is a building of type M_J.

2. If $\delta(x, y) = s_f$ with f not necessarily reduced, show there is a gallery of type f from x to y.

3. Let G be a group, B a subgroup, and for each $i \in I$, let there be a subgroup P_i with $B \leq P_i < G$. Take as chambers the left cosets of B, and set $gB \sim_i hB$ if and only if $gP_i = hP_i$.

   (a) Show that this chamber system is connected if and only if $G = \langle P_i \rangle_{i \in I}$.

   (b) An *automorphism* of a chamber system $\mathcal{C}$ is a bijective map on chambers of $\mathcal{C}$ that preserves i-adjacency for each $i \in I$. Let $\mathcal{C}$ be a chamber system admitting G as a group of automorphisms acting transitively on the set of chambers. Given some chamber $c \in \mathcal{C}$, let B denote its stabilizer in G, and let P_i denote the stabilizer of the i-panel on c. Show that $\mathcal{C}$ is the chamber system given by cosets of B as described above.

4. Construct the $A_2(\mathbb{F}_3)$ building.

5. The group $\mathrm{GL}_{n+1}(k)$ of $(n+1) \times (n+1)$ invertible matrices over a field k acts on $(n+1)$-dimensional vector spaces V over k and hence on the building $A_n(k)$.

   (a) Check that this action preserves i-adjacency for each i.

   (b) Show that the stabilizer of a chamber is the subgroup of upper triangular matrices using a suitable ordered basis.

   (c) Show that the subgroup fixing all the chambers of an apartment is the group of diagonal matrices corresponding to a suitable basis. (An apartment can be described as follows: fix a basis $v_1, v_2, \ldots, v_{n+1}$ of V, and take every subspace spanned by a proper subset of this basis, and all nested sequences of such subspaces. The chambers of the apartment are thus all

   $$\left\langle v_{\sigma(1)} \right\rangle \subset \left\langle v_{\sigma(1)}, v_{\sigma(2)} \right\rangle \subset \cdots \subset \left\langle v_{\sigma(1)}, \ldots, v_{\sigma(n)} \right\rangle$$

   where σ ranges through all permutations of $1, \ldots, n+1$.)

6. Let V be a $2n$-dimensional vector space over a field k, with basis $x_1, \ldots, x_n, y_1, \ldots, y_n$, and a bilinear form $(\cdot, \cdot)$ defined by

$$(x_i, y_j) = \delta_{ij} = -(y_j, x_i)$$
$$(x_i, x_j) = 0 = (y_i, y_j).$$

A subspace S is called totally isotropic (t.i.) if $(v, w) = 0$ for all $v, w \in S$; for example $\langle x_1, y_2, y_3 \rangle$.

(a) For any subspace U, let $U^\perp = \{v \in V \mid (v, u) = 0 \quad \forall u \in U\}$. Show that $\dim U + \dim U^\perp = 2n$ and conclude that all maximal t.i. subspaces have dimension n.

(b) Let $I = \{1, \ldots, n\}$ and for each $i \in I$ let S_i denote a t.i. subspace of dimension i. Build a chamber system Δ by taking maximal nested sequences $S_1 \subset S_2 \subset \cdots \subset S_n$ of t.i. subspaces as chambers. As in our $A_n(k)$ example, two chambers $S_1 \subset S_2 \subset \cdots \subset S_n$ and $S_1' \subset S_2' \subset \cdots \subset S_n'$ are said to be i-adjacent if $S_j = S_j'$; for all $j \neq i$. (This is the building $C_n(k)$ as a chamber system.) As before, its geometric realization can be obtained by taking the t.i. subspaces as vertices, and taking all t.i. flags as simplices.

   i. Construct the (labeled) geometric realization of $C_2(\mathbb{F}_2)$.
   ii. Given the basis above, we obtain an apartment by taking every t.i. subspace spanned by a subset of this basis, and all nested sequences of such subspaces. Describe the geometric realization of an apartment of $C_3(k)$.
   iii. What is the type M of the building $C_n(k)$? (Your answer will certainly depend on n.)

7. Let (W, S) be a Coxeter system.

(a) A *reflection* $t \in W$ is a conjugate in W of an element $s \in S$. Verify that t has order 2.

(b) The wall M_t in a Coxeter complex consists of all simplices fixed by t. (Here we are viewing t as an automorphism of the Coxeter complex obtained by left multiplication by t.) We say a gallery crosses M_t if t interchanges c_{i-1} with c_i for some $1 \leq i \leq k$. Show that a minimal gallery cannot cross a wall twice.

(c) Show that for two given chambers x, y, the number of times mod 2 that a gallery from x to y crosses a given wall is independent of gallery.