A BOUND FOR STIELTJES CONSTANTS

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ABSTRACT. The goal of this note is to improve on the currently available bounds for Stieltjes constants using the method of steepest descent applied by Coffey and Knessl to approximate Stieltjes constants.

1. INTRODUCTION

Let $\zeta(s)$ denote the Riemann zeta function (for $s \in \mathbb{C}$ defined by Riemann [17] in 1859). The function $\zeta(s)$ has an Euler product (Euler [5] of 1737) and also satisfies a functional equation (Euler [6] of 1749). In this paper we consider the related Hurwitz zeta function $\zeta(s,a)$ (see Hurwitz [11] of 1882), which for $0 < a \leq 1$ has the Laurent series expansion:

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)}{n!}(s-1)^n,$$

where, for non-negative integers n, the coefficients $\gamma_n(a)$ are known as the Stieltjes constants ([20]), which were generalized to *fractional* values $\alpha \in \mathbb{R}^+$ by Kreminski [12] in 2003. These constants have several interesting and unexpected applications in the zeta function theory, as was shown recently in [4], [3], [8], and [9]. Moreover, the classical Euler-Maclaurin Summation can be used to prove (see [10]) that, if we set $C_\alpha(a) = \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$ and let $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$, then we have:

$$\begin{aligned} C_\alpha(a) \;=\; & \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} \\ & - \sum_{j=1}^{\lfloor v/2 \rfloor} \frac{B_{2j}}{(2j)!} f_\alpha^{(2j-1)}(m) + (-1)^{v-1} \int_m^\infty P_v(x) f_\alpha^{(v)}(x)\, dx, \end{aligned}$$

where the B_j denote the Bernoulli numbers (introduced by Bernoulli in [2] of 1713), and P_v is the v-th periodic Bernoulli function (see [13]). This expression has many useful applications; in our recent work, we have used it to find zero-free regions for the fractional derivatives of the Riemann zeta function. There one of the key estimates (Lemma 4.1, [9]) was the bound, for $0 < \alpha \leq 1$,

$$\left| \int_1^\infty P_3(x) f_\alpha'''(x)\, dx \right| < 0.013.$$

Here, with different goals in mind, we will consider another special case of the above Euler-Maclaurin summation formula. We set $m = 1$ and $v = 2$ and analyze the expression

$$C_\alpha(a) = \frac{\log^\alpha(1+a)}{1+a} - \frac{\log^{\alpha+1}(1+a)}{\alpha+1} - \frac{\log^\alpha(1+a)}{2(1+a)} - \frac{1}{12} f_\alpha'(1) - \int_1^\infty P_2(x) f_\alpha''(x)\, dx. \tag{1}$$

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Now, bounding the generalized fractional Stieltjes constants $\gamma_\alpha(a)$ (or the functions $C_\alpha(a)$), means finding (this time for $1 \le \alpha \in \mathbb{R}$) effective bounds for:

$$\left| \int_1^\infty P_2(x) f_\alpha''(x)\, dx \right|. \tag{2}$$

Since the Bernoulli periodic function $P_2(x)$ involved in the integrals (1) and (2) has a simple Fourier series expansion, established by Hurwitz in 1890 (see [15]) we have

$$P_2(x) = \frac{2}{(2\pi)^2} \sum_{k\neq 0} \frac{e^{2\pi i k x}}{k^2} = \frac{2}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{e^{2\pi i k x} + e^{-2\pi i k x}}{k^2} = \frac{4}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{\Re(e^{2\pi i k x})}{k^2}.$$

It follows that, if we set $S_k := \int_1^\infty e^{2\pi i k x} f_\alpha''(x) dx$ and $S^* := \sup_{k\in\mathbb{N}} |S_k|$, then we can write

$$\left| \int_1^\infty P_2(x) f_\alpha''(x)\, dx \right| = \left| \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} S_k \right| \le \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} |S_k| \le \frac{1}{6} S^*. \tag{3}$$

We obtain effective bounds on $|S_k|$ and S^* by choosing a suitable integration path as done in the method of steepest decent for approximating integrals, see [22, Section II 4] for example. This path will originate at b and then follow a level curve with constant imaginary part that crosses a saddle point and on the right half plane has the asymptote $\frac{\pi}{2} i$.

We find the location of this important saddle point using the Lambert W function (Lambert [14] of 1758), which is the solution of the special case $x = W(x)e^{W(x)}$ of the so-called Lambert transcendental equation (also investigated by Euler in [7] of 1783). Some of these properties will be discussed in the next section.

2. Utility of the Lambert W function

We first rewrite S_k so that it becomes easier to find the saddle point mentioned above. Recall that $f_\alpha(x)$ was defined as $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$, which means that for its first two derivatives we have $f_\alpha'(x) = \frac{\log^{\alpha-1}(x+a)}{(x+a)^2}(\alpha - \log(x+a))$ and

$$f_\alpha''(x) = \frac{\log^{\alpha-2}(x+a)}{(x+a)^3} \left((\alpha-1) - 3\alpha \log(a+x) + 2\log^2(a+x) \right). \tag{4}$$

Now, since $S_k := \int_1^\infty e^{2\pi i k x} f_\alpha''(x) dx$, with the change of variables $y = \log(x+a)$ and $b = \log(1+a)$ we can write

$$\begin{aligned} S_k &= \int_1^\infty e^{2\pi i k x} \frac{\log^{\alpha-2}(x+a)}{(x+a)^3} \left((\alpha-1) - 3\alpha \log(a+x) + 2\log^2(a+x) \right) dx \\ &= \int_b^\infty e^{2\pi i k (e^y - a)} \frac{y^{\alpha-2}}{e^{3y}} \left((\alpha-1) - 3\alpha \cdot y + 2 \cdot y^2 \right) e^y\, dy \\ &= \int_b^\infty e^{2\pi i k (e^y - a) + \alpha \log y} e^{-2y} \frac{(\alpha-1) - 3\alpha \cdot y + 2\cdot y^2}{y^2}\, dy. \end{aligned}$$

Let $h_k(y) = 2\pi i k(e^y - a) + \alpha \log y$ and $q(y) = \frac{(\alpha-1) - 3\alpha\cdot y + 2\cdot y^2}{y^2}$. Then

$$S_k = \int_b^\infty e^{h_k(y)} e^{-2y} q(y)\, dy. \tag{5}$$

The function $h_k(y)$ defined above has a saddle point where $h_k'(y_1) = 2\pi i k e^{y_1} + \alpha / y_1 = 0$ and the Lambert W function tells us that this happens at $y_1 = W\left(\frac{\alpha i}{2\pi k}\right)$. We use the principal branch of

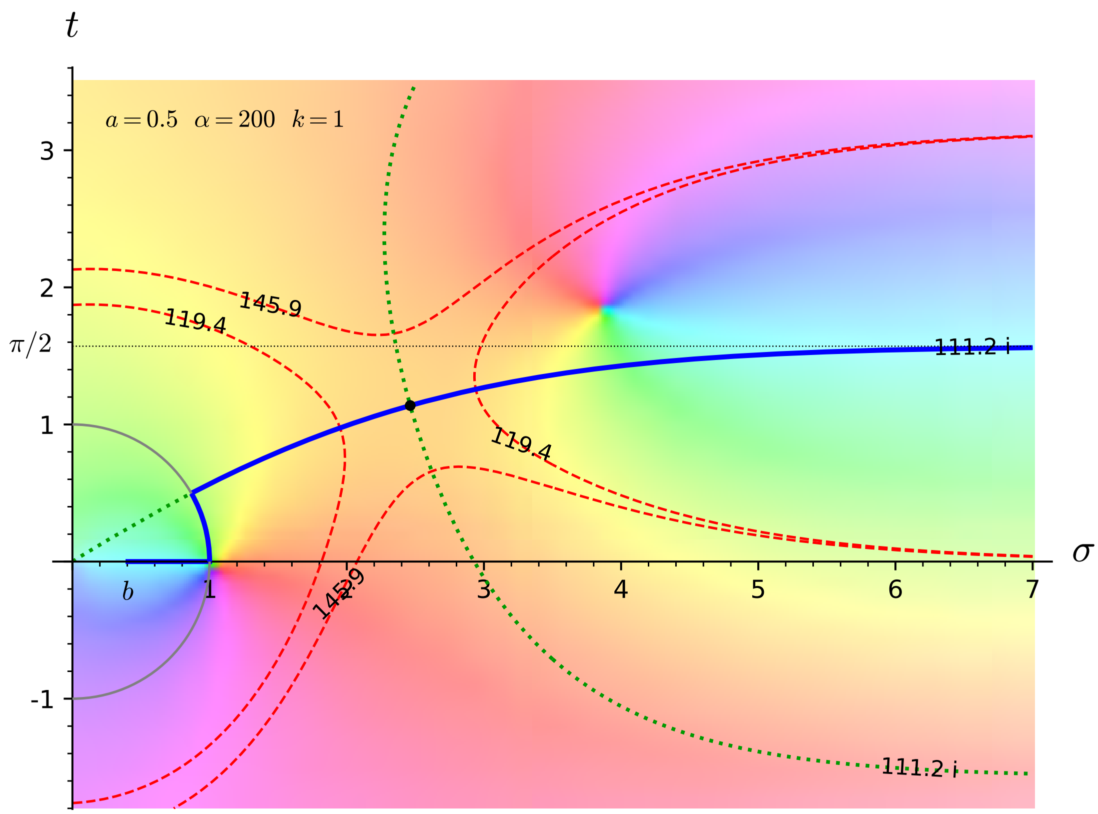

FIGURE 1. The function h_1 for $\alpha = 200$ with the saddle point $\bullet$ near $2.46 + 1.14i$. The dashed red lines are level lines of $\Re(h_1(\sigma + it))$ and the dotted green lines are the level lines of $\Im(h_1(\sigma + it)) = \Im(h_1(w_k(200)))$. The solid blue line is our path of integration.

the Lambert W function and set

$$w_k(\alpha) = W_0\left(\frac{\alpha i}{2\pi k}\right).$$

We make a couple of observations concerning $W_0(it)$ that will be useful later.

Lemma 1. *Let W_0 be the principal branch of the Lambert W function. For $t \in (0, \infty)$, the inverse of $I(t) := \Im(W_0(it))$ is the function $T(y)$, where for $y \in (0, \pi/2)$:*

$$T(y) = \frac{y}{\cos y} \cdot e^{y \cdot \tan(y)}. \tag{6}$$

Proof. Considering the real part of $W_0(it) \cdot e^{W_0(it)} = it$ we get $\Re(W_0(it)) = I(t) \cdot \tan(I(t))$, so that $W_0(it) = I(t)(\tan(I(t)) + i)$. Using this in $W_0(it) \cdot e^{W_0(it)} = it$, and considering only the imaginary parts, we obtain $I(t) \cdot e^{I(t)\tan(I(t))} \frac{1}{\cos(I(t))} = t$. This shows that, for $y \in [0, \pi/2)$, if we set $T(y) = \frac{y}{\cos y} \cdot e^{y \cdot \tan(y)}$, then T is the inverse of I. $\square$

It follows immediately from Lemma 1 that $T(0) = 0$ and $\lim_{y \to \pi/2} T(y) = \infty$ and that

$$T'(y) = \left(\left((\tan^2(y) + 1) \cdot y + \tan(y)\right) \cdot y + y \tan y + 1\right) \frac{e^{y \tan(y)}}{\cos(y)}.$$

Hence $T'(y) > 0$ for $y \in [0, \pi/2)$, and therefore $I(t) > 0$, for $t \in (0, \infty]$. Thus for $t > 0$ we have

$$0 < I(t) < \frac{\pi}{2} \tag{7}$$

and $\lim_{t\to\infty} I(t) = \pi/2$. This implies

$$\Re(W_0(it)) = I(t) \cdot \tan(I(t)) > 0. \tag{8}$$

Taking the logarithm of $W_0(it) \cdot e^{W_0(it)} = it$ and only considering real parts we obtain $\Re(W_0(it)) = \log(t) - \Re(\log W_0(it))$. Thus, for $t > 1.97$ where $|W_0(it)| > 1$,

$$\Re(W_0(it)) < \log(t). \tag{9}$$

We will use the following two lemmas to show that we can set $S^* = |S_1|$.

Lemma 2. *For $t > 0$ we have $\frac{d}{dt}\left(\Re\left(\log(W_0(it)) - \frac{1}{W_0(it)}\right)\right) > 0$.*

Proof. With $W_0'(x) = \frac{W_0(x)}{x\cdot(1+W_0(x))}$ we get $\frac{d}{dt}\Re\left(\log(W_0(it) - \frac{1}{W_0(it)}\right) = \frac{1}{t}\frac{\Re W_0(it)}{|W_0(it)|} > 0$, as wanted. ☐

Lemma 3. *For $k \in \mathbb{N}$ and $\alpha \in [2, \infty)$ we have $k \cdot \sin(\Im(w_k(\alpha))) \geq \frac{1}{2}$.*

Proof. Representing the cosine and tangent functions in (6) by the sine function, and using the well-known fact that $\sin^{-1} x \leq \frac{2}{\pi}x$, for all $x \in \left[0, \frac{\pi}{2}\right]$, we can deduce:

$$\begin{aligned}
T\left(\sin^{-1}\frac{1}{2k}\right) &= \frac{\sin^{-1}\frac{1}{2k}}{\cos\left(\sin^{-1}\frac{1}{2k}\right)} \cdot e^{\left(\sin^{-1}\frac{1}{2k}\right)\cdot\tan\left(\sin^{-1}\frac{1}{2k}\right)} \\
&= \frac{\sin^{-1}\frac{1}{2k}}{\sqrt{1-\sin^2\left(\sin^{-1}\frac{1}{2k}\right)}} \cdot e^{\left(\sin^{-1}\frac{1}{2k}\right)\cdot\frac{\sin\left(\sin^{-1}\frac{1}{2k}\right)}{\sqrt{1-\sin^2\left(\sin^{-1}\frac{1}{2k}\right)}}} \\
&\leq \frac{\frac{2}{\pi}\frac{1}{2k}}{\sqrt{1-\frac{1}{4k^2}}} \cdot e^{\left(\frac{2}{\pi}\frac{1}{2k}\right)\cdot\frac{\frac{1}{2k}}{\sqrt{1-\frac{1}{4k^2}}}} = \frac{1}{2\pi k\sqrt{1-\frac{1}{4k^2}}} \cdot e^{\frac{1}{2\pi k\sqrt{4k^2-1}}} \leq \frac{1}{\pi k}.
\end{aligned}$$

Applying the functions I and sine to both sides of this inequality yields

$$\frac{1}{2k} \leq \sin\left(I\left(\frac{1}{\pi k}\right)\right).$$

Because of the monotonicity of I for $\alpha \geq 2$, we get $\frac{1}{2} \leq k\sin\left(I\left(\frac{\alpha}{2\pi k}\right)\right) = k \cdot \sin\left(\Im(w_k(\alpha))\right)$. ☐

3. Bounding the Integrals

First, recall the definitions of the quantities $|S_k|$ we are interested in:

$$|S_k| := \left|\int_b^\infty e^{h_k(y)} e^{-2y} q(y)\, dy\right|,$$

where (as in Lemma 1) $h_k(y) = 2\pi ik(e^y - a) + \alpha\log y$, $q(y) = \frac{(\alpha-1)-3\alpha\cdot y+2\cdot y^2}{y^2}$, and $b = \log(1+a)$.

We evaluate S_k by integrating along the contour that starts at b, goes along the positive real axis to 1, follows the unit circle until the level line $\Im(h_k(y)) = \Im(h_k(w_k(\alpha))$ is reached, crosses the saddle at $w_k(\alpha)$ (provided it is not inside the unit circle) and continues on the level line. From our observations (7) and (8) we know that $0 < \Im(w_k(\alpha)) < \pi/2$ and $\Re(w_k(\alpha)) > 0$. Setting $\Im(h_k(y)) = \Im(h_k(w_k(\alpha)))$ we see that as $\Re(y) \to \infty$ we have $\Im(y) \to \frac{\pi}{2}$. On the level line, from the origin to the saddle $\Re(h_k(y))$ strictly increases and after crossing it $\Re(h_k(y))$ strictly decreases. See Figures 1 and 2.

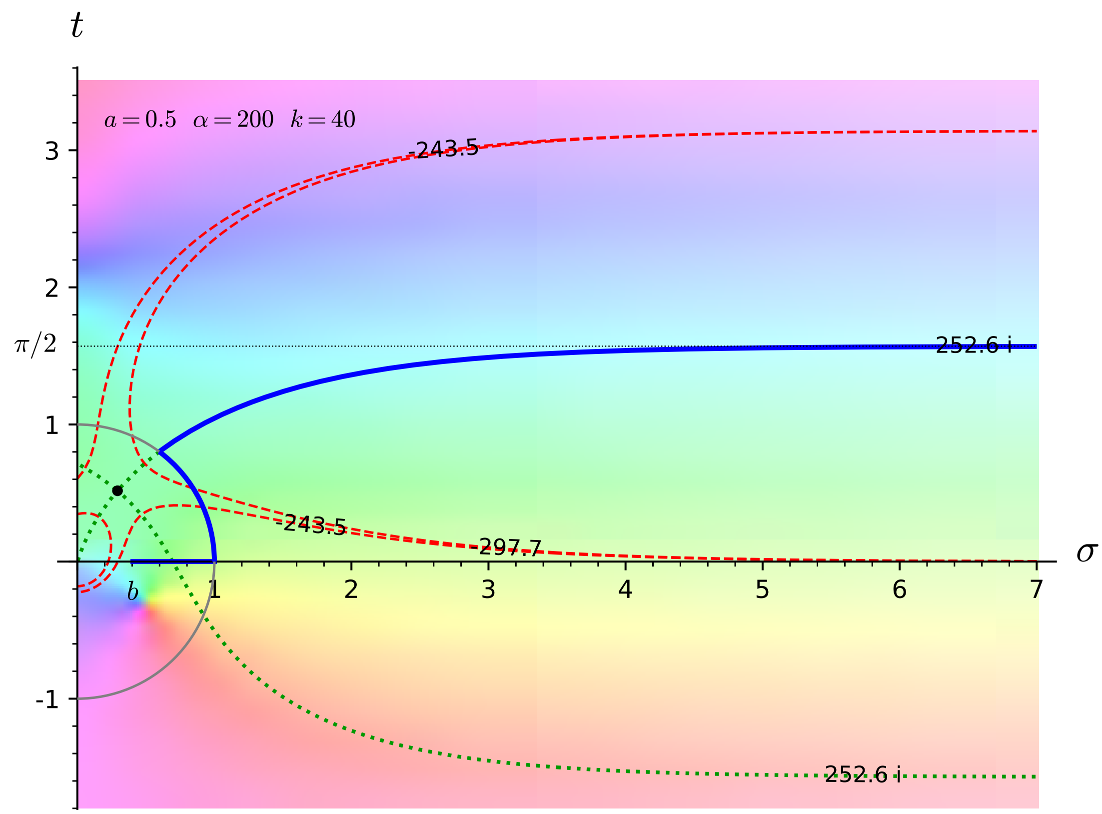

FIGURE 2. The function h_{40} for $\alpha = 200$ with the saddle point $\bullet$ near $0.29 + 0.52i$. The dashed red lines are level lines of $\Re(h_{40}(\sigma + it))$ and the dotted green line is the level line of $\Im(h_{40}(\sigma + it)) = \Im(h_{40}(w_{40}(200)))$. The solid blue line is our path of integration.

Following the segments of the contour we split up the integral into four parts, $S_k = L_1 + L_2 + L_3 + L_4$, where

- $L_1 = \int_b^1 e^{h_k(y)} e^{-2y} q(y) dy$,
- $L_2 = \int_1^u e^{h_k(y)} e^{-2y} q(y) dy$ along the unit circle from 1 to the point u where the unit circle and the level line $\Im(y) = \Im(h_k(w_k(\alpha))$ meet,
- $L_3 = \int_u^v e^{h_k(y)} e^{-2y} q(y) dy$ along the level line until the point v with $\Re(v) = 2\log\alpha$, if $\Re(h_k(v)) > \Re(h_k(u))$, otherwise $v = u$ and $L_3 = 0$,
- $L_4 = \int_v^\infty e^{h_k(y)} e^{-2y} q(y) dy$

In the following we estimate each of these four components separately.

Lemma 4. *With L_1 defined as above, we have*

$$|L_1| < 5.$$

Proof. Let $y \in \mathbb{R}^+$. First, let us note that

$$\Re(h_k(y)) = \Re(2\pi ik(e^y - a) + \alpha \log(y)) = -2\pi k e^{\Re(y)} \sin \Im(y) + \alpha \log |y| \tag{10}$$

and because $|e^{-2y}| \leq 1$, we can write:

$$|L_1| = \left| \int_b^1 e^{h_k(y)} e^{-2y} q(y)\, dy \right| \leq \int_b^1 |e^{h_k(y)}| \cdot |q(y)|\, dy$$

$$\le \int_b^1 e^{-2\pi k e^{\Re(y)}\sin\Im(y)+\alpha\log|y|}\cdot|q(y)|\,dy \;\le\; \int_b^1 e^{\alpha\log y}\cdot|q(y)|\,dy$$
$$\le \int_b^1 y^\alpha\left(\frac{\alpha-1}{y^2}+\frac{3\alpha}{y}+2\right)dy \;=\; \left[y^{\alpha-1}+3y^\alpha+\frac{2y^{\alpha+1}}{\alpha+1}\right]_b^1$$
$$= \left(4+\frac{2}{\alpha+1}\right)-\left(\log(a+1)^{\alpha-1}+3\log(a+1)^\alpha+\frac{2\log(a+1)^{\alpha+1}}{\alpha+1}\right) \;<\; 5. \qquad \square$$

Lemma 5. *With L_2 defined as above, we have*

$$|L_2| < (4\alpha+1)\frac{\pi}{2}.$$

Proof. First observe that on the unit circle we have $|y|=1$ and $0\le\Im(y)\le 1$, and thus, by (10),

$$\Re(h_k(y)) = -2\pi k e^{\Re(y)}\sin\Im(y)+\alpha\log|y| = -2\pi k e^{\Re(y)}\sin\Im(y)\le 0. \tag{11}$$

Moreover, we can write

$$|q(y)| \;\le\; \left|\frac{\alpha-1}{y^2}\right|+\left|\frac{3\alpha}{y}\right|+2 \;=\; \frac{\alpha-1}{|y^2|}+\frac{3\alpha}{|y|}+2 \;=\; 4\alpha+1.$$

Since replacing u by i can only extend the path of integration, the last bound directly gives:

$$|L_2| \;\le\; \int_1^i \left|e^{h_k(y)}e^{-2y}q(y)\right|\,dy \;\le\; \int_1^i |e^{h_k(y)}|\cdot|e^{-2y}|\cdot|q(y)|\,dy \;=\; (4\alpha+1)\frac{\pi}{2}. \qquad \square$$

Lemma 6. *With L_4 defined as above, we have*

$$|L_4| < 4\alpha+1.$$

Proof. The remainder of our integration path follows the level line with $\Im(y)=\Im(w_k(\alpha))$. Here we have

$$\left|\int e^{h_k(y)}e^{-2y}q(y)dy\right| \le (4\alpha+1)\int |e^{h_k(y)-2y}|dy = (4\alpha+1)\int e^{\Re(h_k(y))}dy \tag{12}$$

By (9) we have $\Re(w_k(\alpha))<\log(\alpha)$. So v lies to the right of the saddle. For $\Re(y)\ge 2\log\alpha$ we have

$$\Re(h_k(y)) = -2\pi k e^{\Re(y)}\sin\Im(y)+\alpha\log|y| \le -\pi e^{\Re(y)}+\alpha\log|y| < 0. \tag{13}$$

To see this, just note that in our region $\Re(y) > 4\log\log\Re(y)$ and thus also $\Re(y)/2+\log\pi > 2\log\log\Re(y)+\log 2 > \log\log(2\Re(y)^2) = \log\log(\Re(y)^2+\Im(y)^2) = \log\log|y|$, due to the concavity of both the logarithmic and the Lambert W functions. Plugging (13) into (12) yields:

$$|L_4| \le (4\alpha+1)\int_c \left|e^{-\pi e^{\Re(y)}+\alpha\log|y|-2\Re(y)}\right| e^{\Re(-2y)}dy \le (4\alpha+1)\int_c e^{-2y}dy < 4\alpha+1. \qquad \square$$

Lemma 7. *With L_3 defined as above, we have*

$$|L_3| < e^{\Re\left(-\frac{\alpha}{w_k(\alpha)}+\alpha\log w_k(\alpha)\right)}(4\alpha+1)\sqrt{4\log^2\alpha+\pi^2/4},$$

Proof. Here the curve c has its (real) maximum at the saddle point $w_k(\alpha)$, where $h_k(w_k(\alpha)) = 2\pi i k e^{w_k(\alpha)}+\alpha/w_k(\alpha)=0$. This allows us to bound the real part of $h_k(y)$ as:

(14)
$$\Re(h_k(y)) \le \Re(h_k(w_k(\alpha))) = \Re\left(2\pi i k(e^{w_(\alpha)}-a)+\alpha\log w_k(\alpha)\right) = \Re\left(-\frac{\alpha}{w_k(\alpha)}+\alpha\log w_k(\alpha)\right),$$

which means that we can estimate the integral as:

$$|L_4| = \left|\int_u^v e^{h_k(y)}e^{-2y}q(y)\,dy\right| \le e^{\Re\left(-\frac{\alpha}{w_k(\alpha)}+\alpha\log w_k(\alpha)\right)}(4\alpha+1)\int_u^v dy$$

$$\le e^{\Re\left(-\frac{\alpha}{w_k(\alpha)}+\alpha\log w_k(\alpha)\right)}(4\alpha+1)\int_0^{2\log\alpha+\pi/2} dy$$
$$= e^{\Re\left(-\frac{\alpha}{w_k(\alpha)}+\alpha\log w_k(\alpha)\right)}(4\alpha+1)\sqrt{4\log^2\alpha+\pi^2/4}. \qquad \square$$

Putting these four Lemmas together, we immediately get the following bound:

$$\begin{aligned} |S_k| = & \ |L_1|+|L_2|+|L_3|+|L_4| \\ < & \ 4\alpha+6+(4\alpha+1)\frac{\pi}{2}+e^{\Re\left(-\frac{\alpha}{w_k(\alpha)}+\alpha\log w_k(\alpha)\right)}(4\alpha+1)\sqrt{4\log^2\alpha+\pi^2/4}. \end{aligned} \tag{15}$$

4. The Final Bound

Now we can prove the following general result:

Theorem 8. *For $\alpha \ge 3$ and $a \in (0,1]$ let us denote by $\gamma_\alpha(a)$ the fractional Stieltjes constants and write $C_\alpha(a) = \gamma_\alpha(a) - \frac{\log^\alpha a}{a}$. If we set $w(\alpha) := W_0\left(\frac{\alpha i}{2\pi}\right)$, where W_0 is the principal branch of the Lambert W function, then*

$$|C_\alpha(a)| \ < \ 2+\alpha+3\alpha\log\alpha\cdot\left|e^{\alpha(\log w(\alpha)-1/w(\alpha))}\right|.$$

Proof. It follows from Lemmas 2 and 6 that the quantities S_k decrease as k increases. Thus we can set $S^* = |S_1|$ in (3), and with the help of the bound (15) we can rewrite (1) as:

$$\begin{aligned} |C_\alpha(a)| &< \log^\alpha(1+a)\left|\frac{1}{2}\frac{1}{1+a}-\frac{\log(1+a)}{\alpha+1}+\frac{1}{12(1+a)^2}\right|+\left|\frac{1}{12}\frac{\log^{\alpha-1}(1+a)}{(1+a)^2}\alpha\right|+\frac{1}{6}|S_1| \\ &< 1+\frac{1}{12}\alpha+\frac{1}{6}\left(4\alpha+6+(4\alpha+1)\left(2\log\alpha+\frac{\pi}{2}\right)\cdot e^{\Re\left(-\frac{\alpha}{w_1(\alpha)}+\alpha\log w_1(\alpha)\right)}\right) \\ &< 2+\alpha+\frac{1}{6}\left(\frac{13}{3}\alpha\right)\left(\frac{7}{2}\log\alpha\right)\cdot e^{\Re\left(-\frac{\alpha}{w_k(\alpha)}+\alpha\log w_k(\alpha)\right)}. \qquad \square \end{aligned}$$

Note that the main term of the bound in Theorem 8 differs only by a factor of $\alpha\log\alpha$ from the conjectured bound given in [10]:

$$|C_\alpha(a)| \le 2\left|e^{\alpha(\log w(\alpha)-1/w(\alpha))}\right|. \tag{16}$$

In Figure 3 we compare Theorem 8 and (16) with previously known bounds. For $m \in \mathbb{N}$ we have:

(1) the bound by Berndt [1]: $|\gamma_m| \le \frac{(3+(-1)^m)(m-1)!}{\pi^m}$
(2) the bound by Williams and Zhang [21]: $|\gamma_m| \le \frac{(3+(-1)^m)(2m)!}{m^{m+1}(2\pi)^m}$
(3) the bound by Matsuoka [16] which holds for $m > 4$: $|\gamma_m| < 10^{-4}(\log m)^m$
(4) the bound by Saad Eddin [19]: Let $\theta(m) = \frac{m+1}{\log\frac{2(m+1)}{\pi}} - 1$ then
$$|\gamma_m| \le m!\cdot 2\sqrt{2}e^{-(n+1)\log\theta(m)+\theta(m)\left(\log\theta(m)+\log\frac{2}{\pi e}\right)}\left(1+2^{-\theta(m)-1}\frac{\theta(m)+1}{\theta(m)-1}\right).$$
(5) our bound from [10]: For $\alpha > 0$ let $x = \frac{\pi}{2}e^{W_0\left(\frac{2(\alpha+1)}{\pi}\right)}$ then
$$|\gamma_\alpha| \le \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}}\frac{(2(n+1))!}{(n+1)!} \text{ where } n = \begin{cases} \lfloor x \rceil & \text{if } x < \alpha \\ \lceil \alpha - 1 \rceil & \text{otherwise} \end{cases}$$

Remark. With (9) we get $\Re(\log(w(\alpha)) - 1/w(\alpha)) < \Re(\log w(\alpha)) < \log\log(\alpha)$. Hence

$$\left|e^{\alpha(\log w(\alpha)-1/w(\alpha))}\right| = e^{\alpha\Re(\log w(\alpha)-1/w(\alpha))} < (\log\alpha)^\alpha$$

Thus the main term of Matsuoka's bound follows from (16).

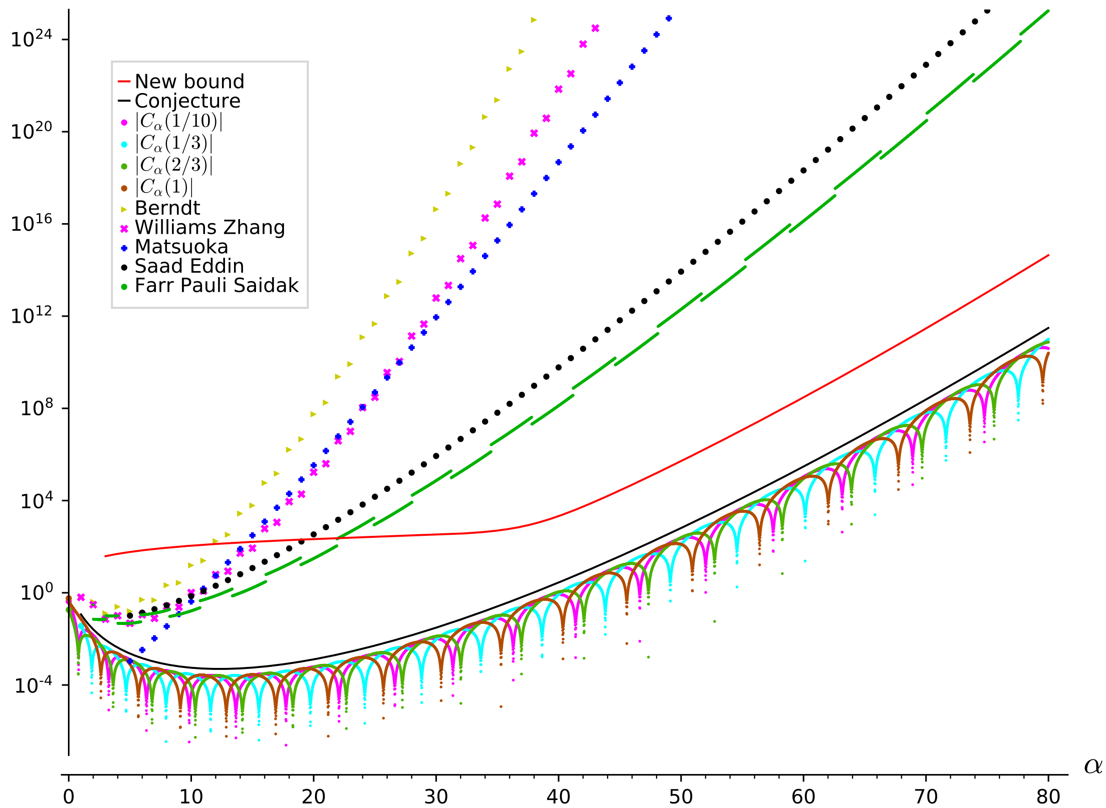

FIGURE 3. On a logarithmic scale we show the absolute values of the Stieltjes constants $\gamma_\alpha = C_\alpha(1)$, $C_\alpha(2/3)$, $C_\alpha(1/3)$ and $C_\alpha(1/10)$ along with the bounds by Berndt [1], Williams and Zhang [21], Matsuoka [16] and Saad Eddin [19], our bound and the conjecture from [10] as well as our new bound from Theorem 8.

5. Acknowledgments

All plots were created with the computer algebra system SageMath [18].

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