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Zero-Free Regions of the Fractional Derivatives of the Riemann Zeta Function

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Dedicated to the memory of Andrzej Schinzel (1937–2021)

Abstract. We generalize our zero-free regions of the integral derivatives for the Riemann zeta function to the general fractional derivatives case, and then we apply them to formulate a more precise description of the previously observed chains of zeros of derivatives.

1 Introduction

Let us start by briefly recalling some simple facts. In 1737, Euler [6] showed that, for real $s > 1$, one can write:

$$\zeta(s) \;:=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the Euler product on the right side is extended over all primes. In 1859, Riemann [23] generalized the definition of $\zeta(s)$ to complex values of s, and showed how, by a process of analytic continuation, it can be extended to a meromorphic function, with a single pole at $s = 1$.

The idea was made more precise by Stieltjes [27], who explicitly computed the very versatile Laurent series expansion of $\zeta(s)$:

$$\zeta(s) \;=\; \frac{1}{s-1} \;+\; \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n, \tag{1.1}$$

where $\gamma_0 := \lim_{x\to\infty}\left(\sum_{n\le x} \frac{1}{n} - \log x\right) = 0.57721\ldots$ is the well-known Euler constant (see Euler's [5] of 1734) and for $n \geq 1$, the Stieltjes constants γ_n can be written as (see Berndt's [1])

$$\gamma_n = \lim_{m\to\infty}\left\{\sum_{k=1}^{m+1} \frac{\log^n k}{k} - \frac{\log^{n+1}(m+1)}{n+1}\right\}. \tag{1.2}$$

Derivatives

Now, for all $k \in \mathbb{N}$, the derivatives $\zeta^{(k)}(s)$ of the Riemann zeta function, for $s \in \mathbb{C}$ with $\Re(s) > 1$, are

$$\zeta^{(k)}(s) \;=\; (-1)^k \sum_{n=1}^{\infty} \frac{(\log n)^k}{n^s}, \tag{1.3}$$

since

$$\frac{d(1/n^s)}{ds} = \frac{d(e^{-s\log n})}{ds} = \frac{d(-s\log n)}{ds} e^{-s\log n} = \frac{-\log n}{n^s},$$

so that every new derivative with respect to s introduces an extra factor of $(-\log n)$. Similar to the Riemann zeta function itself, all $\zeta^{(k)}(s)$ can be extended to meromorphic functions with a single pole at $s = 1$; however, unlike $\zeta(s)$, these derivatives have neither Euler products nor functional equations. As a result, their nontrivial zeros do not lie on a line, but appear to be distributed seemingly at random, the majority of them located to the right of the critical line $\sigma = \frac{1}{2}$ (cf. [26]).

However, within the apparent randomness of the distribution of zeros of $\zeta^{(k)}(s)$, certain intriguing patterns and structures can be detected. As we have shown in our [2], for sufficiently large values of k we have: a) an increasing number of zero-free regions in the right half-plane, with surprising vertical periodicity of the zeros located in the strips between them; and b) with the increasing integer-valued k, the zeros seem to transition (in an almost periodic fashion, see Figure 1) to the left, creating a lattice-like grid. There seems little doubt that this ‘movement’ between the zeros of high derivatives is continuous (as we have conjectured in [2]), however that means that, in order to describe and investigate this intriguing phenomenon, the behavior of the fractional derivatives needs to be understood first.

Fractional Derivatives

There are several definitions of fractional derivatives, some of which have been applied in the theory of zeta functions. In 1975 Keiper [16] proved that the Hurwitz zeta functions can be expressed as fractional derivatives (see [24] and [20]) of the logarithmic derivative of $\Gamma(s)$, also known as the digamma function; and this work that was recently generalized to the Lerch zeta functions by Fernandez [12]. Both authors work with the Riemann-Liouville definition of fractional derivatives. In 2015 Guariglia [14] considered Caputo fractional derivatives (see [4]) of the Riemann zeta function, but in later work [15] employed the Grünwald-Letnikov fractional derivative.

Independently, we found that the Grünwald-Letnikov fractional derivative was best suited for proving a conjecture of Kreminski [17], originally formulated in terms of the Weyl fractional derivative (see [29]). It is shown (in [7] and [8]) that the fractional Stieltjes constants, defined as the generalization of (1.2) to $\alpha \in (0, \infty)$, via

$$\gamma_\alpha = \lim_{m\to\infty} \left\{ \sum_{k=1}^{m+1} \frac{\log^\alpha k}{k} - \frac{\log^{\alpha+1}(m+1)}{\alpha+1} \right\},$$

are the coefficients of a natural generalization of the Laurent expansion (1.1) to the Grünwald-Letnikov fractional derivatives

$$D_s^\alpha \left[\zeta(s)\right] = (-1)^\alpha \left(\frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=1}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}}{n!} (s-1)^n \right). \tag{1.4}$$

With this choice of a fractional derivative we have found (in [11] and [22]) new bounds for the fractional Stieltjes constants that also yield improved bounds for the classical Stieltjes constants. With these bounds we established a zero-free region of the fractional derivative of ζ near the pole $s = 1$ (see [9], Theorem 5.1): For all $\alpha \geq 0$, $D_s^{\alpha}[\zeta(s)] \neq 0$ in the region $|s-1| < 1$.

Now, continuing our work, we investigate the Grünwald-Letnikov fractional derivatives $D_s^{\alpha}[\zeta(s)]$, (with continuous $\alpha \in \mathbb{R}$) on the right half-plane, with the goal of generalizing the zero-free regions from [2] (see Figure 2). The main result is proved by generalizing the rectangular regions, that contain exactly one zero (see Figure 3). As a corollary we obtain the existence of continuous curves of zeros of fractional derivatives.

The Grünwald-Letnikov Fractional Derivative

Everywhere below, we employ the *reverse* α^{th} *Grünwald-Letnikov derivative* of a function $f(z)$, which is defined, for any $\alpha \in \mathbb{C}$, as

$$D_s^{\alpha}[f(s)] = \lim_{h \to 0^+} \frac{(-1)^{\alpha} \sum\limits_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(s+kh)}{h^{\alpha}}, \tag{1.5}$$

whenever the limit exists, where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$, and the gamma function $\Gamma(z) := \int_0^{\infty} x^{z-1} e^{-x}\, dx$, for all $\Re(z) > 0$. This derivative was introduced by Grünwald [13] in 1867 and simplified by Letnikov in 1869 (see [19] and [18]). Defined this way, the fractional derivatives $D_s^{\alpha}[f(s)]$ coincide with the standard derivatives for all $\alpha \in \mathbb{N}$ and one has:

(a) $D_s^{\alpha}[c] = 0$, for all constants $c \in \mathbb{C}$.

(b) $D_s^{0}[f(s)] = f(s)$.

(c) $D_s^{\alpha}\left[D_s^{\beta}[f(s)]\right] = D_s^{\alpha+\beta}[f(s)]$, for all $\alpha, \beta \in \mathbb{C}$.

(d) $D_s^{\alpha}[e^{ms}] = m^{\alpha} e^{ms}$, for $m \neq 0$.

Properties (a) and (d) yield a fractional generalization of (1.3) to all $\alpha > 0$ for any $s \in \mathbb{C}$ with $\Re(s) > 1$:

$$\zeta^{(\alpha)}(s) = D_s^{\alpha}[\zeta(s)] = (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n+1)}{n^s}. \tag{1.6}$$

As a direct consequence of the Laurent expansion (1.4) of the fractional derivatives we obtain:

(a) The branch cut of the complex logarithm creates a discontinuity in $D_s^{\alpha}[\zeta(s)]$ along $(-\infty, 1]$, for all $\alpha \notin \mathbb{N}$.

(b) $D_s^{\alpha}[\zeta(s)]$ is analytic on $\mathbb{C} \setminus (-\infty, 1]$; it is a continuous function of both s and $\alpha > 0$.

(c) If $\sigma \in (1, \infty)$ and $\alpha \notin \mathbb{N}$, then $D_{\sigma}^{\alpha}[\zeta(\sigma)]$ is non-real.

(d) For $s \in \mathbb{C} \setminus (-\infty, 1]$, we have $D_{\sigma}^{\alpha}[\zeta(\overline{s})] = (-1)^{2\alpha}\overline{D_{\sigma}^{\alpha}[\zeta(s)]}$.

Properties (c) and (d) describe the symmetry of *locations* of the zeros of $D_{\sigma}^{\alpha}[\zeta(s)]$ in $\mathbb{C}$, with respect to the real axis, but not the actual mirroring of properties or the related dynamics.

Note

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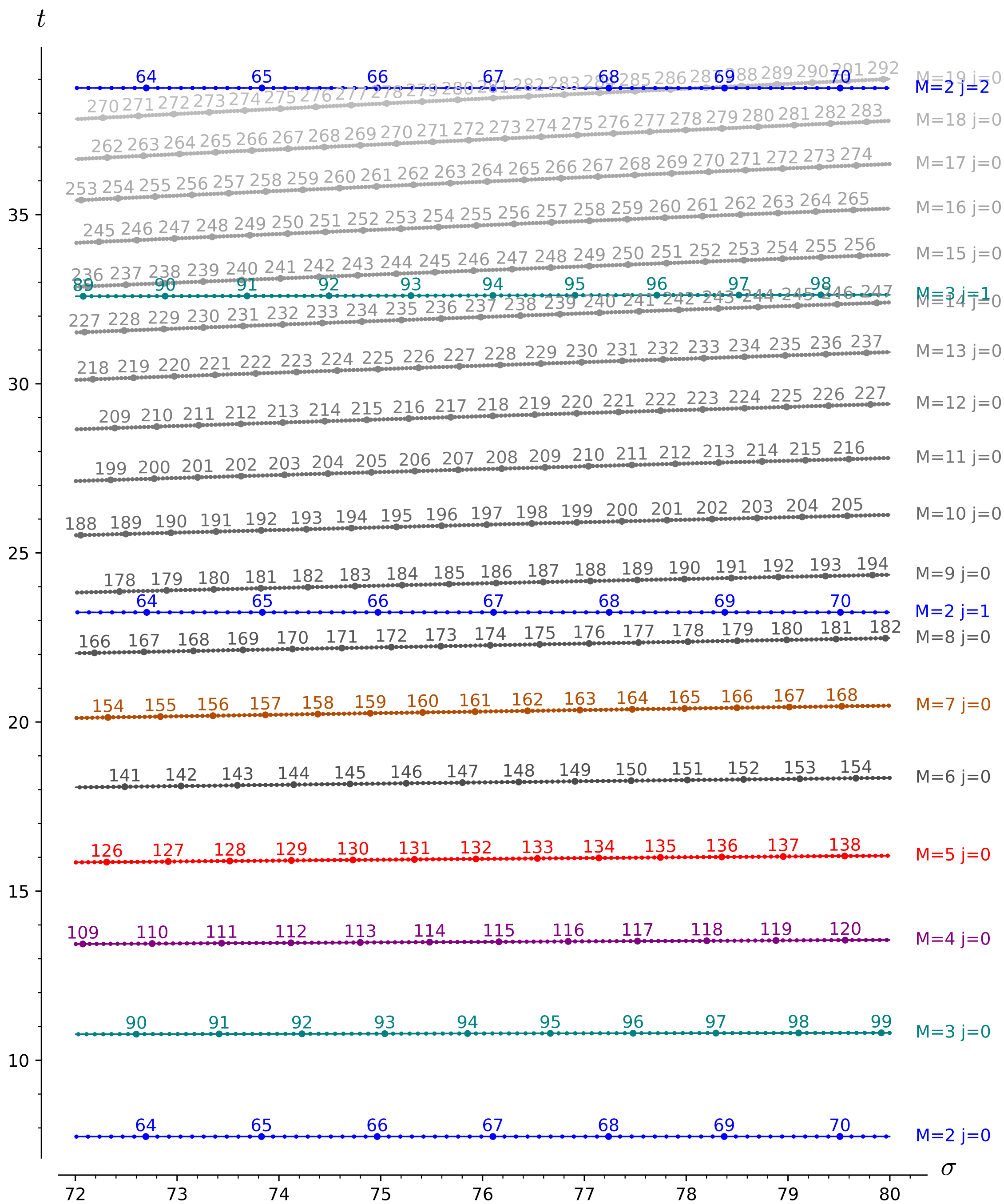

Figure 1. Paths of the zeros of the fractional derivatives $\zeta^{(\alpha)}(\sigma + it)$, for $72 < \sigma < 80$ and $0 < t < 40$. The zeros of the integral derivatives $\zeta^{(k)}$ are denoted by $\bullet^k$ and the labels of the curves reflect the rectangles in which the zeros can be found for sufficiently large α (see Theorem 2).

2 Statement of Main Results

Let $Q_n^\alpha(s) := (\log n)^\alpha / n^s$ denote the n-th term of the Dirichlet series for $(-1)^\alpha \zeta^{(\alpha)}(s)$, so that

$$(-1)^\alpha \zeta^{(\alpha)}(s) = \sum_{n=2}^{\infty} \frac{\log^\alpha n}{n^s} = \sum_{n=2}^{\infty} Q_n^\alpha(s). \tag{2.1}$$

We prove the existence of zero-free regions where one of the terms of (2.1), say $Q_M^\alpha(\sigma)$, dominates the rest of the series, that is, when

$$Q_M^\alpha(\sigma) > \sum_{n \neq M} Q_n^\alpha(\sigma), \tag{2.2}$$

and, in a complementary fashion, we look for the zeros of $\zeta^{(\alpha)}(s)$ near the regions of the complex plane where $Q_M^\alpha(s) = Q_{M+1}^\alpha(s)$, in other words where no term of the series can attain dominance and, in fact, where the cancellation of terms might happen. This occurs at

$$q_M := \frac{\log\left(\frac{\log M}{\log(M+1)}\right)}{\log\left(\frac{M}{M+1}\right)}. \tag{2.3}$$

Our main goal is to prove a generalization of [2, Theorem 2.1]:

Theorem 1. *Let* $\alpha > 0$. *We have (see Figure 2):*

(a) For all $\sigma > q_2\alpha + 2.6$, *we have* $\zeta^{(\alpha)}(s) \neq 0$.

(b) If $q_3\alpha + 4\log 3 < q_2\alpha - 2$, *then* $\zeta^{(\alpha)}(s) \neq 0$ *for*

$$q_3\alpha + 4\log 3 \leq \sigma \leq q_2\alpha - 2.$$

(c) If $M \in \mathbb{N}$, $M > 3$, *and* $q_M\alpha + (M+1)u \leq q_{M-1}\alpha - Mu$, *then* $\zeta^{(\alpha)}(s) \neq 0$ *in the regions*

$$q_M\alpha + (M+1)u \leq \sigma \leq q_{M-1}\alpha - Mu,$$

where $u \in (0, \infty)$ *is a solution of* $1 - \frac{1}{e^u - 1} - \frac{1}{e^u}(1 + \frac{1}{u}) \geq 0$.

Note: The value of $u \in (0, \infty)$ that gives us the widest zero-free regions is $u = 1.1879426249\ldots$, which is the solution of the equation

$$1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) = 0. \tag{2.4}$$

Let S_M^α be the vertical strip between the zero-free regions obtained from the dominance of $Q_M^\alpha(q_M\alpha)$ and $Q_{M+1}^\alpha(q_M\alpha)$ in (2.1), respectively, as described in Theorem 1. The strip S_M^α exists when α reaches

$$A_M := \begin{cases} \frac{4\log 3 + 2}{q_2 - q_3} & \text{if } M = 2 \\ \frac{(2M+3)u}{q_M - q_{M+1}} & \text{if } M > 2. \end{cases}$$

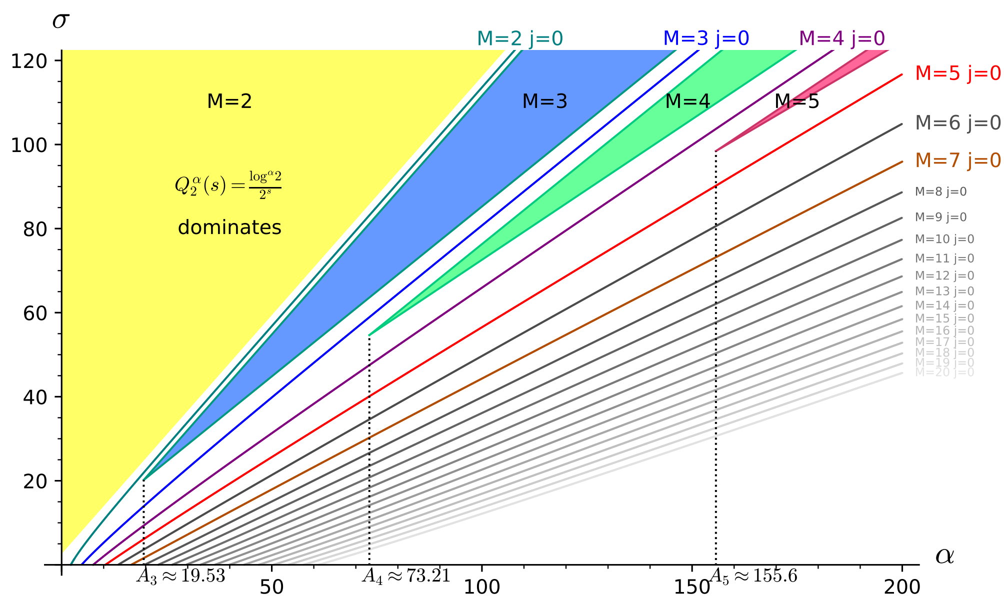

Figure 2. Graphical representation of the main results of Theorem 1. The triangles are the zero-free regions where $Q_M^\alpha(\sigma+it)$ dominates $\zeta^{(\alpha)}(\sigma+it)$. The curves are made up of zeros of the fractional derivatives of ζ and are labeled by the zero free region to their right.

Recall that $Q_M^\alpha(q_M\alpha) = Q_{M+1}^\alpha(q_M\alpha)$. Considering the imaginary parts of the solutions of $Q_M^\alpha(q_M\alpha + it) = Q_{M+1}^\alpha(q_M\alpha+it)$ we find that $\zeta^{(\alpha)}(\sigma+it) \neq 0$ for $\sigma \in S_M^\alpha$ and

$$t = \frac{2\pi J}{\log(M+1)-\log(M)} \tag{2.5}$$

for $J \in \mathbb{Z}$. Together with the border of the zero free regions to the left and right of S_M^α the lines from (2.5), for $J = j$ and $J = j+1$, where $j \in \mathbb{Z}$ form a contour around the zero

$$q_M \cdot \alpha + \frac{\pi(2j+1)}{\log(M+1)-\log(M)}\, i \tag{2.6}$$

of $Q_M^\alpha(q_M\alpha+it) + Q_{M+1}^\alpha(q_M\alpha+it)$. Exactly as in [2], Rouché's theorem immediately shows that there is exactly one zero of $\zeta^{(\alpha)}$ in the rectangular area shown in Figure 3. In other words, a natural generalization of [2, Theorem 2.2] can be quickly obtained, *mutatis mutandis*, replacing integer values of k by positive real numbers α:

Theorem 2. *Let $M \geq 2$ denote a natural number, $j \in \mathbb{Z}$, and $\alpha > A_M$. Let $F_{M,j}^\alpha \subset S_M^\alpha$ be given by*

$$\frac{2\pi j}{\log(M+1)-\log(M)} < t < \frac{2\pi(j+1)}{\log(M+1)-\log(M)}. \tag{2.7}$$

Then $F_{M,j}^\alpha$ contains exactly one zero of $\zeta^{(\alpha)}(s)$, and the zero is simple.

Computations conducted with the methods from [10] suggest that the zeros in the regions $F_{M,j}^\alpha$ form continuous, mostly horizontal curves. We observe that the curves of zeros of fractional derivatives passing

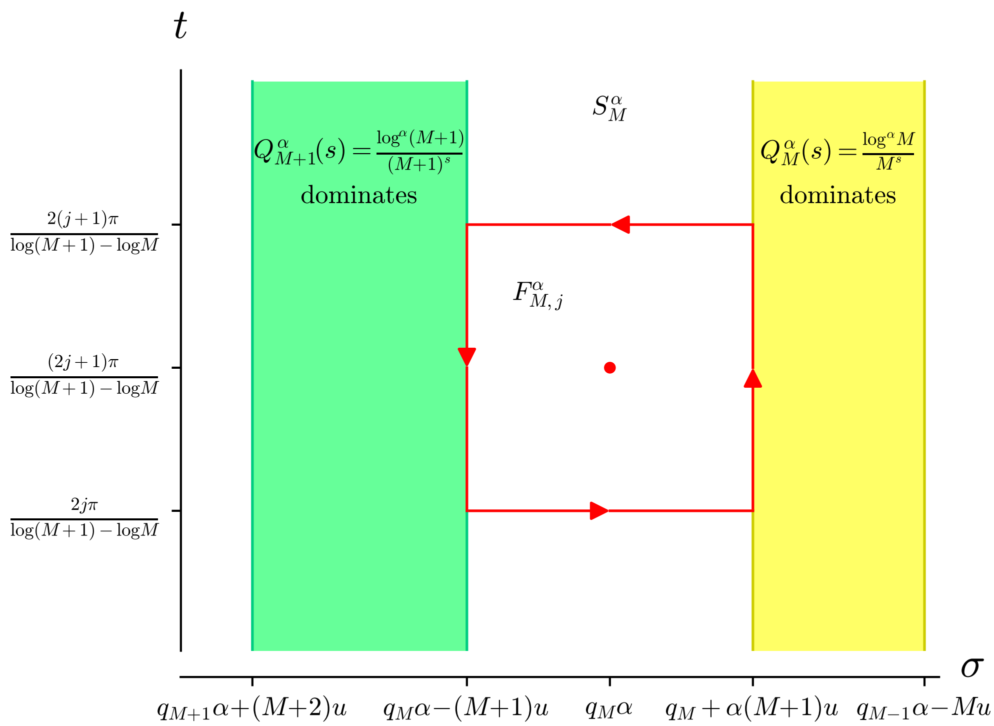

Figure 3. Regions $F_{M,j}^\alpha$ that contains exactly one zero of $\zeta^{(\alpha)}(\sigma + it)$. Rouché's theorem can be used to establish simplicity of the zero using the zero of $Q_M^\alpha(s) + Q_{M+1}^\alpha(s)$ at $\bullet$.

through the regions $F_{M,j}^\alpha$ with $j > 0$ end at a zeros of $\zeta(s) - 1$ where $-\frac{1}{2} < \Re(s) < 1.9402$ ([3, Proposition 7] and [25, Theorem 1] respectively), while curves passing through the regions $F_{M,0}^\alpha$ appear to continue over to the left half plane – see Figure 4.

Far enough to the right the existence of these curves follows from Theorem 2: Let $M \in \mathbb{Z}$, $M \geq 2$ and $\alpha > A_M$ so that S_M^α is non empty. Then for each $j \in \mathbb{Z}$ there is $s = \sigma + it \in F_{M,j}^\alpha$ such that $\zeta^{(\alpha)}(s) = 0$. As s is a simple zero of $\zeta^{(\alpha)}(s)$ we have that $\zeta^{(\alpha+1)}(s) \neq 0$. By the implicit function theorem there is an analytic function z defined on an open neighborhood $U \subset \mathbb{C}$ of α such that $\zeta^{(\beta)}(z(\beta)) = 0$ for $\beta \in U$. As this holds for all $\alpha > A_M$ we obtain a function z that is analytic on an open neighborhood of (A_M, ∞) in $\mathbb{C}$ and thus analytic on (A_M, ∞).

Corollary 1. *Let $M \in \mathbb{N}$ with $M \geq 2$ and $j \in \mathbb{Z}$. The zeros $s = \sigma + it$ of $\zeta^{(\alpha)}(s)$ for $\alpha > A_M$ with*

$$\frac{2\pi j}{\log(M+1) - \log(M)} < t < \frac{2\pi(j+1)}{\log(M+1) - \log(M)}$$

are images of an analytic function $z : (A_M, \infty) \to \mathbb{C}$.

3 Preliminary Lemmas

In our proof of Theorem 1 we follow, with some modifications, the general approach developed in order to establish [2, Theorem 2.1]. We show that $\zeta^{(\alpha)}(s)$ has no zeros if (α, σ) in the $\alpha\sigma$-plane lies in one of the wedges given by

$$q_M\alpha + b_1 \le \sigma \le q_{M-1}\alpha + b_2$$

for constants $b_1, b_2 \in \mathbb{R}$, chosen in a way that guarantees the dominance (in the modulus) of the term $Q_M^\alpha(s) = \frac{\log^\alpha M}{M^s}$ of the series for $\zeta^{(\alpha)}(s)$, see Figure 2. We call the remaining terms of the series the 'head'

$$H_M^\alpha(s) := \sum_{n=2}^{M-1} Q_n^\alpha(s) = \sum_{n=2}^{M-1} \frac{\log^\alpha n}{n^s}$$

and the 'tail'

$$T_M^\alpha(s) := \sum_{n=M+1}^{\infty} Q_n^\alpha(s) = \sum_{n=M+1}^{\infty} \frac{\log^\alpha n}{n^s}.$$

The key idea is to show that in our well-defined regions

$$\begin{aligned} |\zeta^{(\alpha)}(s)| &\ge Q_M^\alpha(\sigma) - H_M^\alpha(\sigma) - T_M^\alpha(\sigma) \\ &= Q_M^\alpha(\sigma)\left(1 - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{T_M^\alpha}{Q_M^\alpha}(\sigma)\right) > 0, \end{aligned} \tag{3.1}$$

thus proving that $\zeta^{(\alpha)}(s)$ does not vanish.

In order to find suitable upper bounds to the tails $T_M^\alpha(\sigma)$, a couple of preliminary bounds are needed. We begin with the following lemma:

Lemma 1. *Fix $2 \le M \in \mathbb{N}$, and assume $\alpha < (\sigma - 1)\log M$. Then*

$$T_M^\alpha(\sigma) = \sum_{n=M+1}^{\infty} \frac{\log^\alpha n}{n^\sigma} \le \int_M^\infty \frac{\log^\alpha x}{x^\sigma} dx \le Q_M^\alpha(\sigma) R_M^\alpha(\sigma), \tag{3.2}$$

where

$$R_M^\alpha(\sigma) = \frac{M}{\sigma - 1}\left(1 + \frac{\alpha}{(\sigma-1)\log M - \alpha}\right).$$

Proof First, for the upper incomplete Gamma function we have the bound (see [21, (3.2)]): $\Gamma(a, x) < Bx^{a-1}e^{-x}$, valid for all $B > 1, a > 1$ and $x > \frac{B(1-a)}{1-B}$. This means that we can write:

$$\begin{aligned} T_M^\alpha(\sigma) &= \sum_{n=M+1}^{\infty} \frac{\log^\alpha n}{n^\sigma} \le \int_M^\infty \frac{\log^\alpha x}{x^\sigma} dx = \frac{\Gamma(\alpha+1, (\sigma-1)\log(M))}{(\sigma-1)^{\alpha+1}} \\ &< \frac{B((\sigma-1)\log(M))^{\alpha+1-1} e^{-(\sigma-1)\log(M)}}{(\sigma-1)^{\alpha+1}} = \frac{\log^\alpha M}{M^\sigma}\frac{M}{\sigma-1}B. \end{aligned}$$

Here, with the choice of $x = (\sigma-1)\log(m)$ and $a = \alpha + 1$ in $x > \frac{B(1-a)}{1-B}$, we can obtain a lower bound for B:

$$B > \frac{(\sigma-1)\log m}{(\sigma-1)\log m - \alpha} = 1 + \frac{\alpha}{(\sigma-1)\log m - \alpha},$$

and if we set $B := 1 + \epsilon + \frac{\alpha}{(\sigma-1)\log M - \alpha}$, for any $\epsilon > 0$, then we get:

$$T_M^\alpha(\sigma) < \frac{\log^\alpha M}{M^\sigma} \frac{M}{\sigma - 1} \left(1 + \epsilon + \frac{\alpha}{(\sigma - 1)\log M - \alpha}\right).$$

Letting $\epsilon \to 0$ this bound becomes

$$T_M^\alpha(\sigma) \leq \frac{\log^\alpha M}{M^\sigma} \frac{M}{\sigma - 1} \left(1 + \frac{\alpha}{(\sigma - 1)\log M - \alpha}\right),$$

which proves the lemma. □

Next, we find a bound for $R_M^\alpha(\sigma)$. We have:

Lemma 2. *If* $a_1\alpha + b_1 \leq \sigma$ *and* $A \leq \alpha$ *and* $a_1 > \frac{1}{\log M}$, *then*

$$R_M^\alpha(\sigma) \leq R_M^\alpha(a_1\alpha + b_1) \leq R_M^A(a_1\alpha + b_1) \leq R_M^A(a_1 A + b_1), \tag{3.3}$$

Proof The left inequality of (3.3) is evident from the fact that $R_M^\alpha(\sigma)$ is decreasing when viewed as a function of σ alone. The right inequality is equivalent to $R_M^\alpha(\sigma)$ being decreasing as a function of α. To see this we set $c := a_1 \log M - 1 \geq 0$ and $d := (b_1 - 1)\log M$, and get

$$\begin{aligned} y(\alpha) &:= \frac{1}{M\log M} R_M^\alpha(a_1\alpha + b_1) \\ &= \frac{1}{M\log M} \frac{M}{a_1\alpha + b_1 - 1} \left(1 + \frac{\alpha}{(a_1\alpha + b_1 - 1)\log M - \alpha}\right) \\ &= \frac{1}{(c+1)\alpha + d} \frac{(c+1)\alpha + d}{c\alpha + d} = \frac{1}{c\alpha + d} \end{aligned}$$

But since $y'(\alpha) = \dfrac{-c}{(c\alpha + d)^2} < 0$, it follows that $y(\alpha)$ is decreasing. □

Note: In what follows, we apply the estimates for $T_M^\alpha(\sigma)$ from Lemma 1 in the proof of Theorem 1 via the useful separation

$$\begin{aligned} T_M^\alpha(\sigma) &= Q_{M+1}^\alpha(\sigma) + T_{M+1}^\alpha(\sigma) \\ &\leq Q_{M+1}^\alpha(\sigma)(1 + R_{M+1}^\alpha(\sigma)) \\ &\leq Q_M^\alpha(q_M\alpha + b_1)(1 + R_{M+1}^\alpha(q_M\alpha + b_1)), \end{aligned}$$

which holds since $Q_{M+1}^\alpha(\sigma) \leq Q_M^\alpha(\sigma)$. The series with these $R_{M+1}^\alpha(q_M\alpha + b_1)$ converges because, by [2, Lemma 3.1], $q_M > 1/\log(M+1)$.

4 Proof of Theorem 1

We conclude with the proof of Theorem 1 and some immediate consequences.

Proof of Theorem 1 (a) We consider the case where $Q_2^\alpha(\sigma) = \frac{\log^\alpha(2)}{2^\sigma}$ is the dominant term of $\zeta^{(\alpha)}(s)$, that is in (3.1) we have $M = 2$. We show that, for all real $\alpha > 0$ and all $\sigma > q_2\alpha + 2.6$, we have $\zeta^{(\alpha)}(s) \neq 0$.

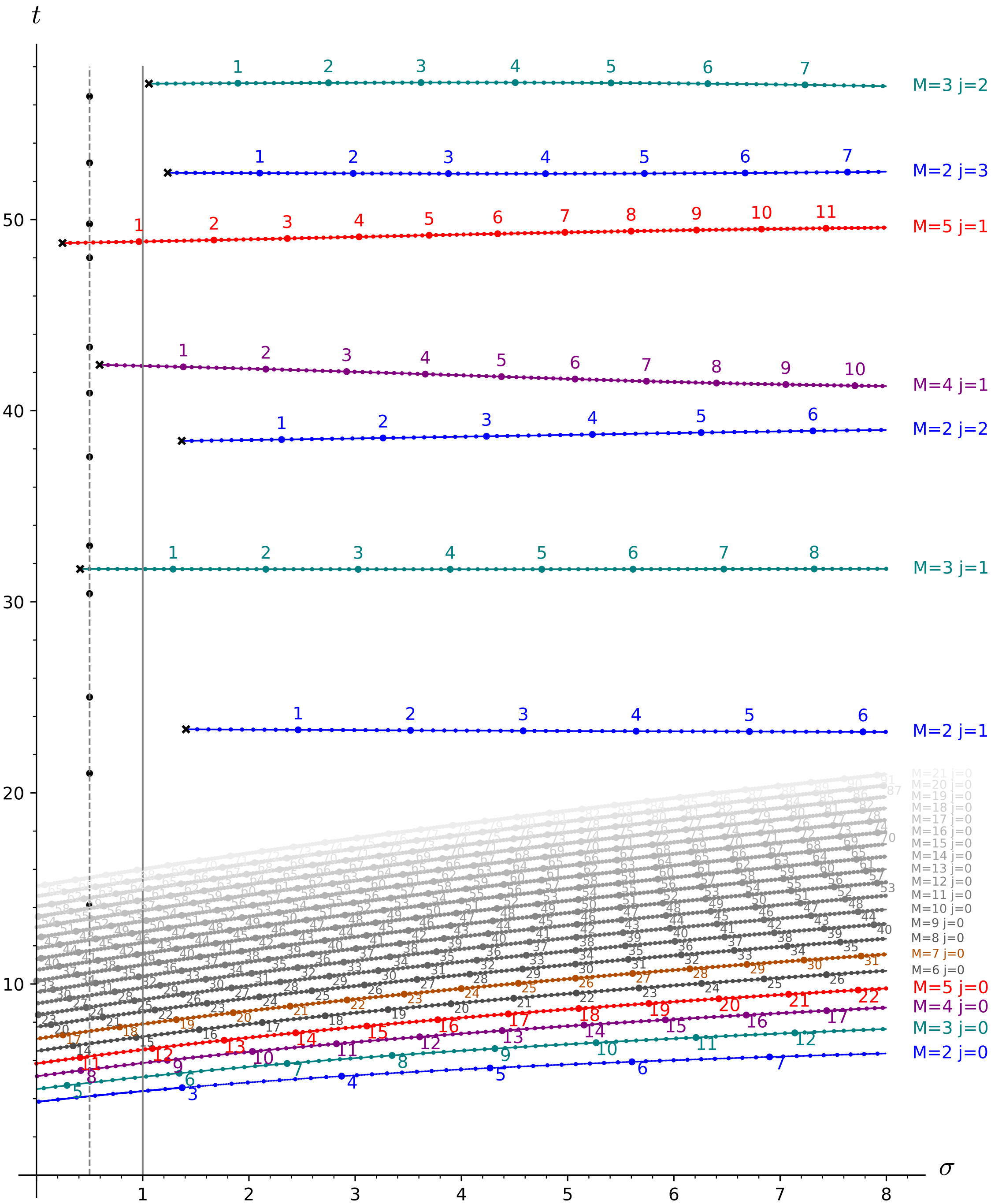

Figure 4. Selected curves of zeros of the fractional derivatives $\zeta^{(\alpha)}(\sigma+it)$. Zeros of $\zeta(\sigma+it)$ are denoted by $\bullet$, zeros of $\zeta(\sigma+it)-1$ are denoted by **x** and zeros of the integral derivatives $\zeta^{(k)}(\sigma+it)$ are denoted by $\bullet^k$.

First, write

$$\begin{aligned}|\zeta^{(\alpha)}(s)| &\geq \frac{\log^\alpha 2}{2^\sigma} - T_2^\alpha(\sigma)\\ &\geq Q_2^\alpha(\sigma)\left(1 - \frac{Q_3^\alpha}{Q_2^\alpha}(\sigma) - \frac{Q_4^\alpha}{Q_2^\alpha}(\sigma)\left(1 + R_4^\alpha(\sigma)\right)\right).\end{aligned}$$

By Lemma 2 for $A \geq \alpha$ we have

$$\begin{aligned}R_4^\alpha(\sigma) &\leq R_4^\alpha(q_2 A + b)\\ &\leq R_4^A(q_2 A + b) = \frac{4}{q_2 A + b - 1}\left(1 + \frac{A}{(q_2 A + b - 1)\log 4 - A}\right)\end{aligned}$$

Furthermore,

$$\frac{Q_4^\alpha}{Q_2^\alpha}(\sigma) = \frac{2^\sigma (\log 4)^\alpha}{4^\sigma (\log 2)^\alpha} = \frac{2^\sigma (2\log 2)^\alpha}{2^{2\sigma}(\log 2)^\alpha} = 2^{\alpha-\sigma} \leq 2^{\alpha - q_2\alpha - b} \leq 2^{(1-q_2)A-b}.$$

Now, the quotient $\frac{Q_3^\alpha}{Q_2^\alpha}(\sigma)$ is decreasing in σ, and as one can easily verify

$$\frac{Q_{M+1}^\alpha}{Q_M^\alpha}(q_M\alpha + b_1) = \left(\frac{M}{M+1}\right)^{b_1}$$

and

$$\frac{Q_{M-1}^\alpha}{Q_M^\alpha}(q_{M-1}\alpha + b_2) = \left(\frac{M}{M-1}\right)^{b_2}.$$

for all $M \geq 2$ and real numbers b_1 and b_2. Therefore,

$$\frac{Q_3^\alpha}{Q_2^\alpha}(\sigma) \leq \frac{Q_3^\alpha}{Q_2^\alpha}(q_2\alpha + b) = \left(\frac{2}{3}\right)^b.$$

For $A = 0$ and $\alpha > A$ and $b = 2.6$ and $\sigma \geq q_2\alpha + b$ we get

$$1 - \frac{Q_3^\alpha}{Q_2^\alpha}(\sigma) - \frac{Q_4^\alpha}{Q_2^\alpha}(\sigma)\left(1 + R_4^\alpha(\sigma)\right) \geq 1 - 0.349 - 0.165(1 + 2.501) > 0.$$

Thus for all real $\alpha > 0$ and all $\sigma \geq q_2\alpha + 2.6$ we have $\zeta^{(\alpha)}(s) \neq 0$. □

Theorem 1 (a) generalizes Verma & Kaur's bound [28] to fractional derivatives. Our bound is a bit weaker than theirs, as we consider any $\alpha > 0$ instead of $\alpha \geq 3$. Smaller values of b in the proof of Theorem 1 (a) yield tighter bounds that hold for greater α. In particular, any $b > 0$ yields a bound that holds for all sufficiently large values of α. With $b = 2$ we obtain the bound proved in [28] for $\alpha \geq 3$.

Corollary 2. *For any $b > 0$ there is an $A \in \mathbb{R}$ such that for all $\alpha > A$ we have $\zeta^{(\alpha)}(s) \neq 0$, for all $s = \sigma + it$ with $\sigma \geq q_2\alpha + b$.*

Proof Let $b > 0$. For estimating $R_4^\alpha(q_2\alpha + b)$ we set $A := 0$ and $\alpha = 1/q_2$. We obtain $R_4^\alpha(q_2\alpha + b) \leq \frac{4}{b}$. We use the bounds from the proof of Theorem 1 (a). We have $\zeta^{(\alpha)}(s) \neq 0$ for $\sigma \geq q_2\alpha + b$ when

$$\left(\frac{2}{3}\right)^b + 2^{(1-q_2)\alpha - b}\left(1 + \frac{4}{b}\right) < 1$$

Solving for α we obtain

$$\alpha > \frac{b + \log_2 \frac{1-(2/3)^b}{1+4/b}}{1 - q_2}.$$

as desired. □

Proof of Theorem 1 (b) In the case $M = 3$ we have $\zeta^{(\alpha)}(s) \neq 0$ for

$$q_3\alpha + 4\log 3 \leq \sigma \leq q_2\alpha - 2.$$

For this zero-free region we require $q_3\alpha + 4\log 3 \leq q_2\alpha - 2$ which implies $\alpha \geq 19.5311\ldots$. Separating the dominant term $Q_3^\alpha(\sigma)$, we get

$$\begin{aligned} |\zeta^{(\alpha)}(s)| &\geq Q_3^\alpha(\sigma) - Q_2^\alpha(\sigma) - T_3^\alpha(\sigma) \\ &\geq Q_3^\alpha(\sigma)\left(1 - \frac{Q_2^\alpha}{Q_3^\alpha}(\sigma) - \frac{Q_4^\alpha}{Q_3^\alpha}(\sigma)\left(1 + R_4^\alpha(\sigma)\right)\right). \end{aligned}$$

Therefore we only need to show that

$$1 - \frac{Q_2^\alpha}{Q_3^\alpha}(\sigma) - \frac{Q_4^\alpha}{Q_3^\alpha}(\sigma)\left(1 + R_4^\alpha(\sigma)\right) > 0.$$

But notice that by Lemma 2,

$$R_4^\alpha(\sigma) \leq R_4^\alpha(\alpha_3\alpha + 4\log 3) \leq R_4^{\alpha_3}(q_3\alpha + 4\log 3) < 0.7848,$$

for $\sigma \geq q_3\alpha + 4\log 3$ and $\alpha \geq \alpha_3 = \frac{4\log 3 + 2}{q_2 - q_3} = 19.5311\ldots$. Also,

$$\frac{Q_4^\alpha}{Q_3^\alpha}(\sigma) \leq \frac{Q_4^\alpha}{Q_3^\alpha}(q_3\alpha + 4\log 3) < 0.29 \text{ and } \frac{Q_2^\alpha}{Q_3^\alpha}(\sigma) \leq \frac{Q_2^\alpha}{Q_3^\alpha}(q_2\alpha - 2) < 0.45.$$

Putting this together we obtain

$$1 - \frac{Q_2^\alpha}{Q_3^\alpha}(\sigma) - \frac{Q_4^\alpha}{Q_3^\alpha}(\sigma)\left((1 + R_4^\alpha(\sigma)\right) > 1 - 0.45 - 0.29(1 + 0.75) > 0,$$

which concludes our proof. □

Before we get to the main argument of the proof of Theorem 1(c), let us perform a technical transformation. We rewrite the series (1.6) as

$$\begin{aligned} H_M^\alpha(\sigma) &= Q_M^\alpha(\sigma)\left(\frac{Q_{M-1}^\alpha}{Q_M^\alpha}(\sigma) + \frac{Q_{M-2}^\alpha}{Q_M^\alpha}(\sigma) + \cdots + \frac{Q_2^\alpha}{Q_M^\alpha}(\sigma)\right) \\ &= Q_M^\alpha(\sigma)\left(\frac{Q_{M-1}^\alpha}{Q_M^\alpha}(\sigma)\left(1+\frac{Q_{M-2}^\alpha}{Q_{M-1}^\alpha}(\sigma)\left(1+\ldots\left(1+\frac{Q_2^\alpha}{Q_3^\alpha}(\sigma)\right)\ldots\right)\right)\right). \end{aligned} \tag{4.1}$$

with the hope of finding bounds for $\frac{Q_{n-1}^\alpha}{Q_n^\alpha}(\sigma)$. Observe that because

$$\frac{Q_{n-1}^\alpha}{Q_n^\alpha}(\sigma) = \left(\frac{\log(n-1)}{\log n}\right)^\alpha \left(\frac{n}{n-1}\right)^\sigma$$

the quotient $\frac{H_M^\alpha}{Q_M^\alpha}(\sigma)$ increases with σ. That means that, for $2 \le n \le M$ and $\sigma \le q_{M-1}\alpha + b_2$, we can write

$$\frac{Q_{n-1}^\alpha}{Q_n^\alpha}(\sigma) \le \frac{Q_{n-1}^\alpha}{Q_n^\alpha}(q_{M-1}\alpha + b_2) \le \frac{Q_{n-1}^\alpha}{Q_n^\alpha}(q_{n-1}\alpha + b_2) = \left(\frac{n}{n-1}\right)^{b_2},$$

where the second inequality holds since $q_{M-1} < q_n$ for $n \le M$ and the equality holds because $\sigma = q_{n-1}\alpha$ is the solution of $Q_n^\alpha(\sigma) = Q_{n-1}^\alpha(\sigma)$. Thus, in order for $\frac{H_M^\alpha}{Q_M^\alpha}(\sigma)$ to stay bounded, we must choose $b_2 < 0$.

By [2, Lemma 4.4] we have, for $2 \le n \le M$ and $\sigma \le q_{M-1}\alpha - uM$,

$$\frac{Q_{n-1}^\alpha}{Q_n^\alpha}(\sigma) \le \left(\frac{n}{n-1}\right)^{-uM} \le \left(\frac{M}{M-1}\right)^{-uM} \le \frac{1}{e^u}.$$

Combined with the equation (4.1), this yields

$$\frac{H_M^\alpha}{Q_M^\alpha}(\sigma) \le \sum_{n=1}^{\infty} \frac{1}{(e^u)^n} = \frac{1}{1 - \frac{1}{e^u}} - 1 = \frac{1}{e^u - 1}. \tag{4.2}$$

We are now ready to prove the final part (c) of Theorem 1.

Proof of Theorem 1 (c) Let $\alpha > 0$. We show that if $M \in \mathbb{N}$, $M > 3$, and $q_M k + (M+1)u \le q_{M-1}k - Mu$ then $\zeta^{(\alpha)}(s) \neq 0$ for

$$q_M\alpha + (M+1)u \le \sigma \le q_{M-1}\alpha - Mu.$$

where $u \in (0, \infty)$ is a solution of $1 - \frac{1}{e^u-1} - \frac{1}{e^u}(1 + \frac{1}{u}) \ge 0$. Similar to the proof of Theorem 1 (b) we write

$$\begin{aligned} \left|\zeta^{(\alpha)}(s)\right| &\ge Q_M^\alpha(\sigma) - H_M^\alpha(\sigma) - T_M^\alpha(\sigma) \\ &\ge Q_M^\alpha(\sigma)\left(1 - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{Q_{M+1}^\alpha}{Q_M^\alpha}(\sigma)\left(1 + R_{M+1}^\alpha(\sigma)\right)\right). \end{aligned}$$

Now, notice that

$$R_M^\alpha(\sigma) := \frac{M}{\sigma - 1}\left(1 + \frac{\alpha}{(\sigma-1)\log M - \alpha}\right) < \frac{1}{u}$$

is equivalent to $(\sigma-1)^2 \log M - (\sigma-1)(uM\log M + \alpha) > 0$ and this quadratic inequality is satisfied whenever $\sigma > 1 + uM + \frac{\alpha}{\log M}$. Thus, by Lemma 2, for $\sigma \ge q_M\alpha + u(M+1)$, $\alpha \ge \alpha_M := \frac{(2M+1)u}{q_{M-1} - q_M}$, and $M \ge 4$, we have

$$R_{M+1}^\alpha(\sigma) \le R_{M+1}^{\alpha_M}(q_M\alpha_M + u(M+1)) < \frac{1}{u}.$$

But by [2, Lemma 4.4] $\left(\frac{n-1}{n}\right)^{cn}$ is monotonously increasing with the asymptote $1/e^c$. And therefore

$$\frac{Q_{M+1}^\alpha}{Q_M^\alpha}(q_M\alpha + u(M+1)) = \left(\frac{M}{M+1}\right)^{u(M+1)} < \frac{1}{e^u}.$$

Finally, with the help of the bound (4.2), we can see, that for $M \ge 4$ and $q_M\alpha + u(M+1) \le \sigma \le q_{M-1}\alpha + uM$, we have

$$1 - \frac{H_M^\alpha}{Q_M^\alpha}(\sigma) - \frac{Q_{M+1}^\alpha}{Q_M^\alpha}(\sigma)\left(1 + R_M^\alpha(\sigma)\right) > 1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) \ge 0,$$

which completes the proof of the theorem. □

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