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HALF DERIVATIVES OF THE RIEMANN ZETA FUNCTION

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Abstract. For the real parts of fractional derivatives of the Riemann zeta function $\zeta(s)$ we prove:

$$\Re\left(\zeta^{(k+\frac{1}{2})}(\sigma)\right) \;=\; -\frac{\Gamma(k+\frac{1}{2}+1)}{(1-\sigma)^{k+\frac{1}{2}+1}},$$

an expression valid for all $k \in \mathbb{N}_0$ and $\sigma < 1$, where $\Gamma(s)$ denotes the gamma function.

1. Introduction: special values of $\Gamma(s)$ and $\zeta(s)$

The topic of special values and exact formulas for the functions $\Gamma(s)$ and $\zeta(s)$ has a long and interesting history. Before we discuss our contribution to the subject, we briefly summarize some of its main highlights, so that the relevance of our results can be understood in proper context.

Let $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$ be the Legendre [23] integral for the Γ-function, defined by Euler [8] in a 1730 letter to Goldbach. Clearly, $\Gamma(1) = \int_0^\infty e^{-x}\,dx = [-e^{-x}]_0^\infty = 1$ and, integrating by parts,

$$\Gamma(s+1) \;=\; \int_0^\infty x^s e^{-x}\,dx \;=\; \left[-x^s e^{-x}\right]_0^\infty + \int_0^\infty s x^{s-1}e^{-x}\,dx \;=\; s\Gamma(s), \tag{1}$$

the so-called *functional equation* of $\Gamma(s)$ (the uniqueness of which is known as the Bohr-Mollerup theorem [4]); it readily implies that $\Gamma(s) = (s-1)!$, for all $s \in \mathbb{N}$. The integral representation of $\Gamma(s)$ is correct only for complex s with $\Re(s) > 0$, but the function can be analytically continued to the complex plane $\mathbb{C}\backslash\{0,-1,-2,\cdots\}$ (e.g. via the formula of Mittag-Leffler [29]), avoiding the poles of $\Gamma(s)$ at the non-positive integers. Moreover, $\Gamma(s)$ satisfies Euler's *reflection formula* $\Gamma(s)\Gamma(1-s) = \pi s/\sin(\pi s)$, valid as long as $s \notin \mathbb{N}$ (see [2]), a result that implies that $\Gamma(s)$ has no zeros in $\mathbb{C}$. With $s = \frac{1}{2}$, and a little help from the normal distribution integral of Gauss [18] (namely $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$), this shows that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, as well as $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$, and ultimately also (applying the functional equation (1)): $\Gamma(s+\frac{1}{2}) = \frac{(2s)!}{4^s s!}\sqrt{\pi}$, valid for all $s \in \mathbb{N}_0$.

Many other explicit formulas for, and special values of, $\Gamma(s)$ have been discovered (see [35], for example), and its various derivatives have also been studied. We have (Euler [13] of 1765):

$$\Gamma'(1) = \int_0^\infty e^{-x}\log x\,dx = -\gamma \quad \text{and} \quad \Gamma''(1) = \int_0^\infty e^{-x}(\log x)^2\,dx = \frac{\pi^2}{6} + \gamma^2,$$

where γ is the Euler-Mascheroni constant ([10] & [27]): $\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log n\right) = 0.57721\cdots$ and, by taking the logarithmic derivative of the functional equation of $\Gamma(s)$, it easily follows that

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} \;=\; \frac{\Gamma'(s)}{\Gamma(s)} + \frac{1}{s} \quad\Longrightarrow\quad \Gamma'(2) = 1-\gamma. \tag{1'}$$

These formulas play a central role not just in the theory of $\Gamma(s)$, but are closely related to results concerning $\zeta(s)$, defined by Euler [11] in 1737. Euler proved that, for real $s > 1$, $\zeta(s)$ satisfies

$$\zeta(s) \;:=\; \sum_{n=1}^{\infty}\frac{1}{n^s} \;=\; \prod_p\left(1-\frac{1}{p^s}\right)^{-1}, \tag{2}$$

2010 *Mathematics Subject Classification.* 11M35.

with the so-called *Euler product* extended over all primes p. Moreover, in the same region (see [22]):

$$\frac{d(1/n^s)}{ds} = \frac{d(e^{-s\log n})}{ds} = \frac{-\log n}{n^s} \quad \Longrightarrow \quad \zeta^{(k)}(s) \;=\; (-1)^k \sum_{n=1}^{\infty} \frac{(\log n)^k}{n^s}, \tag{$2'$}$$

true for derivatives of all orders $k \in \mathbb{N}$. Now, $\zeta(s)$ can be, by a process of analytic continuation, extended to a meromorphic functions, with a pole at $s = 1$; and , as Stieltjes [32] proved in 1885,

$$\zeta(s) \;=\; \frac{1}{s-1} \;+\; \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n, \quad \text{with} \quad \gamma_n := \lim_{s\to 1}\left|(-1)^n \zeta^{(n)}(s) - \frac{n!}{(s-1)^{n+1}}\right|,$$

where γ_n are the Stieltjes constants, with $\gamma_0 = \gamma$. Euler's (2) continued his earlier work, on the Basel problem (Euler [9] of 1734), which he solved by proving: $\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}$, and most spectacularly: $\zeta(2m) = (-1)^{m+1}\frac{2^{2m-1}B_{2m}}{(2m)!}\pi^{2m}$, valid for all $m \in \mathbb{N}$, where B_{2m} are the Bernoulli numbers (defined in Bernoulli [3] of 1713): $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, \cdots$ all of which are rational numbers that can be computed via their generating function

$$\frac{x}{e^x - 1} \;=\; \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

And, as Abel [1] noticed in 1823, since $e^{-x} + e^{-2x} + e^{-3x} + \cdots = 1/(e^x - 1)$, we can write

$$\frac{1}{n^s}\Gamma(s) \;=\; \frac{1}{n^s}\int_0^{\infty} (nx)^{s-1} n e^{-nx}\, dx \;=\; \int_0^{\infty} x^{s-1} e^{-nx}\, dx \quad \Longrightarrow \quad \zeta(s)\Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx.$$

This formula is a starting point for at least 3 proofs (see Titchmarsh [34], §2.4, §2.7 and §2.8) of the *functional equation* (conjectured by Euler [12] in 1749, and proved by Riemann [31] in 1859):

$$\zeta(s) = \zeta(1-s) 2^s \pi^{s-1} \Gamma(s-1) \sin\left(\frac{\pi s}{2}\right), \tag{3}$$

valid for $1 \neq s \in \mathbb{C}$. The quasi-symmetry of (3) implies that (letting $s \to 1$): $\zeta(0) = -\frac{1}{2}$ and (using the aforementioned special values $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2})$ and $\zeta(2) = \frac{\pi^2}{6}$) it also gives: $\zeta(-1) = -\frac{1}{12}$. Similarly, one can apply (3) to prove that $\zeta(-3) = \frac{1}{120}, \zeta(-5) = -\frac{1}{252}, \zeta(-7) = \frac{1}{240}$ and, for all $m \in \mathbb{N}$, $\zeta(-2m+1) = (-1)^m \frac{B_{2m}}{2m}$ and $\zeta(-2m) = 0$, the last result establishing that the even negative integers are the *trivial* zeros of $\zeta(s)$. Taking the logarithmic derivative of (3), we get

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} \;=\; -\log 2\pi + \frac{\zeta'(s)}{\zeta(s)} + \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right) \quad \xrightarrow{s\to 1} \quad \zeta'(0) = -\frac{1}{2}\log 2\pi.$$

The alternating Dirichlet [6] eta function $\eta(s) := (1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty}(-1)^{n+1} n^{-s}$ can be used to give an even simpler proof of this result, as well as a justification of the following general formula: $\zeta'(-2m) = (-1)^m \frac{(2m)!}{2(2\pi)^{2m}}\zeta(2m+1)$, for all $m \in \mathbb{N}$.

2. New Extensions: Fractional Derivatives

There are several ways one can generalize ($2'$), and investigate the *fractional derivatives* of the Riemann zeta function $\zeta(s)$. Like in most of our previous work (see [14], [15], [17] and [30]), we will employ the Grünwald-Letnikov fractional derivatives (originally introduced in [19] and [25] & [26]), which, for $\alpha > 0$, define the α-th derivative of the Riemann zeta function as:

$$\zeta^{(\alpha)}(s) \;:=\; (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{\log^{\alpha}(n+1)}{(n+1)^s}, \tag{$2''$}$$

an expression that is valid only for $\Re(s) > 1$, but – like in the classical case of $\zeta(s)$ – admits an analytic continuation to $\mathbb{C}$ via the following Laurent series expansion:

$$\zeta^{(\alpha)}(s) \;=\; (-1)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} \;+\; \sum_{n=0}^{\infty}\frac{(-1)^{n}\gamma_{n+\alpha}}{n!}(s-1)^{n}\right), \tag{4}$$

where $\gamma_{n+\alpha}$ are the *fractional* Stieltjes constants, for $\alpha \in \mathbb{R}^{+}$ defined by Kreminski [21] in 2003. Moreover, with the help the Euler-Maclaurin summation (see our [16]), (4) can be rewritten as:

$$\zeta^{(\alpha)}(s) = (-1)^{\alpha}\left(\sum_{k=2}^{m-1}\frac{\log^{\alpha}k}{k^{s}} \;+\; \frac{1}{2}\frac{\log^{\alpha}m}{m^{s}} \;-\; \sum_{j=1}^{\lfloor v/2\rfloor}\frac{B_{2j}}{(2j)!}\left(\frac{\log^{\alpha}m}{m^{s}}\right)^{(2j-1)} \;+\; \frac{\Gamma(\alpha+1,(s-1)\log m)}{(s-1)^{\alpha+1}} \right.$$
$$\left. +\; \frac{1}{v!}\int_{m}^{\infty}P_{v}(x)g_{\alpha,s}^{(v)}(x)\,dx\right),$$

where $m \geq 2$ and v are positive integers, $g_{\alpha,s}(x) = \frac{\log^{\alpha}x}{x^{s}}$, and $P_{v}(x)$ is the v-th periodic Bernoulli function (see [20]). With these prerequisites we can now state and prove our main result:

Theorem 1. *For all $k \in \mathbb{N}_0$ and real $\sigma < 1$,*

$$\Re\left(\zeta^{(k+\frac{1}{2})}(\sigma)\right) \;=\; -\frac{\Gamma(k+\frac{1}{2}+1)}{(1-\sigma)^{k+\frac{1}{2}+1}}. \tag{$*$}$$

Proof. We consider a special case of the above expression, with $m = 2$, $v = 1$, and $\alpha = k + \frac{1}{2}$,

$$\zeta^{(k+\frac{1}{2})}(\sigma) \;=\; (-1)^{k+\frac{1}{2}}\left(\frac{1}{2}\frac{\log^{k+\frac{1}{2}}2}{2^{\sigma}} \;+\; \frac{\Gamma(k+\frac{1}{2}+1,(\sigma-1)\log 2)}{(\sigma-1)^{k+\frac{1}{2}+1}} \;+\; \int_{2}^{\infty}P_{1}(x)g'_{k+\frac{1}{2},\sigma}(x)\,dx\right),$$

where, for the incomplete gamma function, we note that:

$$\Gamma\left(n+\tfrac{1}{2}+1,(\sigma-1)\log 2\right) \;=\; \Gamma\left(n+\tfrac{1}{2}+1\right) \;-\; \gamma\left(n+\tfrac{1}{2}+1,(\sigma-1)\log 2\right),$$

and $\gamma(n+\frac{1}{2}+1,(\sigma-1)\log 2)$ is purely imaginary (e.g. [33]). Also, $\Re((-1)^{k+\frac{1}{2}}r) = \Re(\pm ir) = 0$ for all $r \in \mathbb{R}$, and both $\frac{1}{2}\frac{\log^{\frac{1}{2}}2}{2^{\sigma}} \in \mathbb{R}$, and the integrand $P_1(x)g'_{k+\frac{1}{2},\sigma}(x) \in \mathbb{R}$, for $x \in \mathbb{R}$. Therefore

$$\Re\left(\zeta^{(k+\frac{1}{2})}(\sigma)\right) \;=\; \Re\left((-1)^{k+\frac{1}{2}}\frac{\Gamma(k+\frac{1}{2}+1)}{(\sigma-1)^{k+\frac{1}{2}+1}}\right) \;=\; -\frac{\Gamma(k+\frac{1}{2}+1)}{(1-\sigma)^{k+\frac{1}{2}+1}},$$

which confirms the statement of the theorem, and finishes its proof. $\square$

3. Auxiliary Remarks

Remark 1. As noted in the Introduction, special values of $\Gamma(s)$ and $\zeta(s)$ and their derivatives have a special place in the history of mathematics. Simple formulas that yield new insights are rare, which makes their existence for the "complicated" fractional derivatives that much more surprising, even though in their complex environment the discovered patterns only concern their real values.

Remark 2. The special values of $\Gamma(s)$ involved in $(*)$ have a closed form (see above), and imply:

$$\Re\left(\zeta^{(k+\frac{1}{2})}(\sigma)\right) \;=\; -\frac{(2k+2)!}{4^{k+1}(k+1)!(1-\sigma)^{k+\frac{1}{2}+1}}\sqrt{\pi}.$$

Since this formula is valid for all real σ, it provides a variety of new interesting special cases.

Remark 3. Fractional half derivatives of ζ at the origin ($\sigma = 0$):

$$\Re(\zeta^{(\frac{1}{2})}(0)) = -\Gamma\left(1+\frac{1}{2}\right) = -\frac{1}{2}\sqrt{\pi}$$
$$\Re(\zeta^{(3/2)}(0)) = -\Gamma\left(1+\frac{3}{2}\right) = -\frac{3}{4}\sqrt{\pi}$$
$$\Re(\zeta^{(5/2)}(0)) = -\Gamma\left(1+\frac{5}{2}\right) = -\frac{15}{8}\sqrt{\pi}$$
$$\Re(\zeta^{(7/2)}(0)) = -\Gamma\left(1+\frac{7}{2}\right) = -\frac{105}{16}\sqrt{\pi}$$

Remark 4. Fractional half derivatives of ζ at the negative integers ($\sigma = -n$, with $n \in \mathbb{N}$):

$$\Re(\zeta^{(1/2)}(-1)) = -\frac{\sqrt{2\pi}}{8}$$
$$\Re(\zeta^{(1/2)}(-2)) = -\frac{\sqrt{3\pi}}{18}$$
$$\Re(\zeta^{(1/2)}(-3)) = -\frac{\sqrt{4\pi}}{32}$$
$$\Re(\zeta^{(1/2)}(-4)) = -\frac{\sqrt{5\pi}}{50}$$

References

[1] Abel, N. H. – *Solution de quelques problèmes à l'aide d'intégrales définies*, Mag. Naturvidenskaberne, 2, 1823 (see also Abel's Oeuvres, Vol. 1, 11–27, Christiania, 1881).

[2] Artin, E. – *The Gamma Function*, Holt, Rinehart & Winston, New York, 1964.

[3] Bernoulli, J. – *Ars Conjectandi*, Part II, Chap. III, Thurneysen Bros., Basel, 1713.

[4] Bohr, H. & Mollerup, J. – *Laerebog i Kompleks Analyse*, Vol. III, Copenhagen, 1922.

[5] Coffey, M. W., Knessl, C. – *An effective asymptotic formula for the Stieltjes constants*, Math. Comp., 80 (2011), no. 273, 379–386.

[6] Dririchlet, P. L. J. – *Sur l'usage des séries infinies dans la théorie des nombres*, Crelle: Journal für die reine und angewandte Math., Bd. 18, 259–274, 1838 (see also Dirichlet's Werke, Vol. 1, 359–374).

[7] Dutka, J. – The early history of the factorial function, Archive Hist. Exact Sci., Vol. 43, 225–249, 1991.

[8] Euler, L. – Letter to Goldbach, January 8, 1730 (see Fuss, P. H. - *Correspondance mathématique et physique … du XVIII. siècle*, l'Academie imperiale des sciences, St.-Pétersbourg, Vol. 1, 1843).

[9] Euler, L. – *De summis serierum reciprocarum* (1734), Comm. Acad. Sci. Petropol., Vol. 7, 123–134, 1740 (see also Euler's Opera Omnia, Series 1, Vol. 14, 73–86).

[10] Euler, L. – *De progressionibus harmonicis observationes* (1734), Comm. Acad. Sci. Petropol., Vol. 7, 150–161, 1740 (see also Euler's Opera Omnia, Series 1, Vol. 14, 87–100).

[11] Euler, L. – *Variae observationes circa series infinitas* (1737), Comm. Acad. Scient. Petropol., Vol. 9, 160–188, 1744 (see also Euler's Opera Omnia, Series 1, Vol. 14, 217–244).

[12] Euler, L. – *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques* (1749), Mémoires de l'Acad. Sci. Berlin, Vol. 17, 83–106, 1768 (see also Euler's Opera Omnia, Series 1, Vol. 15, 70–90).

[13] Euler, L. – *De curva hypergeometrica hac aequatione expressa* $y = 1 \cdot 2 \cdot 3 \cdots$ (1765), Novi Comm. Acad. Sci. Petropol., Vol. 13, 3–66, 1769 (see also Euler's Opera Omnia, Series 1, Vol. 28, 41–98).

[14] Farr, R., Pauli, S., Saidak, F. – *On Fractional Stieltjes Constants*, Indag. Math., Vol. 29, No. 5, 1425–1431, 2018.

[15] Farr, R., Pauli, S., Saidak, F. – *A Zero free region for the fractional derivatives of the Riemann zeta function*, New Zealand J. Math., Vol. 50, 1–9, 2020.

[16] Farr, R., Pauli, S., Saidak, F. – *Evaluating Fractional Derivatives of the Riemann Zeta Function*, ICMS 2020, 94–101, Springer, 2020.

[17] Farr, R., Pauli, S., Saidak, F. – *Approximating and Bounding fractional Stieltjes constants*, Funct. Approx. Comment. Math, Vol. 64, No. 1, 7–22, 2021.

[18] Gauss, C. F. – *Theoria Motus Corporum Celestium*, Hamburg, Perthes et Besser, 1809.

[19] Grünwald, A. K. – *Über begrenzte Derivation und deren Anwendung*, Z. Angew. Math. Phys., 12, 1867.

[20] Kimball, B. – *A generalization of the Bernoulli polynomial of order one*, Bull. Amer. Math. Soc. Vol. 41, No. 12, 894–900, 1935.

[21] Kreminski, R. – *Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants*, Math. Comp., Vol. 72, 1379–1397, 2003.

[22] Landau, E. – *Handbuch der Lehre von der Verteilung der Primzahlen,* Vol. 1, §30, B. G. Teubner, Leipzig, 1909.

[23] Legendre, A. M. – *Mémoires de la classe des sciences math. et physiques*, l'Institut de France, Paris, 1809.

[24] Lehmer, D. H. – *A New Approach to Bernoulli Polynomials*, Amer. Math. Monthly, Vol. 95, 10, 905–911, 1988.

[25] Letnikov, A. V. – *Theory of differentiation of fractional order*, Mat. Sbornik, 3, 1–68, 1868.

[26] Letnikov, A. V. – *Historical development of the theory of differentiation of fractional order*, Mat. Sbornik, 3, 85–119, 1868.

[27] Mascheroni, L. – *Adnotationes ad calculum integrale Euleri*, Ticini (ed.), Pavia, 1790.

[28] Matsuoka, Y. – *Generalized Euler constants associated with the Riemann zeta function*, Number Theory and Combinatorics: Japan 1984, World Scientific, Singapore, 279–295, 1985.

[29] Mittag-Leffler, G. M. – *Sur la représentation analytique d'une branche uniforme d'une fonction monogène*, Acta Math.. Vol. 29, 101–181, 1905.

[30] Pauli, S. & Saidak, F. – *Zero-free regions of the fractional derivatives of the Riemann zeta function*, Lith. Math. J., Vol. 62, no. 1, 99–112, 2022.

[31] Riemann B. – *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, 6 pp., November 1859.

[32] Stieltjes, T. J. – *Correspondence d'Hermite et de Stieltjes*, Tomes I, II, Gauthier-Villars, Paris, 1905.

[33] Thompson, I. – *Algorithm 926: Incomplete Gamma Functions with Negative Arguments*, ACM Transactions on Mathematical Software, Vol. 39, No. 2, Article 14, 2013.

[34] Titchmarsh, E. C. – The Theory of the Riemann Zeta-function, 2nd Edition. Clarendon Press, Oxford, 1986.

[35] Vidunas, R. – *Expressions for the values of the Gamma Function*, Kyushu J. Math, Vol. 59, 2, 267–283, 2005.

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