# ON THE ZEROS OF $\zeta(s)-c$

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Abstract. Let $\zeta(s)$ be the Riemann zeta function and $z_0 \in \mathbb{C}\setminus\mathbb{R}$ a zero of $\zeta(s)$. We investigate the graphs of the implicit functions $z : [0,1) \to \mathbb{C}$ with $z(0) = z_0$ given by $\zeta(z(c)) - c = 0$ by giving zero free regions for $\zeta(s) - c$ where $c \in [0,1)$.

## 1. Introduction

For $\sigma = \Re(s) > 1$ the Riemann zeta function can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

By analytic continuation, $\zeta(s)$ may be extended to the whole complex plane, with the exception of the simple pole $s = 1$. This analytic continuation is characterized by the functional equation

$$\zeta(1-s) = 2\Gamma(s)\zeta(s)(2\pi)^{-s}\cos\frac{s\cdot\pi}{2}. \tag{2}$$

The existence of the class $\zeta(-2n) = 0$, $n \in \mathbb{N}$ of zeros follows directly from the functional equation. These zeros are called trivial. The Riemann hypothesis states that all non-trivial zeros of $\zeta(s)$ are located on the critical line $\sigma = \frac{1}{2}$.

In order to understand the Riemann zeta function better various mathematicians have investigated the behavior of its derivatives. In 1934 Speiser [7] showed that the Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros for $0 < \Re(s) < \frac{1}{2}$.

Spira [8] computed zeros of the first and second derivative of $\zeta(s)$ and noticed that they occur in pairs. Skorokhodov [6] went further in his computation and noticed that the zeros of derivatives seem to form chains, that is for each zero $s_k$ of $\zeta^{(k)}(s)$ there is a corresponding zero $s_{k+1}$ of $\zeta^{(k+1)}(s)$. Indeed for sufficiently large $k$ the existence of these chains is a direct consequence of the following theorem:

**Theorem 1** (Binder, Pauli, Saidak [3]). *Let $u \in \mathbb{R}^{>0}$ be a solution of $1-\frac{1}{e^u-1}-\frac{1}{e^u}(1+\frac{1}{u}) \geq 0$. Let $M \in \mathbb{N}$, $M \geq 2$, and $j \in \mathbb{Z}$. Let*

$$q_M := \frac{\log\left(\frac{\log M}{\log(M+1)}\right)}{\log\left(\frac{M}{M+1}\right)}.$$

*If there is $k \in \mathbb{N}$ with*

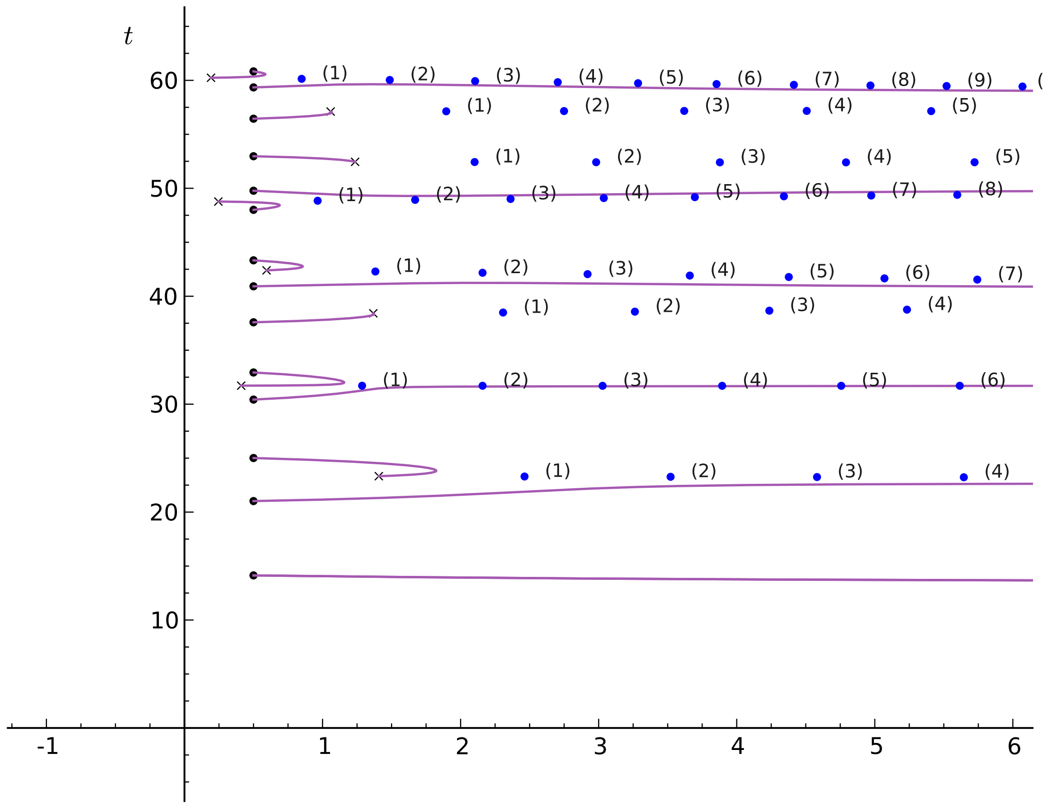
$$q_{M+1}k + (M+2)u \leq q_M k - (M+1)u$$

*then each rectangle $R_j \subset S_M^k$, consisting of all $s = \sigma + it$ with*

$$q_M k - (M+1)u < \sigma < q_M k + (M+1)u$$

*and*

FIGURE 1. Zeros of derivatives of $\zeta^{(k)}(s)$ (denoted by $\bullet^{(k)}$) and the paths from zeros of $\zeta(s)$ (denoted by $\bullet$) to the zeros of $\zeta(s)-1$ (denoted by $\times$).




$$\frac{2\pi j}{\log(M+1)-\log(M)} < t < \frac{2\pi(j+1)}{\log(M+1)-\log(M)},$$

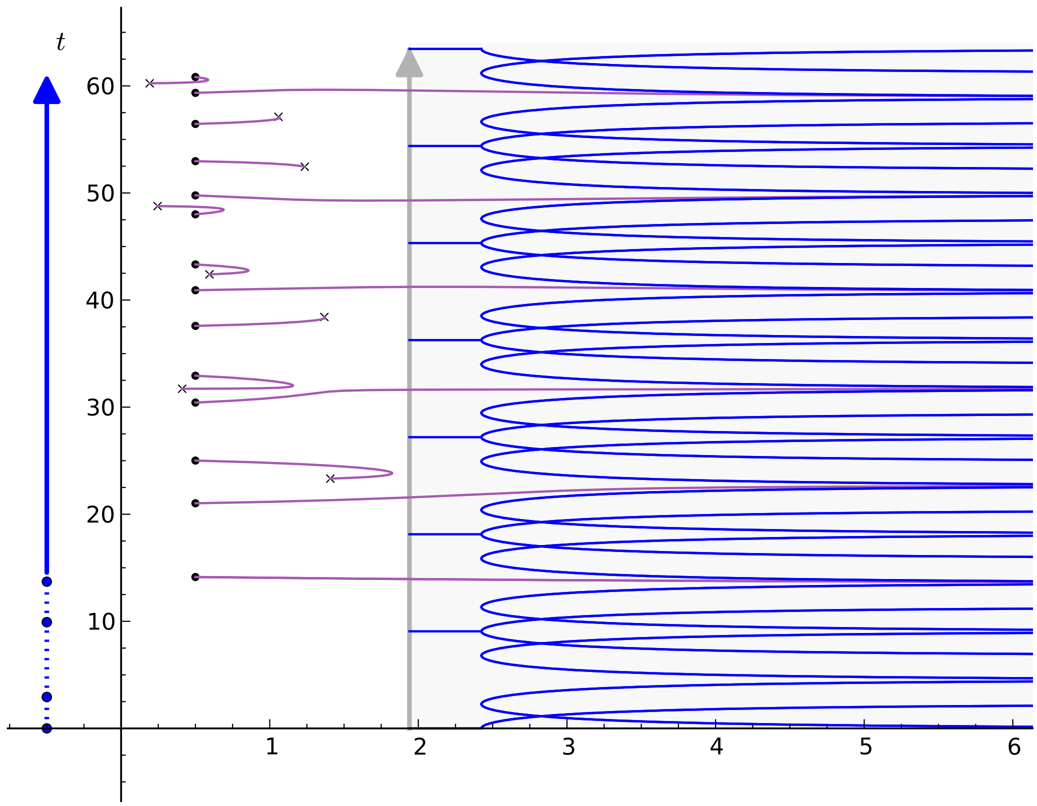
*contains exactly one zero of* $\zeta^{(k)}(s)$. *This zero is simple.*

The existence of the chains of zeros of derivatives can be seen as follows. For a given $M \in \mathbb{N}$, $M \geq 2$ there is $K \in \mathbb{N}$ such that $q_{M+1}k + (M+2)u \leq q_M k - (M+1)u$ for all $k \geq K$. By Theorem 1, for each $k \geq K$ and each $j \in \mathbb{Z}$ there is exactly one zero in a rectangular region given by $M$, $k$, and $j$. Again by Theorem 1 there exists a unique corresponding zero of $\zeta^{(k+1)}(s)$ in the rectangular region given by $M$, $k+1$, and $j$, which can be obtained by shifting the first region to the right (and stretching it horizontally). This shows the existence of a chain of zeros of $\zeta^{(K)}(s), \zeta^{(K+1)}(s), \zeta^{(K+2)}(s), \ldots$.

Skorokhodov also noticed that the zeros of $\zeta(s)-1$ can be regarded as the first points in these chains, and that there are curves from some zeros of $\zeta(s)$ to these points given by the zeros of $\zeta(s)-c$ for $c \in [0,1)$ (see Figure 1).

The curves of zeros $s(c)$ of $\zeta(s)-c$ for $c \in [0,1)$ either end at a zero of $\zeta(s)-1$ or go off to the left approaching their asymptote $t = \Re(s) = \frac{(2m+1)\pi}{\log 2}$ for some $m \in \mathbb{Z}$ as $\sigma = \Re(s)$ approaches infinity. If each zero of $\zeta(s)-1$ indeed corresponded to a zero of $\zeta'(s), \zeta''(s), \zeta'''(s), \ldots$, then some zeros of $\zeta(s)$ would not correspond to zeros with derivatives, namely those from which the paths of zeros of $\zeta(s)-c$ for $c \in [0,1)$ goes off to the right.

Figure 2. The paths from zeros of $\zeta(s)$ (denoted by $\bullet$) to the zeros of $\zeta(s)-1$ (denoted by $\times$), the barrier on the left (denoted by $\uparrow$), the zeros of $\Im\left(\zeta\left(-\frac{1}{2}+it\right)\right)$ with $0 \le t < 13.7$ (denoted by $\bullet$), the borders of zero free regions of $\zeta(s)-c$ for $c \in [0,1)$ (denoted by $-$), and the zero free region of $\zeta(s)-1$ on the right in grey.




This agrees with the formulas for the number of non-trivial zeros of $\zeta(s)$ and $\zeta^{(k)}(s)$. Namely, if we let $N(T)$ and $N_k(T)$ denote the number of such zeros $\rho$ with $0 \le \Im(\rho) \le T$ of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively. The classical Riemann-von Mangoldt formula (see [4]) states that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{3}$$

and, according to Berndt [2], we have

$$N_k(T) = N(T) - \frac{T\log 2}{2\pi} + O(\log T). \tag{4}$$

So there are about $\frac{T\log 2}{2\pi}$ less zeros with imaginary part less than $T$ of $\zeta^{(k)}(s)$ than of $\zeta(s)$, which is also about the number of paths of zeros of $\zeta(s)-c$ with imaginary part less than $T$ that go off to the right.

The aim of this paper is to describe better the behavior of paths of zeros of $\zeta(s)-c=0$ for $c \in [0,1)$ by finding new zero free regions for the functions $\zeta(s)-c$. Our results are summarized in Figure 2. Clearly the zeros of $\zeta(s)-c$ lie on the real lines of $\zeta(s)$, that is the lines on which $\Im(\zeta(s)) = 0$. A review of some results about these lines in section 2 is followed by the derivation of the zero free regions for $\zeta(s) - c$ on the right half plane (section 3) and the vertical boundary for the zeros of $\zeta(s) - 1$ for $\Re(s) = \frac{1}{2}$ (section 4).

## 2. Real Lines

Obviously the solutions of the equations $\zeta(s) - c = 0$ where $c \in [0, 1)$ are on the level lines with $\Im(\zeta(s)) = 0$ called real lines. Most of the results described here go back to the work of Speiser and his student Utzinger [7]. Plots of the behavior of the real (and imaginary) lines and some further discussion can be found in [1].

Because the term $1 + 2^{-s}$ dominates the infinite series $\zeta(s) = \sum_{i=0}^{\infty} \frac{1}{n^s}$ for $\sigma = \Re(s) > 3$ the real lines have asymptotes $t = \frac{j\cdot\pi}{\log 2}$ for $j \in \mathbb{Z}$. On the real lines with asymptote $t = \frac{2m\cdot\pi}{\log 2}$ ($m \in \mathbb{Z}$) the function $\zeta(s)$ approaches 1 from above, while on the real lines with asymptote $t = \frac{(2m+1)\pi}{\log 2}$ ($m \in \mathbb{Z}$) the function $\zeta(s)$ approaches 1 from below. The zero free regions for $\zeta(s) - c = 0$ where $c \in [0, 1)$ narrow around these asymptotes as $\sigma$ increases, see Lemma 4 and Lemma 3).

As $\zeta(s)$ is a meromorphic function no two of these real lines can cross, where $\zeta'(s) \neq 0$. Zero free regions for $\zeta'(s)$ have been found on the left of the critical line for $\Im(s) \neq 0$ and $\Re(s) < 0$ [5, Theorem 9] ($\Re(s) < \frac{1}{2}$ under the Riemann hypothesis [7]) and on the right of the critical line for $\sigma > 2.94$ [6, Theorem 2]. Indeed the only point where two real lines coming from the right cross is the first real zero of $\zeta'(s)$ at $s \approx -2.7172628292$ [7]. Here the lines with asymptotes $t = \frac{2\pi}{\log 2}$ and $t = \frac{-2\pi}{\log 2}$ intersect the real axis.

The lines coming from the right continue to the left at least until $\sigma = 1.95$ (compare Lemma 5). If one of the lines coming from the right did not cross the strip $-1 \leq \sigma \leq 2$ it would have go up towards infinity. Because no two real lines coming from the right intersect all following lines would have to do the same. This would contradict the estimate

$$\Im\left(\int_{2+Ti}^{-1+Ti} \frac{\zeta'(s)}{\zeta(s)} ds\right) = O(\log T)$$

used in the proof of the Riemann-van-Mangoldt formula (equation (3)). Thus all real lines coming from the right cross the strip $-1 \leq \sigma \leq 2$ [7].

Hence the zeros of $\zeta(s) - c = 0$ where $c \in [0, 1)$ are either on the real lines described above or on real lines that enter the critical strip from the left half plane and then curve back to the left half plane. The lines coming from the left half plane are the linens on which $\zeta(s) - 1$ is zero. By Proposition, 7 $|\zeta(-\frac{1}{2} + it)| > 1$ for $t \geq 13.7$. Furthermore for $0 < t < 13.7$ there are only two points where $\Re(\zeta(-\frac{1}{2} + it)) = 0$, that is where the real lines with asymptote $t = \frac{2\pi}{\log 2}$ and $t = \frac{3\pi}{\log 2}$ cross the line $\sigma = -\frac{1}{2}$ (see Remark 8). It follows that each of these lines coming from the left contains a zero of $\zeta(s)$ and a zero of $\zeta(s) - 1$ on the left of $\sigma = -\frac{1}{2}$. It is well known that the real part of the zeros of $\zeta(s)$ is between zero and one and equals $\frac{1}{2}$ if one assumes the Riemann Hypothesis. An upper bound for the real part zeros of $\zeta(s) - 1$ was given by Skorokhodov [6], see Lemma 2 below.

## 3. Zero free regions for $\zeta(s) - c$ on the right

A right bound $\sigma = 3$ for the zeros of $\zeta(s) - 1$ can be easily obtained with the triangle inequality and an estimate for $\zeta(\sigma) - \frac{1}{2^\sigma} - 1$. S. Skorokhodov was able to get a better bound by applying the triangle in equality to a real valued function that only considers terms of the zeta function with $n$ odd:

**Lemma 2** (Skorokhodov [6])**.** *The function $\zeta(s)$ is distinct from unity at $\sigma \in (\sigma_0, \infty)$, where*

$$\sigma_0 = 1.940101683745\ldots$$

*is the zero of the function*

$$f(\sigma) = 1 + 2^{-\sigma} - (1 - 2^{-\sigma})\zeta(\sigma), \quad \sigma > 1.$$

For $c \in [0, 1)$ we find zero free regions of $\zeta(s) - c$ that depend on $t$. We obtain them by considering the real and imaginary part of $\zeta(s) - c$ separately.

**Lemma 3.** *If $c \in [0, 1)$ and $|\sin(t \log 2)| \geq 2^\sigma \zeta(\sigma) - 2^\sigma - 1$ then $\zeta(\sigma + it) - c \neq 0$.*

*Proof.* We consider the imaginary part of $\zeta(s) - c$ and obtain

$$|\Im(\zeta(s) - c)| \geq \left|\frac{1}{2^\sigma}\sin(t\log(2))\right| - \left|\sum_{n=3}^{\infty}\frac{1}{n^\sigma}\right| \tag{5}$$

$$= \left|\frac{1}{2^\sigma}\sin(t\log(2))\right| - \left|\zeta(\sigma) - 1 - \frac{1}{2^\sigma}\right|. \tag{6}$$

Which is greater than zero when

$$|\sin(t\log(2))| \geq 2^\sigma\zeta(\sigma) - 2^\sigma - 1.$$

□

**Lemma 4.** *If $c \in [0, 1)$ and $\cos(t\log(2)) \geq 2^\sigma\zeta(\sigma) - 2^\sigma - 1$, then $\zeta(\sigma + it) - c \neq 0$.*

*Proof.* For the real part of $\zeta(s) - c$ we obtain:

$$\begin{aligned}\Re(\zeta(s) - c) &= 1 - c + \frac{1}{2^\sigma}\cos(t\log(2)) + \cdots \\ &\geq \frac{1}{2^\sigma}\cos(t\log(2)) - \left(\zeta(\sigma) - 1 - \frac{1}{2^\sigma}\right), \text{ assuming } c = 1.\end{aligned}$$

Which is greater than zero when

$$\cos(t\log(2)) \geq 2^\sigma\zeta(\sigma) - 2^\sigma - 1.$$

□

These regions can be extended a bit if we restrict ourselves to certain values of $t$.

**Lemma 5.** *If $c \in [0, 1)$, $m \in \mathbb{Z}$, and $t$ is fixed at $\frac{2\pi m}{\log(2)}$, then $\Re(\zeta(s) - c) \neq 0$ for $\sigma > 1.95$.*

*Proof.* $\Re(\zeta(s)-c) = 1-c+\frac{1}{2^\sigma}\cos(t\log(2))+\frac{1}{3^\sigma}\cos(t\log(3))+\dots$ When $t$ is fixed and $t\log(2)=2\pi m$, we get:

$$\begin{aligned}\Re(\zeta(s)-c) &\geq 1-c+\sum_{\nu=0}^{\infty}\frac{1}{(2^\nu)^\sigma}-\left(\sum_{n=2}^{\infty}\frac{1}{n^\sigma}-\sum_{\nu=0}^{\infty}\frac{1}{(2^\nu)^\sigma}\right)\\ &= 2\sum_{\nu=1}^{\infty}\left(\frac{1}{2^\sigma}\right)^\nu-\zeta(\sigma)\\ &= \frac{2}{1-\frac{1}{2^\sigma}}-\zeta(\sigma).\end{aligned}$$

□

## 4. Zero free barrier for $\zeta(s)-c$ on the left

On the left, instead of finding a zero free region, we find a horizontal line where $|\zeta(s)|>1$. The line $\sigma=-\frac{1}{2}$ fulfills this condition with the exception of one point.

First we find a lower bound for the absolute value of $\zeta(s)$ where $\sigma=\frac{3}{2}$.

**Lemma 6.** *$|\zeta(\frac{3}{2}+it)|>0.46$ for all $t\in\mathbb{R}$.*

*Proof.* To get a lower bound for $|\zeta(s)|$ we use the Euler product, let $P$ be the set of the first 1000000 prime numbers and consider the expression $\prod_{p\in P}|1-p^{-s}||\zeta(s)|$.

$$\prod_{p\in P}|1-p^{-s}||\zeta(s)| = \left|1+\sum_{p\nmid n, p\in P}\frac{1}{n^s}\right| \geq \left|1-\left|\sum_{p\nmid n,p\in P}\frac{1}{n^s}\right|\right| \geq 1-\sum_{p\nmid n,p\in P}\frac{1}{n^\sigma} = 2-\prod_{p\in P}(1-p^{-\sigma})\zeta(\sigma)$$

We also have from the triangle inequality that $|1-p^{-s}|\leq 1+p^{-\sigma}$, thus

$$|\zeta(s)| \geq \frac{2-\prod_{p\in P}(1-p^{-\sigma})\zeta(\sigma)}{\prod_{p\in P}(1+p^{-\sigma})} \geq 0.46 \quad \text{for } \sigma=\frac{3}{2}.$$

So we get $|\zeta(s)|\geq\delta>0$ for $\sigma=\frac{3}{2}$ and $\delta=0.46$. □

Now we can use $\delta$ and the functional equation to obtain a barrier for the zeros of $\zeta(s)-c$ on the left.

**Proposition 7.** *$\left|\zeta\left(-\frac{1}{2}+it\right)\right|>1$ for $t\geq 13.7$.*

*Proof.* By the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma(s)\zeta(s) = 2^{1-s}\cdot\pi^{-s}\cdot\cos\left(\frac{s\cdot\pi}{2}\right)\cdot\Gamma(s)\cdot\zeta(s)$$

Taking the absolute value of both sides gives,

$$|\zeta(1-s)| = 2^{1-\sigma}\cdot\pi^{-\sigma}\cdot\left|\cos\left(\frac{s\cdot\pi}{2}\right)\right|\cdot|\Gamma(s)|\cdot|\zeta(s)|.$$

But

$$\begin{aligned}
\left|\cos\left(\frac{s\cdot\pi}{2}\right)\right| &= \frac{1}{2}\left|e^{-\frac{\pi}{2}(\sigma i-t)}+e^{\frac{\pi}{2}(t-\sigma i)}\right| \\
&= \frac{1}{2}\left|e^{-\frac{t\cdot\pi}{2}}(\cos\sigma+i\sin\sigma)+e^{\frac{t\cdot\pi}{2}}(\cos\sigma-i\sin\sigma)\right| \\
&= \frac{1}{2}\left|\cos\sigma\left(e^{\frac{t\cdot\pi}{2}}+e^{-\frac{t\cdot\pi}{2}}\right)+i\sin\sigma\left(e^{-\frac{t\cdot\pi}{2}}-e^{\frac{t\cdot\pi}{2}}\right)\right| \\
&= \frac{1}{2}\left(\cos^2\sigma\left(e^{\pi t}+e^{-\pi t}+2\right)+\sin^2\sigma\left(e^{\pi t}+e^{-\pi t}-2\right)\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(e^{\pi t}+e^{-\pi t}+2(\cos^2\sigma-\sin^2\sigma)\right)^{\frac{1}{2}}
\end{aligned}$$

As $\Gamma(z+1)=z\Gamma(z)$ for $z\in\mathbb{C}$ and as $\left|\Gamma(\frac{1}{2}+it)\right|=\sqrt{\pi\operatorname{sech}(\pi t)}=\sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}$ for $t\in\mathbb{R}$, we get

$$|\Gamma(\frac{3}{2}+it)|=\left|\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}+it\right)\right|=\sqrt{\frac{1}{4}+t^2}\cdot\sqrt{\pi}\cdot\sqrt{\frac{2}{e^{\pi t}+e^{-\pi t}}}.$$

For $\sigma=\frac{3}{2}$ we obtain

$$\left|\zeta\left(-\frac{1}{2}+it\right)\right|\geq 2^{-0.5}\cdot\pi^{-1}\cdot\frac{1}{\sqrt{2}}\left(1+\frac{4\cos^2(\frac{3}{2})-2}{e^{\pi t}+e^{-\pi t}}\right)\cdot\sqrt{\frac{1}{4}+t^2}\cdot\delta,$$

where the right hand is obviously increasing in $t$. With $\delta>0.46$ by Lemma 6 this gives $|\zeta(\frac{1}{2}+it)|>1$ for $t\geq 13.7$. □

**Remark 8.** The zeros of $\Im\left(\zeta\left(-\frac{1}{2}+it\right)\right)$ with $0\leq t<13.7$ are $t_0=0$, $t_1\approx 2.93$, and $t_2\approx 9.92$, where $\zeta\left(-\frac{1}{2}+it_0\right)\approx -0.21$, $\zeta\left(-\frac{1}{2}+it_1\right)\approx 0.35$, and $\zeta\left(-\frac{1}{2}+it_2\right)\approx 2.03$. So the only hole in the barrier is $-\frac{1}{2}+it_1$. This is where the real line with asymptote $\frac{\pi}{\log 2}$ crosses the line $\sigma=-\frac{1}{2}$.

## 5. Outlook

In our work we investigated the behavior of the graphs of the implicit functions $s:[0,1)\to\mathbb{C}$ given by $\zeta(s(c))-c=0$, where $s(0)$ is a zero of $\zeta(s)$. If $s(1)$ exists, they connect the zeros of $\zeta(s)$ and the zeros of $\zeta(s)-1$, which are the first points on the conjectured chains of zeros of derivatives.

A similar approach could also be used to investigate the conjectured chains of zeros of the derivatives of $\zeta(s)$. For each zero $s_0$ of $\zeta(s)-1=\sum_{n=1}^{\infty}\frac{1}{n^s}$ one would consider the implicit function $s:[0,\infty)\to\mathbb{C}$ given by

$$\zeta^{(k)}(s(k))=(-1)^k\sum_{n=1}^{\infty}\frac{\log^k n}{n^{s(k)}}=0$$

with $s(0)=s_0$. This function $s(k)$ should yield the correspondence of zeros of $\zeta^{(k)}(s)$ and $\zeta^{k+1}(s)$ for $k\in\mathbb{Z}$, $k\geq 0$ for two zeros which would be connected by $\{s(x)\mid k\leq x\leq k+1\}$.

Together the two implicit functions could give more detailed insight into the distribution of the zeros of $\zeta(s)$ by relating it to the distribution of higher derivatives (see Theorem 1). Furthermore it will be interesting to see how the conjectured chains of zeros of the derivatives of $\zeta(s)$ fit in with the universality of $\zeta(s)$ found by Voronin [11].

## 6. Acknowledgments

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## References

[1] Arias-de-Reyna, J. - X-Ray of Riemann zeta-function, arXiv:math/0309433v1, 2005
[2] Berndt, B. - The number of zeros for $\zeta^{(k)}(s)$, J. London Math. Soc. (2), 2, 577 – 580, **1970**
[3] Binder, T., Pauli, S., and Saidak, F. - New Zero Free Regions for the derivatives of the Riemann Zeta Function, Preprint 2011
[4] Landau, E. - Handbuch der Lehre von der Verteilung der Primzahlen, Chelsea, New York, **1974**
[5] Levinson, N. & Montgomery, H. L. - Zeros of the derivatives of the Riemann zeta function, Acta Math., 133, (1974), 49 – 65.
[6] Skorokhodov, S. L. - Padé approximants and numerical analysis of the Riemann zeta function, Zh. Vychisl. Mat. Mat. Fiz., 43, No. 9, 1330 – 1352, **2003**
[7] Speiser, A, - Geometrisches zur Riemannschen Zetafunktion, Math. Ann., 110, 514 – 521, **1934**
[8] Spira, R. - Zero-free region for $\zeta^{(k)}(s)$, J. London Math. Soc. 40, 677–682 **1965**
[9] Stein, W. *et al.* - Sage, open-source mathematics software, 2009. `http://www.sagemath.org`
[10] Titchmarsh, E. - The Theory of the Riemann Zeta-Function, 2nd Edition (revised by D. R. Heath-Brown), Oxford U. Press, Oxford, **1986**
[11] Voronin, S. M. - Theorem on the Universality of the Riemann Zeta Function, Izv. Akad. Nauk SSSR, Ser. Matem. 39, 475–486, **1975**