ON FRACTIONAL STIELTJES CONSTANTS

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Abstract. We study the non-integral generalized Stieltjes constants $\gamma_\alpha(a)$ arising from the Laurent series expansions of fractional derivatives of the Hurwitz zeta functions $\zeta^{(\alpha)}(s,a)$, and we prove that if $h_a(s) := \zeta(s,a) - 1/a^s$ and $C_\alpha(a) := \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$, then

$$C_\alpha(a) \ = \ (-1)^{-\alpha} h_a^{(\alpha)}(1),$$

for all real $\alpha \geq 0$, where $h^{(\alpha)}(x)$ denotes the α-th Grünwald-Letnikov fractional derivative of the function h at x. This result confirms the conjecture of Kreminski [8], originally stated in terms of the Weyl fractional derivatives.

1. Introduction

The Hurwitz zeta function is defined, for $\Re(s) > 1$ and $0 < a \leq 1$, as $\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$. In a manner very similar to that of the Riemann zeta function $\zeta(s)$, the function $\zeta(s,a)$ can be extended to a meromorphic function with a simple pole at $s = 1$ with residue 1 (see [2], or [4]). Moreover, the function has a Laurent series expansion about $s = 1$, given by

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)(s-1)^n}{n!}, \tag{1}$$

where $\gamma_n(a)$ are the generalized Stieltjes constants. The original Stieltjes constants $\gamma_n = \gamma_n(1)$ were defined in 1885 ([10]), but are themselves a generalization of Euler's constant γ:

$$\gamma = \gamma_0(1) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{1}{n} - \log m \right) = 0.57721\ 56649\ \cdots$$

In 1972, Berndt [3] showed that for the generalized Stieltjes constants in (1) we have:

$$\gamma_k(a) = \lim_{m\to\infty} \left\{ \sum_{n=0}^{m} \frac{\log^k(n+a)}{n+a} - \frac{\log^{k+1}(m+a)}{k+1} \right\}. \tag{2}$$

This result was sharpened by Williams & Zhang [11], who established, for $m \geq 1$,

$$\gamma_k(a) = \sum_{r=0}^{m} \frac{\log^k(r+a)}{r+a} - \frac{\log^{k+1}(m+a)}{k+1} - \frac{\log^k(m+a)}{2(m+a)} + \int_m^\infty P_1(x) f_k'(x)dx, \tag{3}$$

where $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$ and $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$, with $\lfloor x \rfloor$ denoting the integer part of x.

2010 *Mathematics Subject Classification.* 11M35.

More recently, Kreminski [8] has given a generalization of $\gamma_\nu(a)$ to $\nu > 0$, the so-called *fractional Stieltjes constants*. He computed $C_\nu(a) = \gamma_\nu(a) - \frac{\log^\nu(a)}{a}$ and conjectured that it equals $(-1)^\nu$ times the ν-th Weyl fractional derivative of $\zeta(s,a) - 1/(s-1) - 1/a^s$ at $s = 1$.

The main aim of our paper is to explain how one can employ Grünwald-Letnikov fractional derivatives in order to prove this conjecture of Kreminski [8]. In the process we generalize the results from Berndt [3] and Williams & Zhang [11] to the fractional case.

2. Fractional Derivatives

We begin by giving a brief summary of the most useful basic properties of generalized derivatives. The fractional derivative operators are generalizations of the familiar differential operator D^n to arbitrary (integer, rational, or complex) values of n.

For $N \in \mathbb{N}$ and $h > 0$, let $\Delta_h^N f(z) = (-1)^N \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(z+kh)$ be the standard finite difference of f. Then we have (see [9], for example): $f^{(n)}(z) = \lim_{h\to 0} \frac{\Delta_h^n f(z)}{h^n}$ for all $n \in \mathbb{N}$; and this can be naturally extended to the fractional case (see [5]) via

$$\Delta_h^\alpha f(z) = (-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh),$$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$. For any $\alpha \in \mathbb{C}$, the so-called reverse α^{th} Grünwald-Letnikov derivative of a function $f(z)$ is now defined as (cf. Grünwald [7]):

$$D_z^\alpha\left[f(z)\right] = \lim_{h\to 0^+} \frac{\Delta_h^\alpha f(z)}{h^\alpha} = \lim_{h\to 0^+} \frac{(-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^\alpha}, \tag{4}$$

whenever the limit exists. Thus defined, $D_z^\alpha[f(z)]$ coincide with the standard derivatives for all $\alpha \in \mathbb{N}$. Moreover, they are analytic functions of α and z (as long as the function $f(z)$ is analytic) and they satisfy: $D_z^0[f(z)] = f(z)$ and $D_z^\alpha\left[D_z^\beta\left[f(z)\right]\right] = D_z^{\alpha+\beta}\left[f(z)\right]$.

We note that the Grünwald-Letnikov derivative is defined for all $\alpha \in \mathbb{C}$, but in this paper we only consider $\alpha \in \mathbb{R}$, with $\alpha \geq 0$. The following results can be found in [9].

Lemma 2.1. *Let $\alpha \geq 0$, $a > 0$, and $z \in \mathbb{C}$. Then $D_z^\alpha\left[e^{-az}\right] = (-1)^\alpha a^\alpha e^{-az}$.*

This for $0 < a \leq 1$, and $s \in \mathbb{C}$ with $\Re(s) > 1$ the Grünwald-Letnikov fractional derivative of order $\alpha \geq 0$ with respect to s of $\zeta(s,a) - 1/a^s$ is

$$D_s^\alpha\left[\zeta(s,a) - \frac{1}{a^s}\right] = (-1)^\alpha \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s}. \tag{5}$$

The fractional derivative of $\zeta(s,a) - 1/a^s$ can be analytically continued to all of C with the exception of a pole at 1.

3. Fractional Stieltjes Constants

We study the non-integral generalized Stieltjes constants $\gamma_\alpha(a)$ arising from the Laurent series expansions of fractional derivatives of the Hurwitz zeta functions $\zeta^{(\alpha)}(s,a)$. Our result confirms the conjecture of Kreminski [8], originally stated in terms of the Weyl fractional derivatives.

In the following we find the Lauren series expansion for $D_s^\alpha\left[\zeta(s,a)-\frac{1}{a^s}\right]$ and define the fractional Stieltjes Constants as its coefficients. We use the following form of the Euler-Maclaurin summation formula

$$\sum_{k=m}^{n} g(k)=\int_{m}^{n} g(x)dx+\sum_{k=1}^{\ell}\frac{(-1)^k B_k}{k!}g^{(k+1)}(x)\Big|_{m}^{n}+(-1)^{\ell+1}\int_{m}^{n} P_\ell(x)g^{(\ell)}(x)dx, \tag{6}$$

where $g(x)\in C^\ell\left[m,n\right]$, $\ell\in\mathbb{N}$ and $P_k(x)$ denotes the k^{th} periodic Bernoulli polynomial

$$P_k(x)=\frac{B_k(x-\lfloor x\rfloor)}{k!}.$$

We take $\ell=1$ in (6) and set $g(x):=\frac{\log^\alpha(x+a)}{(x+a)^s}$. Let $n\to\infty$, for $\Re(s)>1$ we obtain

$$\begin{aligned}
D_s^\alpha\left[\zeta(s,a)-\frac{1}{a^s}\right]&=\sum_{r=1}^{\infty}\frac{\log^\alpha(r+a)}{(r+a)^s}\\
&=\sum_{r=1}^{m-1}\frac{\log^\alpha(r+a)}{(r+a)^s}+\int_{m}^{\infty}\frac{\log^\alpha(x+a)}{(x+a)^s}dx+\frac{\log^\alpha(m+a)}{2(m+a)^s}+\int_{m}^{\infty}P_1(x)g'(x)dx\\
&=\sum_{r=1}^{m}\frac{\log^\alpha(r+a)}{(r+a)^s}+\int_{m}^{\infty}\frac{\log^\alpha(x+a)}{(x+a)^s}dx-\frac{\log^\alpha(m+a)}{2(m+a)^s}+\int_{m}^{\infty}P_1(x)g'(x)dx\\
&=:G_{s,a}^\alpha(m)+I_s^\alpha(m)-D(s)+G(s).
\end{aligned}$$

We will estimate each of these four terms separately.

For the first term $G_{s,a}^\alpha(m)$ we have:

$$\begin{aligned}
G_{s,a}^\alpha(m)&=\sum_{r=1}^{m}\frac{\log^\alpha(r+a)}{(r+a)^s}\\
&=\sum_{r=1}^{m}\frac{\log^\alpha(r+a)}{r+a}e^{-(s-1)\log(r+a)}\\
&=\sum_{r=1}^{m}\frac{\log^\alpha(r+a)}{r+a}\sum_{n=0}^{\infty}\frac{(-1)^n\log^n(r+a)}{n!}(s-1)^n\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\sum_{r=1}^{m}\frac{\log^{\alpha+n}(r+a)}{(r+a)}.
\end{aligned}$$

Since $\alpha\geq 0$, $m\in\mathbb{N}$, and $0<a\leq 1$, for all $s\in\mathbb{C}$ with $\Re(s)>1$, the second term $I_s^\alpha(m)$ can be written in terms of the Upper Incomplete Gamma function $\Gamma(\alpha,s)$ as follows (comparing it, for instance, with results found in [6, p. 346] and [1, 6.5.3]):

$$
\begin{aligned}
I_s^\alpha(m) &= \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s} dx \\
&= \frac{\Gamma(\alpha+1, (s-1)\log(m+a))}{(s-1)^{\alpha+1}} \\
&= \frac{1}{(s-1)^{\alpha+1}} \left[\Gamma(\alpha+1) - (s-1)^{\alpha+1} \log^{\alpha+1}(m+a) \sum_{n=0}^\infty \frac{(-1)^n (s-1)^n \log^n(m+a)}{(\alpha+1+n)n!} \right] \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \log^{\alpha+1}(m+a) \sum_{n=0}^\infty \frac{(-1)^n (s-1)^n \log^n(m+a)}{(\alpha+1+n)n!} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \sum_{n=0}^\infty \left(\frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1} \right) \frac{(-1)^n (s-1)^n}{n!}.
\end{aligned}
$$

We write the third term as:

$$
\begin{aligned}
D(s) &= \frac{\log^\alpha(m+a)}{2(m+a)} \sum_{n=0}^\infty \frac{(-1)^n \log^n(m+a)(s-1)^n}{n!} \\
&= \sum_{n=0}^\infty \left(\frac{\log^{\alpha+n}(m+a)}{2(m+a)} \right) \frac{(-1)^n (s-1)^n}{n!}.
\end{aligned}
$$

Combining the above three expressions for $G_{s,a}^\alpha(m)$, $I_s^\alpha(m)$ and $D(s)$ we get:

$$
\begin{aligned}
G_{s,a}^\alpha(m) + I_s^\alpha(m) + D(s)& \\
&= \sum_{r=1}^m \frac{\log^\alpha(r+a)}{(r+a)^s} + \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s} dx - \frac{\log^\alpha(m+a)}{2(m+a)^s} \\
&= \sum_{n=0}^\infty \frac{(-1)^n (s-1)^n}{n!} \sum_{r=1}^m \frac{\log^{\alpha+n}(r+a)}{(r+a)} \\
&\quad + \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \sum_{n=0}^\infty \left(\frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1} \right) \frac{(-1)^n (s-1)^n}{n!} \\
&\quad + \sum_{n=0}^\infty \left(\frac{\log^{\alpha+n}(m+a)}{2(m+a)} \right) \frac{(-1)^n (s-1)^n}{n!} \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^\infty \frac{(-1)^n (s-1)^n}{n!} \left(\sum_{r=1}^m \frac{\log^{\alpha+n}(r+a)}{(r+a)} + \frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1} - \frac{\log^{\alpha+n}(m+a)}{2(m+a)} \right) \\
&= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^\infty H_m^{\alpha+n} \frac{(-1)^n (s-1)^n}{n!}.
\end{aligned}
$$

where

$$H_m^{\alpha+n} := \sum_{r=1}^{m} \frac{\log^{\alpha+n}(r+a)}{r+a} - \frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1} - \frac{\log^{\alpha+n}(m+a)}{2(m+a)},$$

This yields the form of the Laurent series expansion of $D_s^\alpha\left[\zeta(s,a) - \frac{1}{a^s}\right]$.

Definition 3.1. Letting $\alpha \in \mathbb{R}$, $\alpha > 0$ and $0 < a \leq 1$ and $s \neq 1$. For $n \in \mathbb{N}$ we define the fractional Stieltjes constants $C_{\alpha+n}(a)$ to be the coefficients of the expansion

$$D_s^\alpha\left[\zeta(s,a) - \frac{1}{a^s}\right] = \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + (-1)^{-\alpha}\sum_{n=0}^{\infty} \frac{(-1)^n C_{\alpha+n}(a)}{n!}(s-1)^n, \tag{7}$$

We now conclude our consideration of the sum $G_{s,a}^\alpha(m) + I_s^\alpha(m) - D(s) + G(s)$ obtained from the Euler-Maclaurin summation for $D_s^\alpha\left[\zeta(s,a) - \frac{1}{a^s}\right]$. For the last term $G(s)$ we have:

$$G(s) = \int_m^\infty P_1(x)g'(x)dx = \int_m^\infty P_1(x)\left[-s\frac{\log^\alpha(x+a)}{(x+a)^{s+1}} + \alpha\frac{\log^{\alpha-1}(x+a)}{(x+a)^{s+1}}\right]dx.$$

From the definition of the fractional Stieltjes constants it follows that:

$$\sum_{n=0}^{\infty} H_m^{\alpha+n}\frac{(-1)^{\alpha+n}(s-1)^n}{n!} + G(s) = \sum_{n=0}^{\infty}\frac{(-1)^{\alpha+n}C_{\alpha+n}(s-1)^n}{n!}.$$

Therefore, taking successive derivatives with respect to s, of both sides of this equation, and then evaluating them at $s = 1$, we obtain for all $n \in \mathbb{N}\cup\{0\}$:

$$C_{\alpha+n}(a) = H_m^{\alpha+n} + G^{(n)}(1). \tag{8}$$

Setting $n = 0$ in (8) and noting that $g_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$, we obtain

$$\begin{aligned} C_\alpha(a) &= H_m^{\alpha+0} + G(1) \\ &= \sum_{r=1}^{m}\frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x)g_\alpha'(x)dx, \end{aligned}$$

which proves:

Theorem 3.2. *Let $\alpha \in \mathbb{R}$ with $\alpha > 0$, $0 < a \leq 1$ and $m \in \mathbb{N}$. We have*

$$C_\alpha(a) = \sum_{r=1}^{m}\frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x)g_\alpha'(x)dx, \tag{9}$$

where $g_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$ and $P_1(x) = x - \lfloor x\rfloor - \frac{1}{2}$.

Letting $m \to \infty$ immediately yields (for all $\alpha > 0$ and $0 < a \leq 1$) the natural generalization of Berndt's result (2) which Kreminski [8] used to define the fractional $\gamma_\alpha(a)$ for $\alpha \in \mathbb{R}$:

Corollary 3.3.

$$\gamma_\alpha(a) - \frac{\log^\alpha(a)}{a} = C_\alpha(a) = \lim_{m\to\infty}\left\{\sum_{r=1}^{m}\frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1}\right\},$$

Corollary 3.4. *As $\alpha \to 0^+$, $\gamma_\alpha(1) \to \gamma - 1$, where $\gamma = \gamma_0(1)$ is Euler's constant.*

Proof. Observe that, with $a = 1$, the left-hand sum in (7) becomes $\sum_{n=0}^{\infty} \frac{\log^\alpha(n+1)}{(n+1)^s}$, which, with $\alpha \to 0^+$, will converge to

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^s} = \sum_{n=2}^{\infty} \frac{1}{n^s} = \zeta(s) - 1.$$

From the Laurent series expansion of $\zeta(s)$ about $s = 1$, we have

$$\zeta(s) - 1 = \frac{1}{s-1} + \gamma - 1 + \sum_{n=2}^{\infty} \frac{(-1)^n \gamma_n(1) \cdot (s-1)^n}{n!}. \tag{10}$$

Hence, in order to maintain equality in (7), the right hand side of (7) must approach $\zeta(s) - 1$ as $\alpha \to 0^+$. This occurs if and only if $\gamma_\alpha(1) \to \gamma - 1$ as $\alpha \to 0^+$. □

Note: It follows that $\gamma_\alpha(1)$, as a function of α, is discontinuous at $\alpha = 0$.

4. Kreminski's Conjecture

Now we are ready to prove our main result, namely [8, Conjecture (IIIa)]:

Theorem 4.1. *Let $h_a(s) := \zeta(s,a) - \frac{1}{s-1} - \frac{1}{a^s}$ and let $h_a^{(\alpha)}(s) = D_s^\alpha [h_a(s)]$ be the α-th Grünwald-Letnikov fractional derivative of h_a. Then*

$$C_\alpha(a) := \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a} = (-1)^{-\alpha} h_a^{(\alpha)}(1).$$

Note: Kreminski's original statement (in [8]) of the conjecture is slightly different than ours, due to his use of the Weyl fractional derivative W_s^α. For $0 < \alpha < 1$ the Weyl fractional derivatives of the relevant functions are

$$W_s^\alpha \left[\frac{1}{s-1} \right] = \frac{(-1)^{-\alpha} \alpha\pi \csc(\alpha\pi)}{\Gamma(1-\alpha)(s-1)^{\alpha+1}} = (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}}$$

and

$$W_s^\alpha \left[\zeta(s,a) - \frac{1}{a^s} \right] = (-1)^{-\alpha} \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s}.$$

These expressions differ by a factor $(-1)^{2\alpha}$ from the Grünwald-Letnikov fractional of the same functions (see Lemma 2.1), the same factor by which our restatement of Kreminski's conjecture differs from the original.

Proof. We have

$$h_a^{(\alpha)}(s) = D_s^\alpha [h_a(s)] = D_s^\alpha \left[\zeta(s,a) - \frac{1}{a^s} \right] - D_s^\alpha \left[\frac{1}{s-1} \right].$$

Applying formulas from the parts (b) and (c) of Lemma 2.1 we readily obtain

$$h_a^{(\alpha)}(s) = (-1)^\alpha \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} - \frac{(-1)^\alpha \Gamma(\alpha+1)}{(s-1)^{\alpha+1}},$$

or equivalently

$$(-1)^{-\alpha} h_a^{(\alpha)}(s) = \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s} - \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \frac{\log^{\alpha}(a)}{a}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!}(s-1)^n - \frac{\log^{\alpha}(a)}{a}.$$

Evaluating $h_a^{(\alpha)}(s)$ at the point $s=1$, we get:

$$(-1)^{-\alpha} h_a^{(\alpha)}(1) = \gamma_{\alpha}(a) - \frac{\log^{\alpha}(a)}{a} = C_{\alpha}(a),$$

which finishes the proof. □

5. Acknowledgements

The authors would like to thank the referee for a number of helpful suggestions.

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