A ZERO FREE REGION FOR THE FRACTIONAL DERIVATIVES OF THE RIEMANN ZETA FUNCTION

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Abstract. For any $\alpha \in \mathbb{R}$, we denote by $D_s^{\alpha}[\zeta(s)]$ the α-th Grünwald-Letnikov fractional derivative of the Riemann zeta function $\zeta(s)$. We prove that

$$D_s^{\alpha}[\zeta(s)] \neq 0$$

inside the region $|s-1| < 1$. This result is proved by a careful analysis of integrals involving Bernoulli polynomials and bounds for fractional Stieltjes constants.

1. Introduction

The Riemann zeta function $\zeta(s)$ and its derivatives $\zeta^{(k)}(s)$ are defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{(\log n)^k}{n^s},$$

for all $k \in \mathbb{N}$, everywhere in the complex half-plane $\Re(s) > 1$.

In [2], the authors have investigated the zero-free regions of higher derivatives $\zeta^{(k)}(s)$, and have discovered not only that, for all $k \in \mathbb{N}$, all of these derivatives have *identical* counts of zeros in $\Re(s) > 1/2$, but that there exists a dynamics that, with discretely increasing k, moves the non-trivial zeros of $\zeta^{(k)}(s)$, in a one-to-one fashion, to the right, in a virtually periodic manner. Due to increasing density of the zeros in vertical direction, this simple bijective idea is difficult to state quantitatively; however, the observed "flow" suggests that *fractional* derivatives (the Grünwald-Letnikov derivatives $D_s^{\alpha}[\zeta(s)]$, in particular) could provide the missing link needed to establish this property. Despite the incredible amount of research concerning $\zeta(s)$ and its derivatives, the fractional derivatives have been largely neglected.

We will not try to prove the audacious one-to-one conjecture in this paper, but we will establish a zero-free region for fractional derivatives of $\zeta(s)$, which – although modest and far from optimal – is proved in an elementary way, and seems to be the first of its kind.

We start by recalling some basics. First, note that $\zeta(s)$ can be extended to a meromorphic function with a simple pole at $s = 1$, with residue 1, and has a Laurent series expansion:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (s-1)^n, \tag{1}$$

where γ_n are the Stieltjes constants [10]. Bounds for *fractional* Stieltjes constants will be needed in the proof of our zero-free region. Before we define them, let us note that for any

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$\alpha \in \mathbb{C}$, the so-called "reverse α^{th} Grünwald-Letnikov derivative" of $f(z)$ is (see [6]):

$$D_z^\alpha [f(z)] = \lim_{h\to 0^+} \frac{\Delta_h^\alpha f(z)}{h^\alpha} = \lim_{h\to 0^+} \frac{(-1)^\alpha \sum\limits_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^\alpha},$$

whenever the limit exists. Thus defined, $D_z^\alpha[f(z)]$ coincides with the standard derivatives for all $\alpha \in \mathbb{N}$. Also, they satisfy: $D_z^0[f(z)] = f(z)$ and $D_z^\alpha \left[D_z^\beta [f(z)]\right] = D_z^{\alpha+\beta} [f(z)]$. And if $f(z)$ is analytic, then $D_z^\alpha[f(z)]$ is an analytic function of both α and z. (Note: although the Grünwald-Letnikov derivative is defined for all $\alpha \in \mathbb{C}$, in this paper we only consider $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, since these cases are most useful in the theory of the Riemann zeta function.)

Finally, let us note that, in [8] it was shown that for $z \in \mathbb{C}$ we have $D_z^\alpha [e^{-az}] = (-1)^\alpha a^\alpha e^{-az}$, which for $\zeta(s)$ implies the following: For all $s \in \mathbb{C}$ with $\Re(s) > 1$, we have

$$D_s^\alpha [\zeta(s)] = (-1)^\alpha \sum_{n=1}^{\infty} \frac{\log^\alpha(n+1)}{(n+1)^s}. \tag{2}$$

2. Fractional Stieltjes Constants

The *fractional Stieltjes constants* γ_α where $\alpha \in \mathbb{R}^{>0}$ were introduced by Kreminski [7] and can be defined as the coefficients of the Laurent expansion of the α-th Grünwald-Letnikov fractional derivative of $\zeta(s)$ for $s \neq 1$ [4]:

$$D_s^\alpha [\zeta(s)] = (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}}{n!} (s-1)^n. \tag{3}$$

In view of this, it becomes clear that in order to establish regions of non-vanishing of these derivatives (which is the main objective of this paper), one needs to investigate behavior of the fractional Stieltjes constants in more detail. In [4] (in the process of proving a conjecture of Kerminski concerning the special values of the derivatives of Hurwitz zeta functions), we have proved the following useful generalization of a result of Williams & Zhang [11]:

For $\alpha > 0$ and $m \in \mathbb{N}$,

$$\gamma_\alpha = \sum_{r=1}^{m} \frac{\log^\alpha(r)}{r} - \frac{\log^{\alpha+1} m}{\alpha+1} - \frac{\log^\alpha(m)}{2m} + \int\limits_m^\infty P_1(x) f_\alpha'(x) dx, \tag{4}$$

where $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$ and $f_\alpha(x) = \dfrac{\log^\alpha x + 1}{x+1}$. Integrating (4) by parts m times yields

$$\int_m^\infty P_1(x) f_\alpha'(x) dx = \sum_{j=1}^{v} \left[P_j(x) f_\alpha^{(j-1)}(x)\right]_{x=m}^{\infty} + (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) dx$$

$$= -\sum_{j=1}^{v} P_j(m) f_\alpha^{(j-1)}(m) + (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) dx \tag{5}$$

where for $k \in \mathbb{N}$, $P_k(x) = \frac{B_k(x-\lfloor x \rfloor)}{k!}$ is the k^{th} periodic Bernoulli polynomial and B_k is the k^{th} Bernoulli number. What's more, the derivatives of f_α can be written in terms of the (signed) Stirling numbers (see [5, Proposition 2.2]) as follows:

$$f_\alpha^{(n)}(x) = \sum_{i=0}^{n} s(n+1, i+1)(\alpha)_i \frac{\log^{\alpha-i}(x+1)}{(x+1)^{n+1}}, \tag{6}$$

where $(\alpha)_i = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}$ is the falling factorial. This particular result was applied (see [5, Theorem 4.1]) in the proof of an upper bound of the fractional Stieltjes constants:

$$|\gamma_\alpha| \leq \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!}, \tag{7}$$

where $n \in \mathbb{N}$, such that $1 \leq n < \alpha$. These estimates present a natural generalization of the bounds for the so-called generalized Stieltjes constants, see [11, Theorem 3].

3. Three Lemmas

We begin the construction of our proof with the following three lemmas.

Lemma 3.1. *Let* $0 < \alpha \leq 1$ *and* $f_\alpha(x) = \frac{\log^\alpha(x+1)}{x+1}$. *Then* $\left| \int_1^\infty P_3(x) f_\alpha'''(x) dx \right| < 0.013$.

Note: Ostrowski observed, in [9], that for odd $n > 1$ one has: $|P_n(x)| < \frac{2}{(2\pi)^n}$.

Proof. Let us consider the expression (6). With the help of the triangle inequality, and the change of variables for the integral, we are able to write:

$$\begin{aligned} \left| \int_1^\infty P_3(x) f_\alpha'''(x) dx \right| &< \frac{2}{(2\pi)^3} \sum_{i=0}^{3} |s(4, i+1)(\alpha)_i| \int_1^\infty \frac{\log^{\alpha-i}(x+1)}{(x+1)^4} dx \\ &< \frac{2}{(2\pi)^3} \sum_{i=0}^{3} \frac{|s(4, i+1)(\alpha)_i|}{3^{\alpha-i+1}} \int_{3\log(2)}^\infty x^{\alpha-i} e^{-x} dx. \end{aligned} \tag{8}$$

We will estimate each of the four summands on the right side of the inequality separately.

We start with $i = 0$. Since $x^\alpha \leq x$ in the interval $[3\log(2), \infty)$, we can write

$$\frac{|s(4,1)(\alpha)_0|}{3^{\alpha+1}} \int_{3\log(2)}^\infty x^\alpha e^{-x} dx \leq \frac{6}{3^{\alpha+1}} \int_{3\log(2)}^\infty x e^{-x} dx = \frac{1}{4} \frac{3\log(2)+1}{3^\alpha}. \tag{9}$$

For $i = 1$, in the interval $[3\log(2), \infty)$ we have $x^{\alpha-1} \leq 3^{\alpha-1} \log^{\alpha-1}(2)$, for all $\alpha \leq 1$; thus

$$\frac{|s(4,2)(\alpha)_1|}{3^\alpha} \int_{3\log(2)}^\infty x^{\alpha-1} e^{-x} dx \leq \frac{11\alpha}{3^\alpha} 3^{\alpha-1} \log^{\alpha-1}(2) \int_{3\log(2)}^\infty e^{-x} dx \leq \frac{11 \log^{\alpha-1}(2)}{24}. \tag{10}$$

Now, for the summand corresponding to $i = 2$ we have

$$\frac{|s(4,3)(\alpha)_2|}{3^{\alpha-1}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-2}e^{-x}dx = \frac{6|\alpha(\alpha-1)|}{3^{\alpha-1}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-2}e^{-x}dx \tag{11}$$

$$\leq \frac{3}{2}\frac{1}{3^{\alpha-1}}3^{\alpha-2}\log^{\alpha-2}(2) \int\limits_{3\log(2)}^{\infty} e^{-x}dx = \frac{\log^{\alpha-2}(2)}{16},$$

since for $0 < \alpha \leq 1$ we have $|\alpha(\alpha-1)| \leq \frac{1}{4}$ and for $x \in [3\log(2), \infty)$: $x^{\alpha-2} \leq 3^{\alpha-2}\log^{\alpha-2}(2)$.

Finally, for $i = 3$ we can write

$$\frac{|s(4,4)(\alpha)_3|}{3^{\alpha-2}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-3}e^{-x}dx = \frac{|\alpha(\alpha-1)(\alpha-2)|}{3^{\alpha-2}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-3}e^{-x}dx \tag{12}$$

$$\leq \frac{2\sqrt{3}}{9}\frac{3^{\alpha-3}\log^{\alpha-3}(2)}{3^{\alpha-2}} \int\limits_{3\log(2)}^{\infty} e^{-x}dx = \frac{\sqrt{3}\log^{\alpha-3}(2)}{108},$$

since $|\alpha(\alpha-1)(\alpha-2)| \leq \frac{2}{9}\sqrt{3}$ for $\alpha \in (0,1]$ and $x^{\alpha-3} \leq 3^{\alpha-3}\log^{\alpha-3}(2)$ for $x \in [3\log(2), \infty)$.

Combining these four bounds, we conclude:

$$\left|\int\limits_1^{\infty} P_3(x)f_\alpha'''(x)\right| < \tfrac{2}{(2\pi)^3}\left[\tfrac{1}{4}\tfrac{3\log(2)+1}{3^\alpha} + \tfrac{11\log^{\alpha-1}(2)}{24} + \tfrac{\log^{\alpha-2}(2)}{16} + \tfrac{\sqrt{3}\log^{\alpha-3}(2)}{108}\right] < 0.013, \tag{13}$$

as desired. □

Lemma 3.2. *If $0 < \alpha < 1$, then $|\gamma_\alpha| < 0.436$.*

Proof. Taking $m = 2$ in the representation (4), we get

$$\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} + \int\limits_2^{\infty} P_1(x)f_\alpha'(x)dx.$$

But from (5) we know that

$$\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - P_2(1)f_\alpha'(1) + P_3(1)f_\alpha''(1) + \int\limits_2^{\infty} P_3(x)f_\alpha'''(x)dx.$$

So, with $P_2(1) = \frac{B_2}{2!} = \frac{1}{12}$ and $P_3(1) = \frac{B_3}{3!} = 0$ and $f_\alpha'(x) = \alpha\frac{\log^{\alpha-1}(2)}{4} - \frac{\log^\alpha(2)}{4}$ we obtain

$$\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - \frac{1}{12}\left[\alpha\frac{\log^{\alpha-1}(2)}{4} - \frac{\log^\alpha(2)}{4}\right] + \int\limits_1^{\infty} P_3(x)f_\alpha'''(x)dx$$

$$= \frac{13\log^\alpha(2)}{48} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - \frac{\alpha\log^{\alpha-1}(2)}{48} + \int\limits_1^{\infty} P_3(x)f_\alpha'''(x)dx.$$

Now, note that the maxima of the first three terms are attained when $\alpha = 0$. Since the bound obtained in Lemma 3.1 also holds for the absolute value of the integral $\int_2^\infty P_1(x) f'_\alpha(x) dx$, we immediately obtain the wanted bound: $|\gamma_\alpha| \leq 0.436$. $\square$

Lemma 3.3. *For all $\alpha > 0$, we have*

$$(i)\ \frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} < 0.348 \quad \text{and} \quad (ii)\ \frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} \leq 0.323.$$

Proof. Combining the bound for $|\gamma_\alpha|$ proved in Lemma 3.2 and the fact that $\Gamma(\alpha+1) \geq \Gamma(3/2) = \frac{\sqrt{2\pi}}{2}$, for $0 < \alpha \leq 1$, we deduce that $\frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} < \frac{0.436}{\frac{\sqrt{2\pi}}{2}} < 0.348$ in the region $0 < \alpha \leq 1$.

Now, in the complementary region $\alpha > 1$, by (7), for all $1 \leq n < \alpha$, we have

$$\begin{aligned} \frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} &\leq \frac{4}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!} \leq \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \left(\frac{4(n+1)}{e}\right)^{n+1} \\ &= \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)^{\alpha-n}} \left(\frac{4}{e}\right)^{n+1} \leq 4\sqrt{2}\left(\frac{2}{\pi e}\right)^{n+1} \leq 4\sqrt{2}\left(\frac{2}{\pi e}\right)^{2} \leq 0.311, \end{aligned}$$

which is an even sharper bound. Together, these two bounds prove (i) for all $\alpha > 0$.

Similarly, to justify (ii), note that since $\alpha + 1 > 1$, the equation (7) with $n = 1$ yields

$$\frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} \leq \frac{4\Gamma(\alpha+2)4!}{(2\pi)^2 2^{\alpha+2} 2!\Gamma(\alpha+1)} = \frac{12(\alpha+1)}{(2\pi)^2 2^\alpha}. \tag{14}$$

The maximum of $g(\alpha) = \frac{\alpha+1}{2^\alpha}$ is at $\alpha = \frac{1}{\log(2)} - 1$. This immediately yields the result (ii). $\square$

4. A Zero Free Region

We need one more technical lemma before we can prove our main theorem.

Lemma 4.1. *For all $\alpha > 0$ and $n \in \mathbb{N} \cup \{0\}$,*

$$\frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^\alpha} < \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2} < 1.036,$$

where

$$\alpha_1 = \frac{\sqrt{5\log^2(3)+4}}{2\log(3)} + \frac{1}{\log(3)} - \frac{3}{2}.$$

Proof. We proceed by induction on n. For $n = 0$ we have

$$\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)2!3^\alpha} = \frac{\alpha^2+3\alpha+2}{3^\alpha 2}.$$

The maximum of $g(\alpha) = \frac{\alpha^2+3\alpha+2}{3^\alpha 2} = \frac{(\alpha^2+3\alpha+2)e^{-\alpha\log(3)}}{2}$ is at $\alpha_1 = \frac{\sqrt{5\log^2(3)+4}}{2\log(3)} + \frac{1}{\log(3)} - \frac{3}{2}$, with $g(\alpha_1) = 1.0356$. Now, let us assume that, for all integers j with $1 \leq j \leq n$, we have

$$\frac{\Gamma(\alpha+j+3)}{\Gamma(\alpha+1)(j+2)!2^j(j+3)^\alpha} \leq \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}.$$

We will show the assertion is true for $j = n+1$. Applying the induction hypothesis gives

$$
\begin{aligned}
\frac{\Gamma(\alpha+j+3)}{\Gamma(\alpha+1)(j+2)!2^j(j+3)^\alpha} &= \frac{\Gamma(\alpha+n+4)}{\Gamma(\alpha+1)(n+3)!2^{n+1}(n+4)^\alpha} \\
&= \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}\frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^\alpha} \\
&\leq \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}\frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}.
\end{aligned} \tag{15}
$$

Hence, all we need to show is that $\frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3} \leq 1$. However, notice that the function $g(\alpha) = \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}$ is positive for all $\alpha > 0$; and taking the logarithmic derivative we get

$$
\frac{g'(\alpha)}{g(\alpha)} = \log\left(\frac{n+3}{n+4}\right) + \frac{1}{\alpha+n+3} \;\leq\; -\frac{1}{n+4} - \frac{1}{2}\left(\frac{1}{n+4}\right)^2 + \frac{1}{\alpha+n+3},
$$

since, from the Taylor's Theorem, we know that $\log(1-x) \leq -x - \frac{1}{2}x^2$, in the range $0 \leq x < 1$. Moreover, $\frac{1}{\alpha+n+3} \leq \frac{1}{n+4}$, and since $g(\alpha) > 0$, we can conclude that $g'(\alpha) < 0$. Therefore $g(\alpha)$ is decreasing in the interval $[1, \infty)$, with the maximum at $g(1) = \frac{1}{2}$.

On the other hand, if $0 < \alpha < 1$, the maximum of $\left(\frac{n+3}{n+4}\right)^\alpha$ is attained at $\alpha = 0$. And since $\frac{\alpha+n+3}{n+3} < \frac{n+4}{n+3} = 1 + \frac{1}{n+3} \leq \frac{4}{3}$, we have $g(\alpha) < \frac{1}{2}\frac{4}{3} = \frac{2}{3}$, for $\alpha \in (0,1)$. Combining these two results in (15), we deduce the bound for $j = n+1$. This completes the inductive proof. ☐

Now we are ready to prove our main result.

Theorem 4.2. *For all $\alpha \geq 0$, $D_s^\alpha[\zeta(s)] \neq 0$ in the region $|s-1| < 1$.*

Proof. For $\alpha = 0$, the reader is referred to [1]. We prove that $\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)} D_s^\alpha[\zeta(s)] \neq 0$ in the region $|s-1| < 1$. Starting with (3), we are able to write

$$
\begin{aligned}
\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| &= \left|1 + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(s-1)^{\alpha+n+1}}{\Gamma(\alpha+1)n!}\right| \\
&\geq 1 - \frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} - \frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} - \sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}.
\end{aligned}
$$

Applying Lemma 3.3, we see that

$$
\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| \;>\; 1 - 0.492 - 0.323 - \sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}. \tag{16}
$$

We can focus now on finding an upper bound for $\sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}$. By (7) we have

$$
\frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!} \;\leq\; \frac{4\Gamma(\alpha+n+1)(2(n+1))!}{(2\pi)^{n+1}(n+1)^{\alpha+n+1}(n+1)^{\alpha+n+1}(n+1)!n!\Gamma(\alpha+1)}.
$$

It follows from Stirling's formula that $\frac{(2n)!}{n!} \leq \sqrt{2}\left(\frac{4n}{e}\right)^n$ for all integers $n \geq 1$. Therefore

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!} &\leq \sum_{n=2}^{\infty} \frac{4\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)^{\alpha+n+1}n!\Gamma(\alpha+1)}\sqrt{2}\left(\frac{4(n+1)}{e}\right)^{n+1} \\
&= \sum_{n=2}^{\infty} \frac{4\sqrt{2}\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)^{\alpha}n!\Gamma(\alpha+1)}\left(\frac{4}{e}\right)^{n+1} \\
&= 4\sqrt{2}\left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^{\alpha}}\left(\frac{4}{\pi e}\right)^n \\
&\leq 4\sqrt{2}\left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}\left(\frac{4}{\pi e}\right)^n < 0.142,
\end{aligned}$$

by Lemma 4.1 (and with the same notation). Using this bound in (16), we obtain

$$\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| > 1 - 0.492 - 0.323 - 0.142 > 0.$$

We conclude that $D_s^{\alpha}[\zeta(s)] \neq 0$, for all $\alpha > 0$, in the region $|s-1| < 1$. $\square$

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