New zero-free regions for the derivatives of the Riemann Zeta Function

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Abstract

We describe new zero-free regions for the derivatives $\zeta^{(k)}(s)$ of the Riemann zeta function: they take form of vertical strips in the right half-plane; and we show that the zeros located in the narrow complements of these zero-free regions – which, in analogy with the classical case, we call "critical strips" – tend to converge to their central "critical lines" and exhibit surprising vertical periodicities that enable one to give exact formulas for their number. An immediate corollary of the method is the fact that all the zeros contained inside these new critical strips are simple.

1. Introduction

In this paper we investigate the distribution of zeros of higher derivatives of the Riemann zeta function. In order to put our main results in perspective,

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we first give a brief summary of some of the most important results and outstanding conjectures in this area.

Let $s = \sigma + it$. For all $k \in \mathbb{N}$, the k-th derivative of the Riemann zeta function $\zeta^{(k)}(s)$ is

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} \frac{\log^k n}{n^s}, \quad \text{for} \quad \sigma > 1, \tag{1}$$

and can be extended to a meromorphic function on $\mathbb{C}$, with a single pole (of order k) at the point $s = 1$. However, unlike $\zeta(s)$ itself, the functions $\zeta^{(k)}(s)$ have neither Euler products nor functional equations. Thus their nontrivial zeros do not lie on a line, but appear to be distributed (seemingly at random) to the right of the critical line $\sigma = \frac{1}{2}$. Speiser [8] was the first to show, in 1934, that the Riemann Hypothesis (RH) is equivalent to the fact that $\zeta'(s)$ has no zeros with $0 < \sigma < \frac{1}{2}$. Levinson and Montgomery [5] gave a simpler and more instructive proof of this and also showed that $\zeta'(s)$ can vanish on the critical line only at a multiple zero of $\zeta(s)$ if ever such a zero exists. They also showed, assuming the Riemann Hypothesis (RH), that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros with $\sigma < \frac{1}{2}$, for $k \geq 1$. For $k = 1$ they proved unconditionally that $\zeta'(s)$ has only real zeros in the closed left half-plane. For $k = 2$ and $k = 3$, Yıldırım [14] established, assuming RH, that $\zeta^{(k)}(s)$ has no zeros with $0 \leq \sigma < \frac{1}{2}$, and, unconditionally, that both $\zeta''(s)$ and $\zeta'''(s)$ have exactly one pair of nontrivial zeros with $\sigma < 0$. Namely $\zeta''(s)$ has zeros at approximately $s = -0.35508433021 \pm 3.590839324398i$ and $\zeta'''(s)$ at approximately $s = -2.110145792653 \pm 2.58422477204i$.

In regions to the right of the critical line, i.e. for $\sigma \geq \frac{1}{2}$, the total number of zeros of $\zeta^{(k)}(s)$ does not differ by much from the number of zeros of $\zeta(s)$. In fact, if we let $N(T)$ and $N_k(T)$ denote the number of such zeros ρ, with $0 \leq \Im(\rho) \leq T$, of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively, then according to a theorem of Berndt [1]

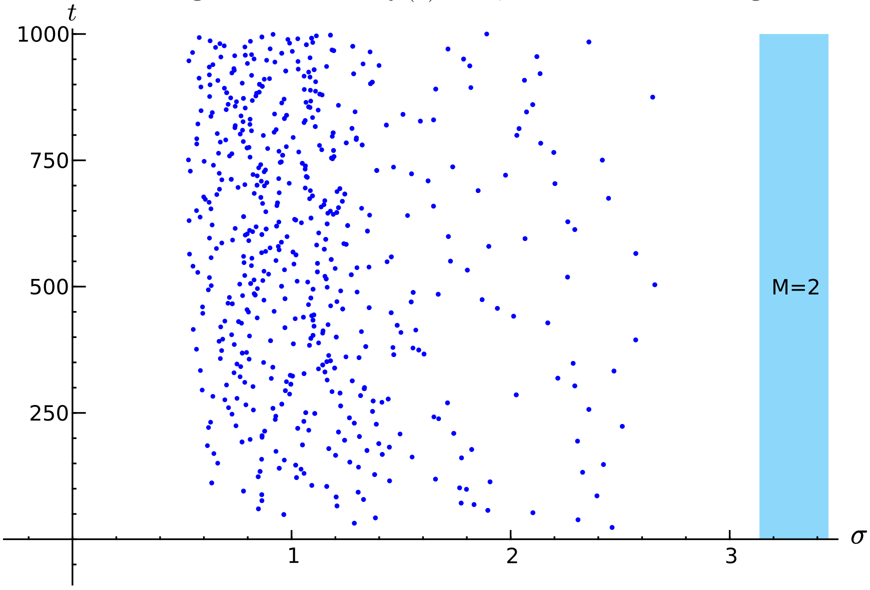
$$N_k(T) = N(T) - \frac{T}{2\pi} \log 2 + O(\log T), \tag{2}$$

where, by the classical Riemann-von Mangoldt formula (see Landau [4]),

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

It should also be noted that most nontrivial zeros of $\zeta^{(k)}(s)$ are located relatively close to the line $s = \frac{1}{2} + it$. In fact, in recent years, in a series

Figure 1: Zeros of $\zeta'(s)$ in $\mathbb{C}$, with the zero-free region.



of improvements, Soundararajan [7], Zhang [15], and Feng [2], succeeded in showing (conditionally) that, for $k = 1$, a positive portion of the zeros ρ of $\zeta'(s)$ satisfies $\Re(\rho) < \frac{1}{2} + c/\log T$. Nevertheless, for all $k \in \mathbb{N}$, many of the zeros of $\zeta^{(k)}(s)$ lie much farther to the right, even though their real parts can still be effectively bounded from above by absolute constants (see Figure 1 for illustration of the bound in the case $k = 1$). For $k \geq 3$ such general upper bounds were first given by Spira [9] in 1965, and they were later improved by Verma and Kaur [13] (see Table 1):

$$\zeta^{(k)}(\sigma + it) \neq 0 \quad \text{for} \quad \sigma > (1.13588\ldots)k + 2$$

In this work we explicate some new, unexpected properties of the location of zeros of $\zeta^{(k)}(s)$ in the intermediate regions $\frac{1}{2} \leq \Re(s) < (1.13588\ldots)k + 2$. In particular we dispel the notion of the apparent randomness of these zeros by proving the existence of a sequence of zero-free regions for $\zeta^{(k)}(s)$, and by showing that all zeros found in the strips between them exhibit a fascinating vertical periodicity. This enables us to give exact formulas for their number, while also proving that all the zeros of $\zeta^{(k)}(s)$ inside them are simple.

Table 1: Lower real bounds for zero-free regions in the right half-plane.

	ζ	ζ'	ζ''	$\zeta^{(k)}$ for $k \geq 3$
Hadamard [3], de la Vallée-Poussin [12]	1			
Titchmarsh [11]		$E < 3$		
Spira [9]				$\frac{7}{4}k + 2$
Verma & Kaur [13]				$(1.13588\ldots)k + 2$
Skorokhodov [6]		2.93938	4.02853	

2. Statement of Main Results

In what follows, we restrict our treatment to the case $k \geq 3$. To state our results precisely, we introduce some notation and definitions. Let

$$Q_n^k(s) := (\log n)^k / n^s$$

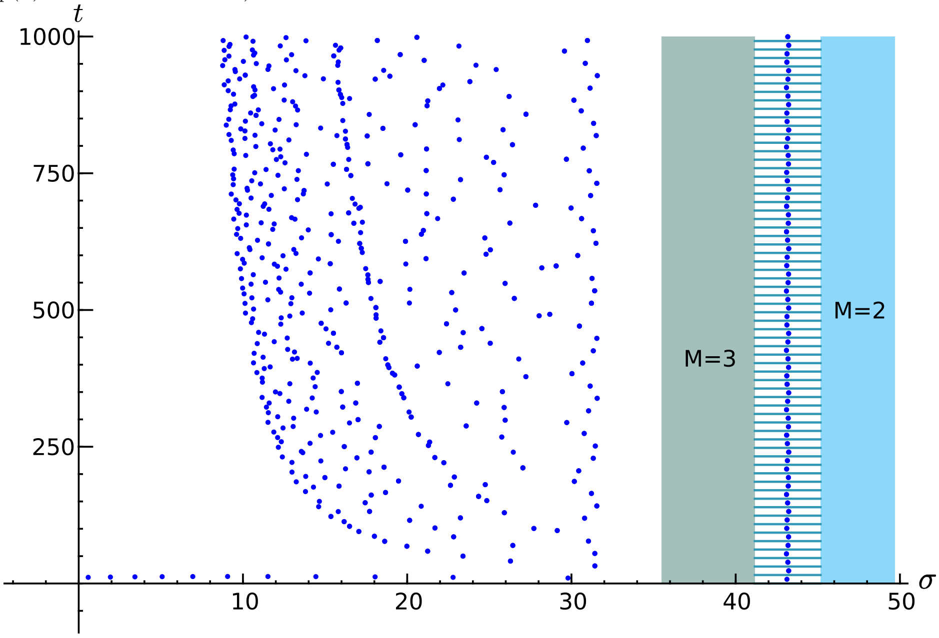
denote the n-th term of the Dirichlet series (eq1) for $(-1)^k \zeta^{(k)}(s)$. All the previously known zero-free regions for $\zeta^{(k)}(s)$ have been obtained by finding solutions to

$$\left|\zeta^{(k)}(s)\right| = \left|\sum_{n=2}^{\infty} Q_n^k(s)\right| \geq Q_2^k(\sigma) - \sum_{n=3}^{\infty} Q_n^k(\sigma) > 0,$$

or some variation thereof (see [6, 11, 13]); that is, by finding the regions of the complex plane where the term $Q_2^k(s)$ *dominates* all the other terms of the expansion (eq1) of $\zeta^{(k)}(s)$ (i.e. $Q_2^k(s)$ is greater in modulus than the rest of the terms combined), because then, evidently, $\zeta^{(k)}(s) \neq 0$. However, $Q_2^k(s)$ is not always the dominant term; any other term can not only be the largest in modulus, but take the dominant role as well. This is clear from the fact that $|Q_n^k(s)| = Q_n^k(\sigma)$, viewed as a function of n, has its global maximum at $n = e^{k/\sigma}$. Using this simple property one can show the existence of regions where $Q_n^k(s)$ (for any $n \geq 2$) becomes the dominant term of (eq1), which then provides us with a new zero-free region of $\zeta^{(k)}(s)$, for each $n \in \mathbb{N}$, for every sufficiently large k.

Let us denote by $Q_M^k(s)$ the term of (eq1) which has the largest modulus. If we fix some such M, then the moduli of the terms of (eq1) will increase for $m < M$ and decrease for $m > M$, in monotone fashion (see section 3). Since no term $Q_M^k(s)$ can attain dominance on a line where its absolute value is

Figure 2: Zeros of $\zeta^{(38)}(s)$ in $\mathbb{C}$, with zero-free regions (characterized by the dominance of $Q_M^{38}(s)$ for $M = 2$ and 3)



equal to that of another term (and by the aforesaid property this can only happen when $Q_M^k(\sigma) = Q_{M+1}^k(\sigma)$ or $Q_M^k(\sigma) = Q_{M-1}^k(\sigma)$), it is reasonable to expect that the zeros of $\zeta^{(k)}(s)$ will be located close to the lines where this equality occurs. Thus we define

$$q_M := \frac{\log\left(\frac{\log M}{\log(M+1)}\right)}{\log\left(\frac{M}{M+1}\right)}, \tag{3}$$

so that $Q_M^k(\sigma) = Q_{M+1}^k(\sigma)$ whenever $\sigma = q_M k$. (Note that $q_2 = 1.13588\ldots$, $q_3 = 0.808484\ldots$, $q_4 = 0.668855\ldots$, where q_2 is the constant that appears in Table 1.) In the $k\sigma$-plane, $\sigma = q_M k$ defines a line of slope q_M which will be called the M-th critical line.

Our first main result describes zero free regions between these critical lines for sufficiently large k:

Theorem 2.1. *Let $k \in \mathbb{N}$ and $u \in \mathbb{R}^{>0}$ a solution of* $1 - \frac{1}{e^u-1} - \frac{1}{e^u}(1+\frac{1}{u}) \geq 0$.

(a) *If* $q_3k + 4\log 3 < q_2k - 2$, *then* $\zeta^{(k)}(s) \neq 0$ *for*

$$q_3k + 4\log 3 \leq \sigma \leq q_2k - 2.$$

(b) *If* $M \in \mathbb{N}$, $M > 3$, *and* $q_Mk + (M+1)u \leq q_{M-1}k - Mu$ *then* $\zeta^{(k)}(s) \neq 0$ *for*

$$q_Mk + (M+1)u \leq \sigma \leq q_{M-1}k - Mu.$$

Remark 2.2. The value for $u \in \mathbb{R}^{>0}$ that gives the widest zero free regions is the solution of

$$1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) = 0,$$

namely $u = 1.1879426249\ldots$.

We call the region between two zero free regions critical strips. Thus the M-th critical strip S_M^k of $\zeta^{(k)}(s)$ is the open set

$$\begin{aligned} S_2^k &:= \{\,\sigma + it \mid q_2k + 2 < \sigma < q_2k - 2\,\}, \\ S_3^k &:= \{\,\sigma + it \mid q_3k - 4u < \sigma < q_3k + 4\log 3\,\}, \\ S_M^k &:= \{\,\sigma + it \mid q_Mk - (M+1)u < \sigma < q_Mk + (M+1)u\,\} \text{ for } M > 3. \end{aligned}$$

So the zero-free regions are the connected components that remain after one removes S_M^k from the right half-plane.

Another way to visualize the critical strips S_M^k is to consider them in the $k\sigma$-plane (see Figure 4). In this representation, the wedges correspond to the zero-free regions, i.e. the regions of dominance of the terms $\frac{\log^k M}{M^s}$ (for $M = 2$ this is treated by Verma and Kaur [13], for $M \geq 3$ it is new), while the critical strips S_M^k are the narrow regions centered around the critical lines that separate the wedges. For $M \geq 3$ the k-coordinates of the tips of the wedges in the k-σ-plane are

$$k_3 = \frac{4\log 3 + 2}{q_2 - q_3} \quad \text{and} \quad k_M = \frac{(2M+1)u}{q_{M-1} - q_M} \quad \text{for } M \geq 4, \tag{4}$$

which immediately implies that the first critical strips S_2^k can be observed for all $k \geq 20$, the second S_2^k for all $k \geq 77$, and the third S_3^k for all $k \geq 163$ and so on. With some extra work, these values can be improved to $k \geq 19$ for S_2^k, $k \geq 58$ for S_3^k, and $k \geq 123$ for S_4^k (see Remark 4.5).

Figure 3: Zero-free regions and horizontal zero-free line segments for $\zeta^{(100)}$, $\zeta^{(200)}$, $\zeta^{(400)}$, and $\zeta^{(800)}$.

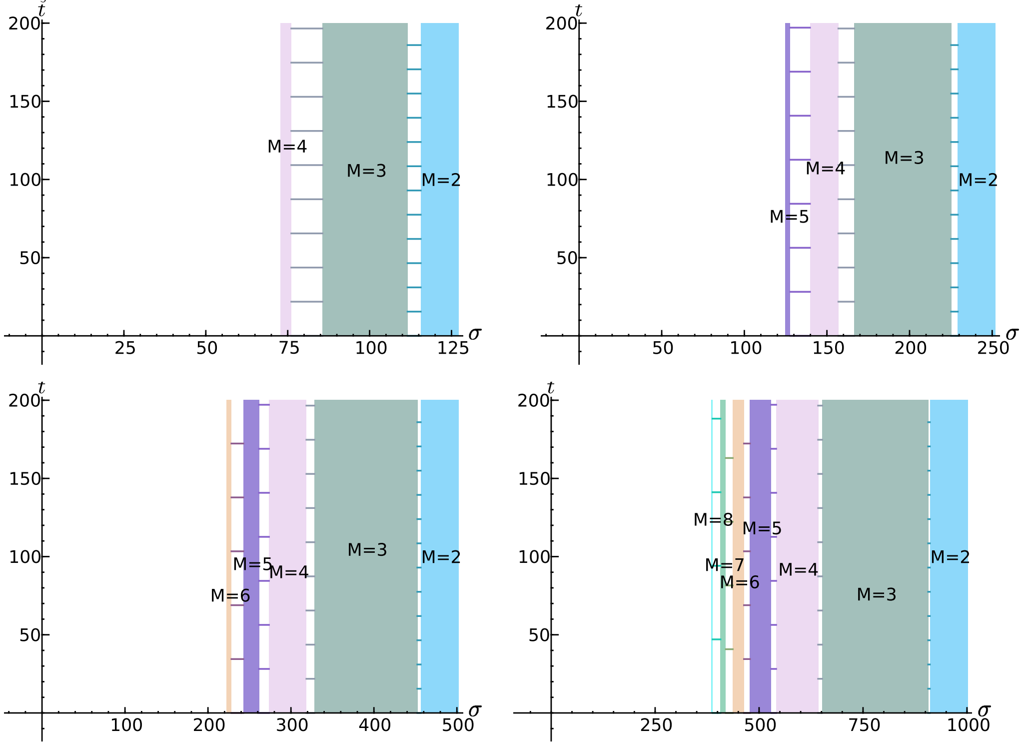
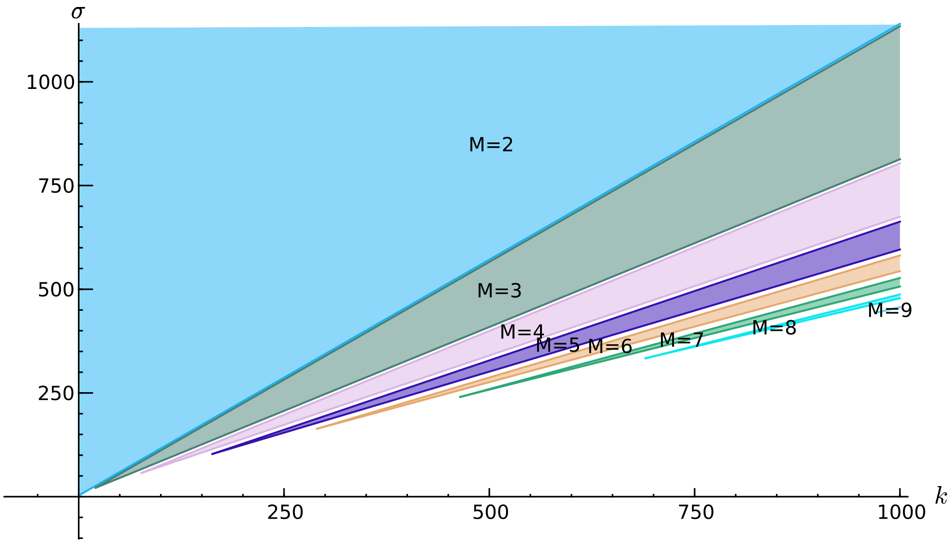


Figure 4: Zero-free regions of $\zeta^{(k)}(\sigma+it)$, for $M=2,\dots,9$.



Moreover, if one also considers the imaginary parts of the solutions of $Q_M^k(q_M k+it)+Q_{M+1}^k(q_M k+it)=0$, then one obtains

$$t=\frac{\pi(2j+1)}{\log(M+1)-\log(M)} \tag{5}$$

for $j\in\mathbb{Z}$, showing that the location of the zeros ρ inside S_M^k is close to

$$k\cdot q_M+\frac{\pi(2j+1)i}{\log\left(\frac{M+1}{M}\right)}$$

for some $j\in\mathbb{N}$. This suggests a vertical periodicity in the limit of the zeros of $\zeta^{(k)}(s)$ at the critical lines. (The computational data confirms that the M-th period equals $\pi/(\log(M+1)-\log(M))$.) With the help of Rouché's theorem, we are able to show that between every two consecutive lines $s=\sigma+\frac{2\pi ji}{\log(M+1)-\log M}$, that horizontally partition the critical strip S_M^k (see Figure 3), there is exactly one zero of $\zeta^{(k)}(s)$.

That is our second main result:

Theorem 2.3. *Let $u\in\mathbb{R}^{>0}$ be a solution of $1-\frac{1}{e^u-1}-\frac{1}{e^u}(1+\frac{1}{u})\geq 0$. Let $M\in\mathbb{N}$, $M>3$, and $j\in\mathbb{N}$. If there is $k\in\mathbb{N}$ with*

$$q_{M+1}k+(M+2)u\leq q_M k-(M+1)u$$

then each rectangle $R_j \subset S_M^k$, consisting of all $s = \sigma + it$ with

$$q_M k - (M+1)u < \sigma < q_M k + (M+1)u$$

and

$$\frac{2\pi j}{\log(M+1) - \log(M)} < t < \frac{2\pi(j+1)}{\log(M+1) - \log(M)},$$

contains exactly one zero of $\zeta^{(k)}(s)$. This zero is simple.

Remark 2.4. The corresponding result also holds for the critical strips S_2^k and S_3^k.

Clearly, Theorem 2.3 can be converted into an exact formula for the number of zeros of $\zeta^{(k)}(s)$ (for carefully chosen values of T) inside any given critical strip.

Corollary 2.5. *Let $N_M^k(T)$ denote the number of zeros ρ of $\zeta^{(k)}(s)$ which are inside S_M^k and satisfy $\Im(\rho) \leq T$. Then, for all $j \geq 1$,*

$$N_M^k\left(\frac{2\pi j}{\log(M+1) - \log(M)}\right) = j.$$

Remark 2.6. An immediate consequence is that for $k \geq 3$ and $T > 0$,

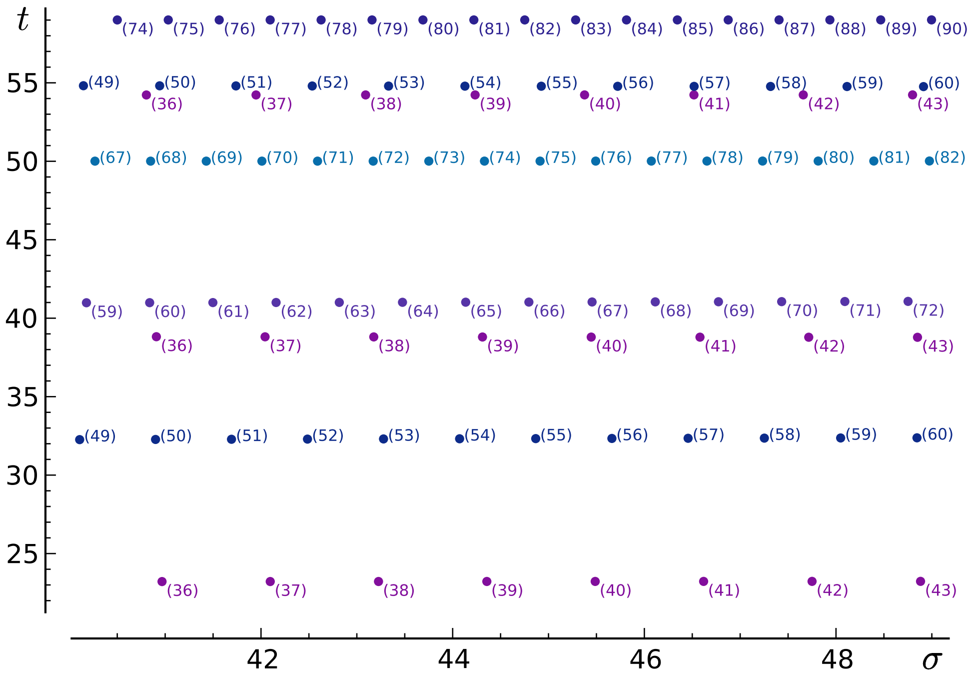
$$N_M^k(T) = \frac{\log(M+1) - \log(M)}{2\pi} T + O(1).$$

This, of course, implies that the total number of zeros contained within any fixed critical strip is $O(T) = o(N_k(T))$.

Spira [9] had already noticed that the zeros of $\zeta'(s)$ and $\zeta''(s)$ seem to come in pairs, where the zero of $\zeta''(s)$ is always located to the right of the zero of $\zeta'(s)$. More recently, with the help of extensive computations, Skorokhodov [6] observed this behavior for higher derivatives as well. Our observations support a straightforward one-to-one correspondence between the zeros of $\zeta^{(k)}(s)$ and $\zeta^{(m)}(s)$ for all $k, m \geq 1$ (Figure 5).

From (eqconjt) we see that the approximation of the imaginary part of a zero of $\zeta^{(k)}(s)$ depends on M but not on k. Therefore, it follows that, with growing k, the critical strips $\{S_M^k\}_{k=2}^{\infty}$ undergo a shift to the right, and the length of this horizontal shift is approximately q_M for each increment of k. In other words, by Theorem 2.3 all zeros of $\zeta^{(k)}(s)$ contained in a given S_M^k will

Figure 5: The consecutive zeros $\bullet^{(k)}$ of the derivatives of $\zeta^{(k)}(\sigma + it)$ in the sample region: $40 < \sigma < 49$ and $20 < t < 60$.



keep shifting (almost) linearly, and with a (almost) fixed shift q_M to the right, as $k \to \infty$ (see Figure 3). An interesting consequence of this observation is that, in addition to the aforementioned vertical quasi-periodicity, we also have a rigid horizontal pattern of zeros. Even more surprisingly (although this is difficult to quantify via counting functions): all derivatives of $\zeta(s)$ have *exactly the same number* of nontrivial zeros.

Conjecture 2.7. *For a positive integer k there is a one-to-one correspondence between the non-trivial zeros $s \in \mathbb{C}$ of $\zeta^{(k)}(s)$ with $\Im(s) > 1$ and $\zeta^{(k+1)}(s)$, such that the index M for two such corresponding zeros is the same, and their difference is approximately q_M.*

Remark 2.8. The zero-free regions obtained in Theorem 2.1 may be generalized to a large class of Dirichlet series. Since we only consider the absolute values of the coefficients, it follows that if $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, and $|a_M| \geq |a_n|$ for some $M \geq 3$ and all $n \geq 2$, then $L^{(k)}(s) \neq 0$ for $q_M k + cM \leq \sigma \leq$

$q_{M-1}k - c(M-1)$, for a suitable constant $c \geq 0$. Extensions and generalization of the remaining results of this paper are more dependent on specific parameters of Dirichlet series (such as the growth of $|a_n|$, as $n \to \infty$); and we relegate those investigations to a future project.

3. An Auxiliary Lemma

We consider the function $z : \mathbb{R}^{>0} \to \mathbb{R}$, $x \mapsto \frac{\log^k x}{x^\sigma}$ for fixed $\sigma > 1$ and $k \in \mathbb{N}$. We have

$$z'(x) = \left(\left(\frac{\log x}{x^\sigma}\right)^k\right)' = k\left(\frac{x^{\sigma-1} - \sigma(\log x)x^{\sigma-1}}{x^{2\sigma}}\right)\left(\frac{\log x}{x^\sigma}\right)^{k-1}.$$

Hence $z'(x) = 0$ if $x^{\sigma-1}(1-\sigma\log x) = 0$, that is, $x = e^{1/\sigma}$. Since $z'(x) > 0$ for $0 < x < e^{1/\sigma}$ and $z'(x) < 0$ for $x > e^{1/\sigma}$, the function $z(x)$ has its maximum at $x = e^{1/\sigma}$.

As we have chosen q_M such that $Q_M^k(\sigma) = Q_{M+1}^k(\sigma)$ for $\sigma = q_M k$, the maximum of $z(x)$ (again for $\sigma = q_M k$) lies between $x = M$ and $x = M+1$. As the maximum of $z(x)$ is at $x = e^{1/\sigma}$, the maximum of $z(x)$ for $\sigma > q_M k$ is to the left of the maximum of $z(x)$ for $\sigma = q_M k$. So the value of σ for which $Q_M^k(\sigma)$ is the largest term in the Dirichlet series representation of $\zeta^{(k)}(\sigma)$ is between $\sigma = q_M k$ and $\sigma = q_{M-1}k$. Thus $Q_M^k(\sigma)$ can dominate $\zeta^{(k)}(\sigma)$ only there.

We will use these monotonicity and dominance considerations implicitly in the proofs of our theorems.

Now, we consider the $k\sigma$-plane interpretation of Theorem 2.1. In general, the wedges in Figure 4 are the sets containing all points (k, σ) that satisfy

$$q_M k + b_1 < \sigma < q_{M-1}k + b_2.$$

for some $M \in \mathbb{N}$ and $b_1, b_2 \in \mathbb{R}$. Thus

$$k \geq \frac{b_1 - b_2}{q_{M-1} - q_M}, \tag{6}$$

with equality holding exactly if $k = k_M$.

The growth properties of q_M play an important role in understanding the critical strips S_M^k.

Lemma 3.1. *For all $n \geq 3$ we have*

$$\frac{1}{\log n} \ < \ q_{n-1} \ < \ \frac{1}{\log(n-1)}.$$

Proof. In order to prove the lower bound, we write

$$\begin{aligned}
\alpha_{n-1} \ &:= \ \frac{\log(n-1)}{\log n} = 1 + \frac{\log(n-1) - \log n}{\log n} = 1 + \frac{\log(\frac{n-1}{n})}{\log n}, \\
\beta_{n-1} \ &:= \ \log(\alpha_{n-1}) = \log\left(1 + \frac{\log(\frac{n-1}{n})}{\log n}\right) < \frac{\log(\frac{n-1}{n})}{\log n},
\end{aligned}$$

where the last inequality holds because $\log(1+x) < x$ whenever $x > -1$. The desired lower bound now immediately follows from $q_{n-1} = \beta_{n-1}/\log((n-1)/n)$.

In order to prove the upper bound, we write $\theta_n := -\log\left(\frac{n-1}{n}\right)$. Then we have:

$$\begin{aligned}
q_{n-1} \ &= \ \frac{\log\left(\frac{\log(n-1)}{\log n}\right)}{\log\left(\frac{n-1}{n}\right)} = \frac{\log\left(1 - \frac{-\log\left(\frac{n-1}{n}\right)}{\log n}\right)}{\log\left(\frac{n-1}{n}\right)} = \frac{\log\left(1 - \frac{\theta_n}{\log n}\right)}{\log\left(\frac{n-1}{n}\right)} \\
&= \ \frac{1}{\log n} + \frac{\theta_n}{2(\log n)^2} + \frac{\theta_n^2}{3(\log n)^3} + \frac{\theta_n^3}{4(\log n)^4} + \cdots \\
&< \ \frac{1}{\log n} + \frac{1}{2\log n}\left(\frac{\theta_n}{\log n} + \left(\frac{\theta_n}{\log n}\right)^2 + \left(\frac{\theta_n}{\log n}\right)^3 + \cdots\right) \\
&= \ \frac{1}{\log n} + \frac{1}{2\log n}\,\frac{\theta_n}{\log n - \theta_n} = \frac{1}{\log n} + \frac{1}{2\log n}\,\frac{\log(1+\frac{1}{n-1})}{\log(n-1)} \\
&< \ \frac{1}{\log n} + \frac{1}{2(\log n)(\log(n-1))(n-1)} < \frac{1}{\log(n-1)},
\end{aligned}$$

where the last inequality holds if and only if

$$\log n - \log(n-1) > \frac{1}{2(n-1)},$$

which is true by the mean value theorem. □

How many distinct critical strips of $\zeta^{(k)}(s)$ are there inside the region $1/2 \leq \sigma < q_2 k + 2$? Let $c(k)$ denote that number. Then in view of Lemma 3.1

it seems reasonable to expect that, for all $k \geq 2$, there exist positive constants A and B, such that

$$A\frac{\sqrt{k}}{\log k} < c(k) < B\frac{\sqrt{k}}{\log k}.$$

Upper bounds of the desired order are easier to prove than lower bounds: obviously, one can just count the number of wedges, with their tips located at points described in (eqs), and then invert the relation. Since the difference $q_{M-1} - q_M$ in the denominator of this fraction can be nicely bounded from above (but not from below), using the estimates in our lemma, effective upper bounds can be obtained.

4. Proof of Theorem 2.1

Now we are ready to prove our first main result. We will show that $\zeta^{(k)}(s)$ has no zeros if (k, σ) in the $k\sigma$-plane lies in one of the wedges given by an inequality of the form

$$q_M k + b_1 \leq \sigma \leq q_{M-1}k + b_2$$

for suitably chosen $b_1, b_2 \in \mathbb{R}$. We choose b_1, b_2 such that these wedges are the regions where $Q_M^k(s) = \frac{\log^k M}{M^s}$ is the dominant term (in the modulus) of $\zeta^{(k)}(s)$. Everywhere hereafter we write $H_M^k(s)$ for the "head" and $T_M^k(s)$ for the "tail" of the series $\zeta^{(k)}(s)$ split by $Q_M^k(s)$:

$$H_M^k(s) := \sum_{n=2}^{M-1} Q_n^k(s) = \sum_{n=2}^{M-1} \frac{\log^k n}{n^s}$$

and

$$T_M^k(s) := \sum_{n=M+1}^{\infty} Q_n^k(s) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^s}.$$

Our goal will be to show that

$$|\zeta^{(k)}(s)| \geq Q_M^k(\sigma) - H_M^k(\sigma) - T_M^k(\sigma) = Q_M^k(\sigma)\left(1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{T_M^k}{Q_M^k}(\sigma)\right) > 0$$

for our choice of b_1 and b_2, keeping in mind that

$$\frac{Q_{M+1}^k}{Q_M^k}(q_M k + b_1) = \left(\frac{M}{M+1}\right)^{b_1} \quad \text{and} \quad \frac{Q_{M-1}^k}{Q_M^k}(q_{M-1}k + b_2) = \left(\frac{M}{M-1}\right)^{b_2},$$

as one can easily verify.

The Tails

We first find an upper bound for the tails $T_M^k(\sigma)$.

Lemma 4.1. *Fix some integer $M \geq 2$, and assume $k-1 < (\sigma-1)\log M$. Then*

$$T_M^k(\sigma) = \sum_{n=M+1}^{\infty} \frac{\log^k n}{n^\sigma} \leq \int_M^\infty \frac{\log^k x}{x^\sigma} dx < Q_M^k(\sigma) R_M^k(\sigma), \tag{7}$$

where

$$R_M^k(\sigma) = \frac{M}{\sigma-1}\left(1 + \frac{k}{(\sigma-1)\log M - k + 1}\right).$$

Proof. For $k \in \mathbb{Z}$, the integral in (intbdi) can be written in a closed form. Applying recursively the general formula (for all $b, -a \neq -1$)

$$\int \frac{(\log x)^a}{x^b}\, dx = -\frac{(\log x)^a}{(b-1)x^{b-1}} + \frac{a}{b-1}\int \frac{(\log x)^{a-1}}{x^b}\, dx,$$

we obtain

$$\begin{aligned}
\int_M^\infty \frac{\log^k x}{x^\sigma} dx &= \frac{\log^k M}{M^\sigma}\frac{M}{\sigma-1}\sum_{r=0}^{k}\frac{k!}{(k-r)!}\frac{\log^{-r} M}{(\sigma-1)^r} \\
&\leq Q_M^k(\sigma)\frac{M}{\sigma-1}\Big(1+\sum_{r=1}^{k} k(k-1)^{r-1}\Big(\frac{1}{(\sigma-1)\log M}\Big)^r\Big) \\
&< Q_M^k(\sigma)\frac{M}{\sigma-1}\Big(1+\frac{k}{(\sigma-1)\log M}\sum_{r=0}^{\infty}\Big(\frac{k-1}{(\sigma-1)\log M}\Big)^r\Big) \\
&= Q_M^k(\sigma)\frac{M}{\sigma-1}\Big(1+\frac{k}{(\sigma-1)\log M - k+1}\Big),
\end{aligned}$$

where the convergence of the geometric series is implied by $k-1 < (\sigma-1)\log M$. □

It is clear why estimating $R_M^k(\sigma)$ will be vital for the proofs of our theorems. We note:

Lemma 4.2. *If $a_1 k + b_1 \leq \sigma$ and $K \leq k$, then*

$$R_M^k(\sigma) \leq R_M^k(a_1 k + b_1) \leq R_M^K(a_1 K + b_1), \tag{8}$$

as long as the following two conditions are satisfied:

$$a_1 > \frac{1}{\log M} \quad \text{and} \quad (a_1 \log M - 1)K + 1 + (b_1 - 1)\log M > 0,$$

and in the case of $b_1 < 1 - 1/\log M$ *also:*

$$K \geq \frac{1}{a_1 \log M}\left(-(b_1 - 1)\log M - 1 + \sqrt{\frac{|(b_1-1)\log M + 1|}{a_1 \log M - 1}}\right).$$

Proof. The left-hand inequality of (9) is evident from the fact that $R^k_M(\sigma)$ is decreasing when viewed as a function of σ alone. The right-hand inequality of (9) is equivalent to saying that $R^k_M(\sigma)$ is decreasing as a function of k. To see this we rewrite $\frac{1}{M\log M} R^k_M(a_1 k + b_1)$ in the form

$$y(k) = \frac{1}{(c+1)k + d - 1} \frac{(c+1)k + d}{ck + d},$$

where $c := a_1 \log M - 1 > 0$ and $d := 1 + (b_1 - 1)\log M$, then clearly

$$y'(k) = -\frac{c(1+c)^2 k^2 + 2cdk(1+c) + d(1+cd)}{((c+1)k + d - 1)^2 (ck+d)^2},$$

from which it is easy to see that $y'(k)$ can change sign only if $d < 0$ (otherwise it remains nonpositive). However, the condition $d < 0$ translates to $b_1 < 1 - 1/\log M$, in which case one requires $K \geq z_0$, where

$$z_0 := -\frac{d}{1+c} + \frac{1}{1+c}\sqrt{\frac{|d|}{c}}$$

is the right zero of the numerator of the above expression for $y'(k)$. □

We will use the estimate for $T^k_M(\sigma)$ from Lemma 4.1 in the proof of Theorem 2.1 via the separation:

$$\begin{aligned} T^k_M(\sigma) &= Q^k_{M+1}(\sigma) + T^k_{M+1}(\sigma) \\ &\leq Q^k_{M+1}(\sigma)(1 + R^k_{M+1}(\sigma)) \\ &\leq Q^k_M(q_M k + b_1)(1 + R^k_{M+1}(q_M k + b_1)), \end{aligned}$$

since $Q^k_{M+1}(\sigma) \leq Q^k_M(\sigma)$. The series with the remainder $R^k_{M+1}(q_M k + b_1)$ will converge because $q_M > 1/\log(M+1)$ by Lemma 3.1, if b_1 is suitably chosen.

Verma and Kaur's bound (see Table 1) follows directly from Lemma 4.1 and Lemma 4.2. We include a proof of their result because it exemplifies several of the important ideas and illustrates key workings of our general method, being the special case of $M = 2$ (representing the dominance of the term $Q_2^k(\sigma)$).

Theorem 4.3 ([13, Theorem (A)]). *For all $\sigma \geq q_2 k + 2$ we have $\zeta^{(k)}(s) \neq 0$.*

Proof. First write

$$|\zeta^{(k)}(s)| \geq \frac{\log^k 2}{s^\sigma} - T_2^k(\sigma) \geq Q_2^k(\sigma)\left(1 - \frac{Q_3^k}{Q_2^k}(\sigma) - \frac{Q_4^k}{Q_2^k}(\sigma)\left(1 + R_4^k(\sigma)\right)\right).$$

By Lemma 4.2 we have $R_4^k(\sigma) \leq R_4^k(q_2 k+2) < 1.57$, for $k \geq 3$. Furthermore,

$$\frac{Q_4^k}{Q_2^k}(\sigma) = 2^{k-\sigma} \leq 2^{k-q_2k+2} \leq 2^{3(1-q_2)+2} \leq 0.19.$$

The quotient $\frac{Q_3^k}{Q_2^k}(\sigma)$ is decreasing in σ, and hence $\frac{Q_3^k}{Q_2^k}(\sigma) \leq \frac{Q_3^k}{Q_2^k}(q_2 k + 2) = \frac{4}{9}$. So we obtain

$$1 - \frac{Q_3^k}{Q_2^k}(\sigma) - \frac{Q_4^k}{Q_2^k}(\sigma)\left(1 + R_4^k(\sigma)\right) \geq 1 - \frac{4}{9} - 0.19(1 + 1.57) > 0,$$

which establishes the result. □

Since Theorem 2.1 (a) deals with the next case of $M = 3$ (corresponding to the dominance of the term $Q_3^k(\sigma)$), and only a little bit of extra effort is needed to prove it, we give a proof of it right now.

Proof of Theorem 2.1 (a). For a zero free region to exist we must have

$$q_3 k + 4\log 3 \leq q_2 k - 2,$$

which implies $k \geq 20$. Separating the dominant term $Q_3^k(\sigma)$, we get

$$\begin{aligned} |\zeta^{(k)}(s)| &\geq Q_3^k(\sigma) - Q_2^k(\sigma) - T_3^k(\sigma) \\ &\geq Q_3^k(\sigma)\left(1 - \frac{Q_2^k}{Q_3^k}(\sigma) - \frac{Q_4^k}{Q_3^k}(\sigma)\left(1 + R_4^k(\sigma)\right)\right). \end{aligned}$$

Therefore we only need to show that

$$1 - \frac{Q_2^k}{Q_3^k}(\sigma) - \frac{Q_4^k}{Q_3^k}(\sigma)\left(1 + R_4^k(\sigma)\right) > 0.$$

By Lemma 4.2, $R_4^k(\sigma) \leq R_4^k(q_3k + 4\log 3) \leq R_4^{k_3}(q_3k_3 + 4\log 3) < 0.72$, for $\sigma \geq q_3k + 4\log 3$ and $k \geq k_3 = \frac{4\log 3+2}{q_2-q_3} = 19.5311\ldots$. Also, $\frac{Q_4^k}{Q_3^k}(\sigma) \leq \frac{Q_4^k}{Q_3^k}(q_3k + 4\log 3) < 0.29$ and $\frac{Q_2^k}{Q_3^k}(\sigma) \leq \frac{Q_2^k}{Q_3^k}(q_2k - 2) < 0.45$. Hence

$$1 - \frac{Q_2^k}{Q_3^k}(\sigma) - \frac{Q_4^k}{Q_3^k}(\sigma)\left((1 + R_4^k(\sigma)\right) > 1 - 0.45 - 0.29(1 + 0.72) > 0,$$

as desired. □

Theorem 2.1 (b) deals with the dominance of the general term $Q_M^k(\sigma)$, and consequently requires knowledge of the behavior of the sum of all the terms preceding it.

The Heads

We rewrite the heads of the series (eq1) in the following form:

$$H_M^k(\sigma) = Q_M^k(\sigma)\left(\frac{Q_{M-1}^k}{Q_M^k}(\sigma) + \frac{Q_{M-2}^k}{Q_M^k}(\sigma) + \ldots + \frac{Q_2^k}{Q_M^k}(\sigma)\right) \tag{9}$$

$$= Q_M^k(\sigma)\left(\frac{Q_{M-1}^k}{Q_M^k}(\sigma)\left(1+\frac{Q_{M-2}^k}{Q_{M-1}^k}(\sigma)\left(1+\ldots\left(1+\frac{Q_2^k}{Q_3^k}(\sigma)\right)\ldots\right)\right)\right) \tag{10}$$

and we will find upper bounds for all the above quotients $\frac{Q_{n-1}^k}{Q_n^k}(\sigma)$ of consecutive terms. Clearly $\frac{Q_{n-1}^k}{Q_n^k}(\sigma) = \left(\frac{\log(n-1)}{\log n}\right)^k \left(\frac{n}{n-1}\right)^\sigma$ and therefore $\frac{H_M^k}{Q_M^k}(\sigma)$ increases with σ. For $2 \leq n \leq M$ and $\sigma \leq q_{M-1}k + b_2$ we get

$$\frac{Q_{n-1}^k}{Q_n^k}(\sigma) \leq \frac{Q_{n-1}^k}{Q_n^k}(q_{M-1}k + b_2) \leq \frac{Q_{n-1}^k}{Q_n^k}(q_{n-1}k + b_2) = \left(\frac{n}{n-1}\right)^{b_2},$$

where the second inequality holds because $q_{M-1} < q_n$ for $n \leq M$, while the equality holds because $\sigma = q_{n-1}k$ is the solution of $Q_n^k(\sigma) = Q_{n-1}^k(\sigma)$. Thus, in order for $\frac{H_M^k}{Q_M^k}(\sigma)$ to stay bounded, we must choose $b_2 < 0$.

Lemma 4.4. *Let $c \in \mathbb{R}$ be positive. Then $y(n) = \left(\frac{n-1}{n}\right)^{cn}$ is monotonously increasing with asymptote $1/e^c$.*

Proof. As $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{cn} = e^c$, we evidently have $\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{cn} = 1/e^c$. Finally,

$$y'(n) = c \cdot y(n)\left(\log\left(1-\frac{1}{n}\right)+\frac{1}{n-1}\right) > 0$$

proves the monotonicity assertion. □

Thus for $2 \le n \le M$ and $\sigma \le q_{M-1}k - uM$ we have

$$\frac{Q^k_{n-1}}{Q^k_n}(\sigma) \le \left(\frac{n}{n-1}\right)^{-uM} \le \left(\frac{M}{M-1}\right)^{-uM} \le \frac{1}{e^u}.$$

Now (eqQquot-prod) yields

$$\frac{H^k_M}{Q^k_M}(\sigma) \le \sum_{n=1}^{\infty}\frac{1}{(e^u)^n} = \frac{1}{1-\frac{1}{e^u}} - 1 = \frac{1}{e^u-1}. \tag{11}$$

Proof of Theorem 2.1 (b). Similar to the proof of Theorem 2.1 (a) we write

$$\begin{aligned} \left|\zeta^{(k)}(s)\right| &\ge Q^k_M(\sigma) - H^k_M(\sigma) - T^k_M(\sigma) \\ &\ge Q^k_M(\sigma)\left(1 - \frac{H^k_M}{Q^k_M}(\sigma) - \frac{Q^k_{M+1}}{Q^k_M}(\sigma)\left(1+R^k_{M+1}(\sigma)\right)\right). \end{aligned}$$

Now, notice that

$$R^k_M(\sigma) := \frac{M}{\sigma-1}\left(1+\frac{k}{(\sigma-1)\log M - k + 1}\right) < \frac{1}{u}$$

is equivalent to: $(\sigma-1)^2\log M - (\sigma-1)(cM\log M + k - 1) - uM > 0$; and this quadratic inequality is satisfied whenever

$$\begin{aligned} \sigma &> 1 + \frac{(uM\log M + k - 1) + \sqrt{(uM\log M + k-1)^2 + 4M\log M}}{2\log M} \\ &> 1 + \frac{2(uM\log M + k - 1)}{2\log M} = 1 + uM + \frac{k-1}{\log M}. \end{aligned}$$

Thus, by Lemma 4.2, for $\sigma \ge q_M k + u(M+1)$, $k \ge k_M = \frac{(2M+1)u}{q_{M-1}-q_M}$, and $M \ge 4$, we have

$$R^k_{M+1}(\sigma) \le R^{k_M}_{M+1}(q_M k_M + u(M+1)) < \frac{1}{u}.$$

By Lemma 4.4 we also have

$$\frac{Q_{M+1}^k}{Q_M^k}(q_M k + u(M+1)) = \left(\frac{M}{M+1}\right)^{u(M+1)} < \frac{1}{e^u},$$

thus, with (eqheadhalf), we obtain, for $M \geq 4$ and $q_M k + u(M+1) \leq \sigma \leq q_{M-1}k + uM$,

$$1 - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{Q_{M+1}^k}{Q_M^k}(\sigma)\left(1 + R_M^k(\sigma)\right) > 1 - \frac{1}{e^u - 1} - \frac{1}{e^u}\left(1 + \frac{1}{u}\right) \geq 0,$$

which proves the theorem. □

Remark 4.5. The zero-free regions we have given are not the largest possible. For example, if one considered the lines $\sigma = \frac{1}{2}\left((q_M + q_{M-1})k + u\right)$ through the centers of the wedges and searches for the lowest k for which there were no zeros on those lines, then one would obtain the following values for k_M (which are lower than the values we have for the tips of the wedge-shaped regions):

M	3	4	5	6	7	8	9	10
k_M on line	19	58	123	220	354	529	748	1014
k_M at the tip	20	77	163	291	465	691	971	1313

5. Proof of Theorem 2.3

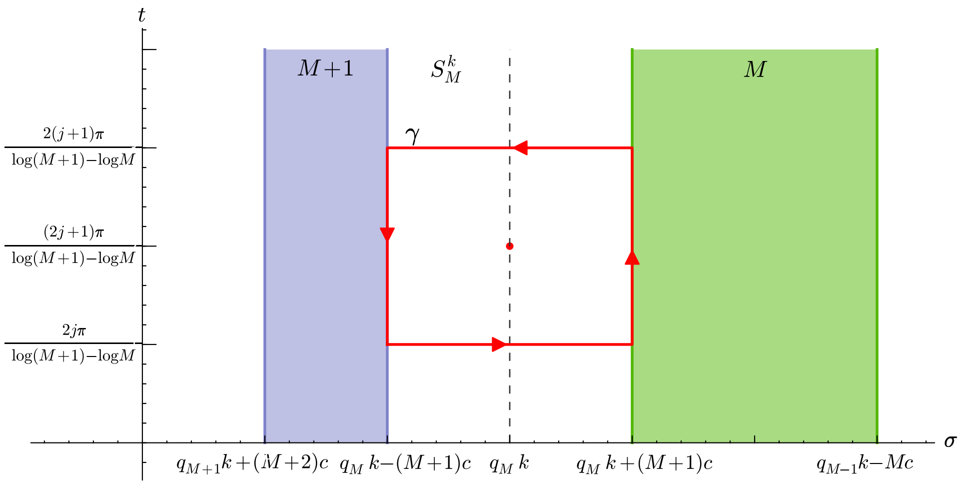
Because of the property of the quasi-periodicity of the zeros of $\zeta^{(k)}(s)$ inside S_M^k we are able to count the zeros by individual separation. In order for our approach to work, we first find horizontal, periodically-spaced zero-free line segments within the critical strips (in Lemma 5.1). Then we show that there is always exactly one zero of $\zeta^{(k)}(s)$ in the rectangles R_j (for $j \in \mathbb{N}$) that are delimited by the vertical edges of two neighboring zero-free regions and two horizontal zero-free lines (see Figure 6).

As already mentioned above, in the critical strips S_M^k, which are located between two consecutive zero-free regions, where the expansion of $\zeta^{(k)}(s)$ is dominated by the terms $Q_M^k(s)$ and $Q_{M+1}^k(s)$ respectively, one can obtain values of the imaginary parts t of expected zeros by solving the equation $Q_M^k(\sigma + it) = Q_{M+1}^k(\sigma + it)$ (an act of balancing the real and imaginary parts of two largest terms), and then choosing the horizontal lines of separation exactly halfway between them, thus managing to avoid even the most irregular of zeros inside S_M^k. That is exactly what we do below. It is a consequence of this that all zeros of $\zeta^{(k)}(s)$ inside S_M^k are simple.

Lemma 5.1. *Let $M \geq 2$ and $k \in \mathbb{N}$. If $s \in S_M^k$, then $\zeta^{(k)}(s) \neq 0$ for*

$$s = \sigma + i \cdot \frac{2\pi j}{\log(M+1) - \log M}.$$

Figure 6: The curve γ is the boundary of the rectangle R_j. The point $\bullet$ represents a zero of $Z(s) = Q_M^k(s) + Q_{M+1}^k(s)$ on the critical line $\sigma = q_M k$.



Proof. In the center of the critical strip S_M^k, that is on the critical line $\sigma = q_M k$ we have $|Q_M^k(s)| = |Q_{M+1}^k(s)|$. We consider the line segments in S_M^k with

$$q_M k - (M+1)u \leq \sigma \leq q_M k + (M+1)u.$$

and

$$t = \frac{2\pi j}{\log(M+1) - \log M}, \text{ where } j \in \mathbb{Z},$$

see Figure 6. Our choice of t gives $Q_M^k(q_M k + it) + Q_{M+1}^k(q_M k + it) = 0$ (compare equation 5) and therefore $\cos(t \log M) = \cos(t \log(M+1))$ and

$\sin(t\log M) = -\sin(t\log(M+1))$. We set $s = \sigma + it$, with t and σ as above, and consider the real and imaginary parts of

$$\zeta^{(k)}(s) = \sum_{n=2}^{\infty} (\cos(t\log n) - i\cdot\sin(t\log n))\, Q_n^k(\sigma).$$

With $|\Im(Q_n^k(s)| \le Q_n^k(\sigma)$ and $|\Re(Q_n^k(s)| \le Q_n^k(\sigma)$ we obtain

$$\begin{aligned} |\Re(\zeta^{(k)}(s))| &\ge |\cos(t\log M)Q_M^k(\sigma) + \cos(t\log(M+1))Q_{M+1}^k(\sigma)| \\ &\quad - H_M^k(\sigma) - T_{M+1}^k(\sigma), \\ |\Im(\zeta^{(k)}(s))| &\ge |\sin(t\log M)Q_M^k(\sigma) + \sin(t\log(M+1))Q_{M+1}^k(\sigma)| \\ &\quad - H_M^k(\sigma) - T_{M+1}^k(\sigma). \end{aligned}$$

If $t = 0$, the situation is trivial. If $t \ne 0$, then we either have $|\sin(t\log M)| \ge \sin(\pi/4) = 1/\sqrt{2}$ or $|\cos(t\log M)| \ge \cos(\pi/4) = 1/\sqrt{2}$. Because $|\zeta^{(k)}(s)| \ge |\Re(\zeta^{(k)}(s))|$ and $|\zeta^{(k)}(s)| \ge |\Im(\zeta^{(k)}(s))|$ we get:

$$\begin{aligned} |\zeta^{(k)}(s)| &\ge \frac{1}{\sqrt{2}}\left(Q_M^k(\sigma) + Q_{M+1}^k(\sigma)\right) - H_M^k(\sigma) - T_{M+1}^k(\sigma) \\ &= Q_M^k(\sigma)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{Q_{M+1}^k}{Q_M^k}(\sigma) - \frac{H_M^k}{Q_M^k}(\sigma) - \frac{Q_{M+2}^k}{Q_M^k}(\sigma) - \frac{T_{M+2}^k}{Q_M^k}(\sigma)\right) \\ &= Q_M^k(\sigma)\left(\frac{1}{\sqrt{2}} - \frac{H_M^k}{Q_M^k}(\sigma) + \frac{Q_{M+1}^k}{Q_M^k}(\sigma)\left(\frac{1}{\sqrt{2}} - \frac{Q_{M+2}^k}{Q_{M+1}^k}(\sigma) - \frac{T_{M+2}^k}{Q_{M+1}^k}(\sigma)\right)\right) \end{aligned}$$

From the proof of Theorem 2.1 (b) we know that for $\sigma \ge q_{M+1}k + (M+2)u$ and $u = 1.1879426249\ldots$ (see Remark 2.2)

$$\begin{aligned} \frac{1}{\sqrt{2}} - \frac{Q_{M+2}^k}{Q_{M+1}^k}(\sigma) - \frac{T_{M+2}^k}{Q_{M+1}^k}(\sigma) &\ge \frac{1}{\sqrt{2}} - \frac{Q_{M+2}^k}{Q_{M+1}^k}(\sigma)\,(1 + R_{M+2}(\sigma)) \\ &\ge \frac{1}{\sqrt{2}} - \frac{1}{e^u}\left(1 + \frac{1}{c}\right) > 0. \end{aligned}$$

Similarly, since $\frac{H_M^k}{Q_M^k}(\sigma)$ is increasing in σ (see equation (eqQquot-prod)) and because $\sigma < q_{M-1}k - Mu$, we get with (eqheadhalf) that

$$\frac{1}{\sqrt{2}} - \frac{H_M^k}{Q_M^k}(\sigma) \ge \frac{1}{\sqrt{2}} - \frac{H_M^k}{Q_M^k}(q_{M-1}k - Mu) \ge \frac{1}{\sqrt{2}} - \frac{1}{e^u - 1} > 0,$$

which concludes the proof of the lemma. $\square$

Proof of Theorem 2.3. Let $Z(s) = Q_M^k(s) + Q_{M+1}^k(s)$. It is easy to check that the function $Z(s)$ has exactly one (simple) zero in R_j, namely

$$s = q_M k + i \cdot \frac{(2j+1)\pi}{\log(M+1) - \log M}.$$

In order to be able to apply Rouché's Theorem we need to show that $|\zeta^{(k)}(s) - Z(s)| < |Z(s)|$ for all s on R_j.

The vertical sides of R_j are in the zero free regions for M and $M+1$. As shown in the proof of Theorem 2.1 the term $Q_M^k(s)$ dominates $\zeta^{(k)}(s)$ on the right vertical side of R_j and the term $Q_{M+1}^k(s)$ dominates $\zeta^{(k)}(s)$ on the left vertical side of R_j. Thus $|\zeta^{(k)}(s) - Z(s)| < |Z(s)|$ on the vertical sides of R_j. Furthermore we have seen in the proof of Lemma 5.1 that $Z(s) = Q_M^k(s) + Q_{M+1}^k(s)$ dominates $\zeta^{(k)}(s)$ on the horizontal sides of R_j. Hence $|\zeta^{(k)}(s) - Z(s)| < |Z(s)|$ on the horizontal sides of R_j.

Therefore, by Rouché's Theorem, $Z(s)$ and $\zeta^{(k)}(s)$ have the same number of zeros inside R_j, for every $j \in \mathbb{N}$. This proves both the simplicity of all zeros of $\zeta^{(k)}(s)$ inside S_M^k, and the sharp formula for $N_M^k(T)$, as given in Corollary 2.5. □

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