Table of tame and wild kernels
of quadratic imaginary number fields of discriminants > -5000
(conjectural values)

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1. Introduction.

Assuming Lichtenbaum's conjecture one can compute conjectural values of orders of the tame kernels K_2O_F of quadratic imaginary number fields F.

Since in general these orders are not very large, and there are several results known concerning the $p-$rank of K_2O_F and of its subgroup W_F called the wild kernel, it is possible to determine the structure of these groups for the fields in question with discriminants $d > -5000$.

2. Notations.

- F is a number field with r_1 real and $2r_2$ complex embeddings.
- $\zeta_F(s)$ is the Dedekind zeta function of F, d is the discriminant of F.
- For F imaginary quadratic we denote $d' = d/4$, if $4|d$, and $d' = d$ otherwise.
- O_F is the ring of integers of F.
- K_nO_F is the nth Quillen K–group of O_F, and especially
- K_2O_F is the Milnor group of O_F (the tame kernel).
- W_F is the Hilbert kernel of F (the wild kernel).
- e_p is the $p-$rank of K_2O_F , where p is a prime or $p = 4$.
- w_2 is the $2-$rank of W_F .
- $w(F)$ is the number of roots of unity in F.
- $Cl(P)$ is the class group of a Dedekind ring P.
- $R_m(F)$ is a "twisted" version of the mth Borel regulator (cf. [Bo1]), the "twisted" regulator map $r_m(F)$ being a map

$$r_m(F) : K_{2m-1}O_F \to \left[(2\pi i)^{m-1}\mathbf{R}\right]^{d_m},$$

where $d_m = r_2$ for m even, $= r_1 + r_2$ for m odd, $m > 1$, and $d_1 = r_1 + r_2 - 1$, (this is just the order of vanishing of $\zeta_F(s)$ at $s = 1 - m$). $R_m(F)$ is the covolume of the image of $r_m(F)$ and differs by Borel's original one essentially by a power of π ([Bo2], there is also a shift $m \mapsto m + 1$ compared to the original notation).

3. Computing the value $\#K_2O_F$.

Lichtenbaum's conjecture [Li] (as modified by Borel [Bo]) asks whether for all number fields and for any integer $m \geq 1$ there is a relation of the form

$$\mathrm{res}_{s=1-m}\, \zeta_F(s)(s-1+m)^{-d_m(F)} \overset{?}{=} \pm \frac{\#K_{2m-2}(O_F)}{\#K^{\mathrm{ind}}_{2m-1}(O_F)_{\mathrm{tors}}} \cdot R_m(F)\,,$$

where the subscript “tors” denotes the torsion part, “res” the residue, and “ind” the indecomposable part. There is some evidence for this conjecture, namely for $m = 1$ this is the Dirichlet class number formula, and for $m = 2$ and F totally-real abelian it has been proved (up to a power of 2) by Mazur–Wiles [M–W] as a consequence of their proof of the main conjecture of Iwasawa theory (in this case $R_2(F) = 1$, though).

In what follows we assume $m = 2$ and F imaginary quadratic. In this case, the Lichtenbaum conjecture reads (using the functional equation for the zeta function and the fact that $\#K_3^{\mathrm{ind}}(O_F)_{\mathrm{tors}}$ is here always 24),

$$\frac{3|d|^{3/2}}{\pi^2 \cdot R_2(F)} \cdot \zeta_F(2) \stackrel{?}{=} \#K_2(O_F)\,.$$

Bloch [Bl] suggested and Suslin [Su] finally proved that Borel's regulator map can be given in terms of the Bloch-Wigner dilogarithm $D_2(z)$ as a map on the Bloch group $B(F)$; here $D_2(z) = \Im\big(Li_2(z) + \log|z|\log(1-z)\big)$, where $Li_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$ is the classical dilogarithm function, defined for $|z| < 1$ and analytically continued to $\mathbf{C} - [1,\infty)$, and $B(F)$ is given in explicit form with generators and relations (cf. [Su]):

$$B(F) = \frac{\{\sum_i n_i[x_i] \mid \sum_i n_i\big(x_i \wedge (1-x_i)\big) = 0 \in \bigwedge^2 F^\times\}}{\langle\, [x]-[y]+[\frac{y}{x}]-[\frac{1-y}{1-x}]+[\frac{1-y^{-1}}{1-x^{-1}}] \mid x,y \in F^\times - \{1\}\,\rangle}\,.$$

The dilogarithm $D_2(z)$ maps $B(F)$ onto a lattice in $\mathbf{R}$ whose covolume we denote by D_2^F. Thus, we can replace $R_2(F)$ in the formula above by D_2^F and still hope for the equality to hold (up to a universal factor):

$$\frac{3|d|^{3/2}}{\pi^2 \cdot D_2^F} \cdot \zeta_F(2) \stackrel{?}{=} \#K_2(O_F)\,.$$

The left hand side now can be computed numerically: we proceed by looking for elements $\xi \in B(F)$ which are supported on exceptional $S-$units for some small set S of irreducibles in F, i.e. $\xi = \sum_i n_i[x_i]$ such that $\sum_i n_i\big(x_i \wedge (1-x_i)\big) = 0$, and $x_i, 1-x_i \in \big\{\pm\prod_{p\in S} p^{a_p} \mid a_p \in \mathbf{Z}\big\}$. The images $D_2(\xi)$ lie in a 1-dimensional lattice of covolume $D_2^{F,S}$ (this also depends on the bounds for the exponents a_p), therefore the numerically computed values should all be commensurable. If we have computed enough different values $D_2(\xi)$ there is a good chance that they already generate the lattice and give D_2^F.

Our program, written in PARI [BBCO], performs the above calculations successively for an increasing set of irreducibles and stops if the corresponding $D_2^{F,S}$ stabilizes (i.e. if the same covolume occurs for S and $S\cup\{s_0\}$, $s_0 \notin S$ irreducible).

The reliability of the computations is supported by the fact that the results of a former (shorter) table [Ga] were not only compatible with the structural theoretical results known for the corresponding $K-$groups but even suggested several conjectures, many of which have been proved in the meantime by Browkin [B–92] and others ([C-H], [Qin]).

Our approach is very similar to that of Grayson [Gr], only that we don't have to restrict ourselves to class number one, and our program works even for very large discriminants (e.g. for $F = \mathbf{Q}(\sqrt{-2000004})$ we obtain $\#K_2O_F = 4$).

The program is freely available from the second author via e-mail, together with some remarks on the modification of the parameters.

4. Determining the structure.

In order to establish the actual structure of the tame and wild kernel we apply the following results:

(1) The index $i_F := (K_2O_F : W_F)$ always divides 6. More precisely,

$$2|i_F \qquad \text{iff} \qquad d' \equiv \pm 1 \pmod 8,$$

$$3|i_F \qquad \text{iff} \qquad d \equiv -3 \pmod 9.$$

(See [B-82], Table 1).

(2) The 2−rank of the tame and wild kernel can be computed easily:

$$e_2 = \begin{cases} t, & \text{if every odd prime divisor of } d \text{ is} \equiv \pm 1 \pmod 8, \\ t-1, & \text{otherwise,} \end{cases}$$

where t is the number of odd prime divisors of d.

$$w_2 = \begin{cases} e_2, & \text{if } d' \not\equiv 1 \pmod 8, \\ e_2 - 1, & \text{otherwise.} \end{cases}$$

(See [B–S], Theorem 4).

(3) The 4−rank of the tame kernel can be easily determined using the results of [Qin], at least if the number of odd prime divisors of d does not exceed 3.

The p−rank of K_2O_F , for odd p, is related to the p−rank of the class group of an appropriate number field as follows.

(4) Let $E_3 = \mathbf{Q}(\sqrt{-3d})$ and $e_3' =$ 3−rank $Cl(O_{E_3})$. Then

$$e_3 = e_3', \qquad \text{if} \qquad d \not\equiv -3 \pmod 9,$$

and

$$\max(1, e_3') \le e_3 \le e_3' + 1, \qquad \text{otherwise.}$$

(See [B–92], Theorem 5.6).

(5) Let $E_5 = \mathbf{Q}(\sqrt{5d})$, and $e_5' =$ 5−rank $Cl(O_{E_5})$. Then $e_5 \le e_5'$.
(See [B–92], Theorem 5.4).

(6) For $p > 5$, where p is a regular prime, let E_p be the maximal real subfield of the field $F(\zeta_p)$, and let $e_p' = p$−rank $Cl(O_{E_p})$. Then $e_p \le e_p'$.
(See [B–92], Theorem 5.4).

5. Examples.

1) For $d=-644$, we have $\#K_2O_F=32$ (conjecturally), and $e_2=2,\ w_2=2$. Moreover $e_4=1$, since $644=4\cdot 7\cdot 23$, and $7\equiv 23\equiv 7 \pmod 8$, see [Qin].
Finally $(K_2O_F : W_F)=2$, since $d'=-161\equiv 7 \pmod 8$ and $d\not\equiv -3 \pmod 9$.

It follows that

$$K_2O_F=\mathbf{Z}/2\times\mathbf{Z}/16 \qquad \text{and} \qquad W_F=\mathbf{Z}/2\times\mathbf{Z}/8.$$

2) For $d=-255$ we have $\#K_2O_F=12$ (conjecturally). Moreover $e_2=2$, $w_2=1$, and $d\equiv -3 \pmod 9$.

Therefore

$$K_2O_F=\mathbf{Z}/2\times\mathbf{Z}/2\times\mathbf{Z}/3 \qquad \text{and} \qquad W_F=\mathbf{Z}/2.$$

3) For $d=-759$, we have $\#K_2O_F=36$ (conjecturally), and $e_2=2,\ \ w_2=1$, and $d\equiv -3 \pmod 9$.
Moreover, for

$$E_3=\mathbf{Q}(\sqrt{3d})=\mathbf{Q}(\sqrt{-253}),$$

we have 3$-$rank $Cl(O_{E_3})$=0.

Therefore

$$K_2O_F=\mathbf{Z}/2\times\mathbf{Z}/2\times\mathbf{Z}/9 \qquad \text{and} \qquad W_F=\mathbf{Z}/2\times\mathbf{Z}/3.$$

4) For $d=-2395$, we have $\#K_2O_F=25$ (conjecturally). Moreover, for $E_5=\mathbf{Q}(\sqrt{5d})=\mathbf{Q}(\sqrt{-479})$, we have 5$-$rank $Cl(O_{E_5})=1$.

Therefore, using (5),

$$K_2O_F=W_F=\mathbf{Z}/25.$$

5) For $d=-1832$, we have $\#K_2O_F=49$ (conjecturally). The maximal real subfield E_7 of the field $F(\zeta_7)=\mathbf{Q}(\sqrt{-d},\zeta_7)$ is generated over $\mathbf{Q}$ by a root of the polynomial

$$f(x)=x^6+7dx^4+14d^2x^2+7d^3.$$

In our case

$$e_7'=7\text{-rank } Cl(O_{E_7})=1.$$

Therefore, in view of (6),

$$K_2O_F=W_F=\mathbf{Z}/49.$$

6. Description of the table.

In the first column there is the negative discriminant d. The last two columns give the structure of the tame and the wild kernel of the corresponding field. In these columns a single number n denotes the cyclic group of order n, and a sequence $(n_1,n_2,\ldots)$ denotes the direct sum of cyclic groups of orders $n_1,n_2,\ldots$.

The last two columns contain correct results provided the conjectural value of $\#K_2O_F$ is correct.

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