ON THE COHOMOLOGY OF LINEAR GROUPS OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. Let Γ be the group $\mathrm{GL}_N(\mathcal{O}_D)$, where $\mathcal{O}_D$ is the ring of integers in the imaginary quadratic field with discriminant $D < 0$. In this paper we investigate the cohomology of Γ for $N = 3, 4$ and for a selection of discriminants: $D \geq -24$ when $N = 3$, and $D = -3, -4$ when $N = 4$. In particular we compute the integral cohomology of Γ up to p-power torsion for small primes p. Our main tool is the polyhedral reduction theory for Γ developed by Ash [4, Ch. II] and Koecher [24]. Our results extend work of Staffeldt [40], who treated the case $N = 3$, $D = -4$. In a sequel [15] to this paper, we will apply some of these results to computations with the K-groups $K_4(\mathcal{O}_D)$, when $D = -3, -4$.

1. INTRODUCTION

1.1. Let F be an imaginary quadratic field of discriminant $D < 0$, let $\mathcal{O} = \mathcal{O}_D$ be its ring of integers, and let Γ be the group $\mathrm{GL}_N(\mathcal{O})$. The homology and cohomology of Γ when $N = 2$ — or rather its close cousin the Bianchi group $\mathrm{PSL}_2(\mathcal{O})$ — have been well studied in the literature. For an (incomplete) selection of results we refer to $[5, 13, 28, 29, 33, 34, 44]$. Today we have a good understanding of a wide range of examples, and one can even compute them for very large discriminants (cf. [46]). For $N > 2$, on the other hand, the group Γ has not received the same attention. The first example known to us is the work of Staffeldt [40]. He treated the case $N = 3$, $D = -4$ with the goal of understanding the 3-torsion in $K_3(\mathbb{Z}[\sqrt{-1}])$. The second example is [10], which investigates the case of the groups $\mathrm{GL}(L)$ for L an $\mathcal{O}$-lattice that is not necessarily a free $\mathcal{O}$-module. This allows the authors to compute the Hermite constants of those rings in case $D \geq -10$ and $\mathrm{rank}(L) \leq 3$. Our methods apply as well to the non-free case, and the corresponding cohomology computations should be useful when investigating automorphic forms over number fields that are not principal ideal domains (cf. [41, Appendix] for more about the connection between cohomology of arithmetic groups and automorphic forms).

In this paper we rectify this situation somewhat by beginning the first systematic computations for higher rank linear groups over $\mathcal{O}$. In particular we investigate the cohomology of Γ for $N = 3, 4$ and for a selection of discriminants: $D \geq -24$ when $N = 3$, and $D = -3, -4$ when $N = 4$. We explicitly compute the polyhedral reduction domains arising from Voronoi's theory of perfect forms, as generalized by Ash [4, Ch. II] and Koecher [24]. This allows us

to compute the integral cohomology of Γ up to p-power torsion for small primes p. In a sequel [15] to this paper, we will apply some of these results to computations with the K-groups $K_4(\mathcal{O}_D)$, when $D = -3, -4$.

1.2.   Here is a guide to the paper (which closely follows the structure of the first five sections of [19]). In Section 2 we recall the explicit reduction theory we need to build our chain complexes to compute cohomology. In Section 3 we define the complexes, and explain the relation between what we compute and the cohomology of Γ. In Section 4 we describe a "mass formula" for the cells in our tessellations that provides a strong computational check on the correctness of our constructions. In Section 5 we give an explicit representative for the non-trivial class in the top cohomological degree; this construction is motivated by a similar construction in [18, 19]. The algorithms we used are described in Section 6. Finally, in Section 7 we give the results of our computations.

2. The polyhedral cones

Fix $N > 0$ and an imaginary quadratic field F of discriminant $D < 0$ with ring of integers $\mathcal{O} = \mathcal{O}_D$ and define the element

$$\omega := \omega_D := \begin{cases} \sqrt{D/4} & \text{if } D \equiv 0 \bmod 4, \\ (1 + \sqrt{D})/2 & \text{if } D \equiv 1 \bmod 4, \end{cases}$$

so that $F = \mathbb{Q}(\omega)$ and $\mathcal{O} = \mathbb{Z}[\omega]$. Throughout we fix a complex embedding $F \hookrightarrow \mathbb{C}$ and consistently identify F with its image in $\mathbb{C}$. We also extend this identification to vectors and matrices with coefficients in F. Furthermore, we put $\Gamma = \mathrm{GL}_N(\mathcal{O})$.

2.1. **Hermitian forms.** Let $\mathcal{H}^N(\mathbb{C})$ denote the N^2-dimensional real vector space of $N \times N$ Hermitian matrices with complex coefficients. Using the chosen complex embedding of F we can view $\mathcal{H}^N(F)$, the Hermitian matrices with coefficients in F, as a subset of $\mathcal{H}^N(\mathbb{C})$. Moreover, this embedding allows us to view $\mathcal{H}^N(\mathbb{C})$ as a $\mathbb{Q}$-vector space such that the rational points of $\mathcal{H}^N(\mathbb{C})$ are exactly $\mathcal{H}^N(F)$.

Define a map $q \colon \mathcal{O}^N \to \mathcal{H}^N(F)$ by the outer product $q(x) = xx^*$, where $*$ denotes conjugate transpose (with conjugation being the non-trivial complex conjugation automorphism of F). Each $A \in \mathcal{H}^N(\mathbb{C})$ defines a Hermitian form $A[x]$ on $\mathbb{C}^N$ by the rule

$$(1) \qquad\qquad A[x] := x^* A x, \quad \text{for } x \in \mathbb{C}^N.$$

Define the non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathcal{H}^N(\mathbb{C}) \times \mathcal{H}^N(\mathbb{C}) \to \mathbb{C}$$

by $\langle A, B \rangle := \mathrm{Tr}(AB)$. For $v \in \mathcal{O}^N$ (identified with its image in $\mathbb{C}^N$) one can easily verify that

$$A[v] = \mathrm{Tr}(Aq(v)) = \langle A, q(v) \rangle. \tag{2}$$

Let $C_N \subset \mathcal{H}^N(\mathbb{C})$ denote the cone of positive definite Hermitian matrices.

Definition 2.1. For $A \in C_N$, we define the *minimum of A* as

$$\min(A) := \min_D(A) := \inf_{x \in \mathcal{O}^N \setminus \{0\}} A[x].$$

Note that $\min(A) > 0$ since A is positive definite. A vector $v \in \mathcal{O}^N$ is called a *minimal vector of A* if $A[v] = \min(A)$. We denote the set of minimal vectors of A by $\mathrm{Min}(A)$.

It should be emphasized that these notions depend on the fixed choice of the imaginary quadratic field F. Since $q(\mathcal{O}^N)$ is discrete in $\mathcal{H}^N(\mathbb{C})$ and the level sets $\{q(x) \mid \langle A, q(x) \rangle = C\}$ are compact, the minimum for each A is attained by only finitely many minimal vectors.

From (2), we see that each vector $v \in \mathcal{O}^N$ gives rise to a linear functional $q(v)$ on $\mathcal{H}^N(\mathbb{C})$, which acts by $A \mapsto \langle A, q(v) \rangle$.

Definition 2.2. We say a Hermitian form $A \in C_N$ is a *perfect Hermitian form over F* if

$$\mathrm{span}_{\mathbb{R}} \{q(v) \mid v \in \mathrm{Min}(A)\} = \mathcal{H}^N(\mathbb{C}).$$

From Definition 2.2 it is clear that a form is perfect if and only if it is uniquely determined by its minimum and its minimal vectors. Equivalently, a form A is perfect when $\mathrm{Min}(A)$ determines A up to a positive real scalar. It is convenient to normalize a perfect form by requiring that $\min(A) = 1$, and we will do so throughout this paper. A priori there is no reason to expect that every perfect form (up to rescaling) can be realized as an F-rational point in $\mathcal{H}^n(\mathbb{C})$; indeed, when one generalizes these concepts to general number fields this is too much to expect (cf. [20]). However, for imaginary quadratic fields this rationality property does hold:

Theorem 2.3. [26, Theorem 3.2] *Suppose F is a CM field. If $A \in C_N$ is a perfect Hermitian form over F such that $\min(A) = 1$, then $A \in \mathcal{H}^N(F)$.*

We now construct a partial compactification of the cone C_N.

Definition 2.4. A matrix $A \in \mathcal{H}^N(\mathbb{C})$ is said to have an *F-rational kernel* when the kernel of A is spanned by vectors in $F^N \subset \mathbb{C}^N$. Let $C_N^* \subset \mathcal{H}^N(\mathbb{C})$ denote the subset of non-zero positive semi-definite Hermitian forms with F-rational kernel.

Note that $C_N \subset C_N^*$, since any $A \in C_N$ has trivial kernel, which is clearly F-rational.

We now describe the connection between the cones $C_N \subset C_N^*$ and symmetric spaces. Let $\mathbf{G}$ be the reductive group over $\mathbb{Q}$ given by the restriction of scalars $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_N)$. Thus $\mathbf{G}(\mathbb{Q}) = \mathrm{GL}_N(F)$ and $\mathbf{G}(\mathbb{Z}) = \Gamma = \mathrm{GL}_N(\mathcal{O})$. The group $\mathbf{G}(\mathbb{R}) = \mathrm{GL}_N(\mathbb{C})$ acts on C_N^* on the left by

$$g \cdot A = gAg^*,$$

where $g \in \mathrm{GL}_N(\mathbb{C})$ and $A \in C_N^*$; one can easily verify that this action preserves C_N. Let $H = R_d\mathbf{G}(\mathbb{R})^0$ be the identity component of the group of real points of the split radical of $\mathbf{G}$. Then $H \simeq \mathbb{R}_+$, and as a subgroup of $\mathbf{G}(\mathbb{R})$, it acts on C_N^* by positive real homotheties. Voronoi's work [45], generalized by Ash [4, Ch. II] and Koecher [24], shows that there are only finitely many perfect Hermitian forms over F modulo the actions of Γ and H.

Let X_N^* denote the quotient $X_N^* = H \backslash C_N^*$, and let $\pi \colon C_N^* \to X_N^*$ denote the projection. Then $X_N = \pi(C_N)$ can be identified with the global Riemannian symmetric space for the

reductive group $H \backslash \mathrm{GL}_N(\mathbb{C})$, or (if one prefers) for the semisimple group $\mathrm{SL}_N(\mathbb{C})$; in terms of the latter we have $X_N \simeq \mathrm{SL}_N(\mathbb{C}) / \mathrm{SU}(N)$.

Moreover, the image X_N^* of C_N^* under homotheties is a partial compactification of the symmetric space X_N and is an example of a *Satake compactification* [30]. The quotient $\Gamma \backslash X_N^*$ is compact with finitely many boundary components. More precisely, let $\mathfrak{g}$ be the Lie algebra of $\mathrm{SL}_N(\mathbb{C})$. Satake defined a collection of partial compactifications of X_N indexed by dominant weights θ of representations of $\mathfrak{g}$. In this collection there are only finitely many topological types: the type of the compactification indexed by θ depends on which walls of the dominant chamber contain θ, and the less regular the weight, the more singular the compactification. For our example θ is the first fundamental weight, and thus the quotient $\Gamma \backslash X_N^*$ is highly singular (in fact, as singular as possible in this family of compactifications). Indeed, it is a *minimal Satake compactification*, and in a sense is the analogue of the *Baily–Borel compactification* (although the latter is not defined for X_N, since X_N is not a Hermitian symmetric space). In the quotient $\Gamma \backslash X_N^*$ there are boundary components of dimensions $m^2 - 1$ for $m = 1, \ldots, N - 1$; the components of dimension $m^2 - 1$ are isomorphic to quotients of the symmetric space X_m for $\mathrm{SL}_m(\mathbb{C})$, i.e. the symmetric space attached to the Levi factor of the standard parabolic subgroup of type $(m, 1, \ldots, 1)$. For more details about the geometry and topology of compactifications of locally symmetric spaces, we refer to [8].

Remark 2.5. The action of $\mathbf{G}(\mathbb{R}) = \mathrm{GL}_N(\mathbb{C})$ on $\mathcal{H}^N(\mathbb{C})$ gives a representation $\rho \colon \mathrm{GL}_N(\mathbb{C}) \to \mathrm{GL}_{N^2}(\mathbb{R})$. Any element g acts on the orientation of $\mathcal{H}^N(\mathbb{C})$ via the sign of $\det(\rho(g))$. Since $\mathrm{GL}_N(\mathbb{C})$ is connected, $\mathrm{GL}_N(\mathbb{C})$ and its subgroup $\mathrm{GL}_N(F)$ act on $\mathcal{H}^N(\mathbb{C})$ via orientation preserving automorphisms.

2.2. **Two cell complexes.** Let M be a finite subset of $\mathcal{O}^N \setminus \{0\}$. The *cone generated by M* is the set of non-zero matrices of the form $\sum_{v \in M} \lambda_v q(v)$, where $\lambda_v \in \mathbb{R}_{\geq 0}$. For a perfect form A, we call the cone $\sigma(A)$ generated by the minimal vectors of A the *perfect cone* attached to A; by abuse of notation we denote the image of $\sigma(A)$ under π, which is a cell, also by $\sigma(A)$. One can show [4, 24] that the cells $\sigma(A)$ and their intersections, as A runs over equivalence classes of perfect forms, define a $\mathrm{GL}_N(\mathcal{O})$-invariant cell decomposition of X_N^*, with finitely many equivalence classes of cells modulo $\mathrm{GL}_N(\mathcal{O})$. We can make the $\mathrm{GL}_N(\mathcal{O})$-action explicit: for a perfect form $A \in C_N$ and an element $\gamma \in \mathrm{GL}_N(\mathcal{O})$, we have

$$
\begin{aligned}
\gamma \cdot \mathrm{Min}(A) &= \{\gamma \cdot v \in \mathcal{O}^N \mid A[v] = \min(A)\} \\
&= \{w \in \mathcal{O}^N \mid A[\gamma^{-1}w] = \min(A)\} = \mathrm{Min}((\gamma^*)^{-1} A \gamma^{-1})
\end{aligned}
$$

and therefore

$$
\gamma \cdot \sigma(A) = \text{perfect cone of } \{\gamma v \mid v \in \mathrm{Min}(A)\} = \sigma((\gamma^*)^{-1} A \gamma^{-1}).
$$

We endow X_N^* with the CW-topology coming from the images of the $\sigma(A)$ and their faces [22, Appendix].

If τ is a closed cell in X_N^* and A is a perfect form with $\tau \subset \sigma(A)$, we let $\mathrm{Min}(\tau)$ denote the set of vectors $v \in \mathrm{Min}(A)$ such that $q(v) \in \tau$. The set $\mathrm{Min}(\tau)$ is independent of the (possible) choice of A. Then τ is the image in X_N^* of the cone C_τ generated by $\{q(v) \mid v \in \mathrm{Min}(\tau)\}$, that is, $C_\tau = \{\sum_{v \in M(\tau)} \lambda_v q(v) \mid \lambda_v \in \mathbb{R}_{\geq 0}\}$. For any two closed cells τ and τ' in X_N^*, we have $\mathrm{Min}(\tau) \cap \mathrm{Min}(\tau') = \mathrm{Min}(\tau \cap \tau')$.

Let $\tilde{\Sigma} \subset C_N^*$ be the (infinite) union of all cones C_σ such that $\sigma \in X_N^*$ has non-trivial intersection with X_N. One can verify that the stabilizer of C_σ in $\mathrm{GL}_N(\mathcal{O})$ is equal to the stabilizer of σ in $\mathrm{GL}_N(\mathcal{O})$. By abuse of notation, we write $\mathrm{Min}(C_\sigma)$ as $\mathrm{Min}(\sigma)$.
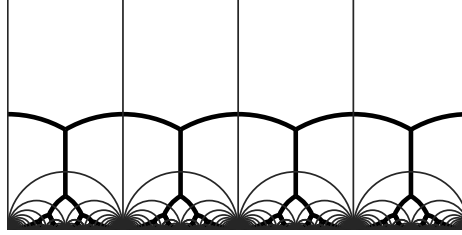
FIGURE 1. The ideal triangles of the Voronoi tessellation of the upper half-plane $\mathfrak{H}$ shown in grey thin lines, with its dual the well-rounded retract shown in heavy black lines.

We now recall the *well-rounded retract* of Ash [2], a contractible $(N^2 - N)-$dimensional cell complex W on which $\mathrm{GL}_N(\mathcal{O})$ acts with finite stabilizers. A non-zero finite set $M \subset \mathcal{O}^N$ is called *well-rounded* if the $\mathbb{C}$-span of M is $\mathbb{C}^N$. For a well-rounded subset M, let $\mu(M)$ denote the set of forms $A \in C_N$ with $\mathrm{Min}(A) = M$ and $\min(A) = 1$. It is easy to prove that if $\mu(M)$ is non-empty then it is convex and thus topologically a cell. The well-rounded retract is then defined to be

$$W = \bigcup_{M \text{ well-rounded}} \mu(M).$$

The space W is dual to the decomposition of X_N induced by the cones $\tilde{\Sigma}$ (see Theorem 2.7 below). Figure 1 shows the relationship between W and the cells induced by $\tilde{\Sigma}$ when $N = 2$ and $F = \mathbb{Q}$. In this case X can be identified with the upper half-plane $\mathfrak{H} = \{x + iy \mid y > 0\}$. The Voronoi tessellation is the tiling of $\mathfrak{H}$ by the $\mathrm{SL}_2(\mathbb{Z})$-translates of the ideal geodesic triangle with vertices $\{0, 1, \infty\}$. The well-rounded retract W is the dual infinite trivalent tree.

Lemma 2.6. *If $C_\sigma \in \tilde{\Sigma}$, then $\mathrm{Min}(\sigma)$ is well-rounded.*

Proof. This is proved by McConnell in [25, Theorem 2.11] when $F = \mathbb{Q}$, and we mimic his proof to yield the analogous result for imaginary quadratic fields. Let $M = \mathrm{Min}(C_\sigma)$ denote the spanning vectors of C_σ. Suppose M is not well-rounded. Then M does not span $\mathbb{C}^N$, and so there exists a non-zero vector $w \in \mathbb{C}^N$ such that $vw^* = 0$ for all $v \in M$. Then each $\theta \in C_\sigma$ can be written as a non-zero Hermitian form $\theta = \sum_{v \in M} a_v q(v)$, where $a_v \geq 0$. It follows that

$$\theta[w] = \sum_{v \in M} a_v \langle q(v), q(w) \rangle = \sum_{v \in M} a_v |vw^*|^2 = 0,$$

where $|\cdot|$ is the usual norm on $\mathbb{C}^N$. In particular, θ is not positive definite, contradicting the assumption that $C_\sigma \in \tilde{\Sigma}$. $\square$

Theorem 2.7. *Let $M \subset \mathcal{O}^N$ be well-rounded and let $\mu(M) \subset W$ be non-empty. Then there is a unique cone $C_\sigma \in \tilde{\Sigma}$ such that $\mathrm{Min}(C_\sigma) = M$. The map $\mu(M) \mapsto C_\sigma$ is a canonical bijection $W \to \tilde{\Sigma}$ that is inclusion-reversing on the face relations.*

Proof. It suffices to prove that a well-rounded subset $M \subset \mathcal{O}^N$ has $\mu(M) \neq \emptyset$ if and only if there is a cone $C_\sigma \in \tilde{\Sigma}$ such that $\mathrm{Min}(\sigma) = M$.

Suppose $C_\sigma \in \tilde{\Sigma}$. By Lemma 2.6, $M = \mathrm{Min}(C_\sigma)$ is well-rounded. Thus it remains to show that $\mu(M)$ is non-empty. We do so by constructing a form $B \in \mu(M)$ with $\mathrm{Min}(B) = \mathrm{Min}(C_\sigma)$. Since $C_\sigma \in \tilde{\Sigma}$, there is a perfect form A such that C_σ is a face of the cone

$S_A = \pi^{-1}(\sigma(A))$. Furthermore, C_σ can be described as the intersection of S_A with some supporting hyperplane $\{\theta \mid \langle H, \theta \rangle = 0\}$ for some $H \in \mathcal{H}^N(\mathbb{C})$. It follows that

$$\langle H, q(v) \rangle = 0 \quad \text{if } v \in M \quad \text{and} \quad \langle H, q(v) \rangle > 0 \quad \text{if } v \in \text{Min}(A) \setminus M.$$

Let $B = A + \rho H$. Since $\langle H, q(v) \rangle = B[v]$, a standard argument for Hermitian forms shows that for sufficiently small positive ρ, B is positive definite, $B[v] = \min(A)$ for $v \in M$ and $B[v] > \min(A)$ for $v \in \text{Min}(A) \setminus M$. Thus $\text{Min}(B) = M$ and so $B \in \mu(M)$. In particular, $\mu(M)$ is non-empty.

Conversely, suppose M is a well-rounded subset of $\mathcal{O}^N$ with $\mu(M)$ non-empty. Choose $B \in \mu(M)$ so that $\text{Min}(B) = M$. If B is perfect, then $M = \text{Min}(B)$ and we are done. Otherwise, we can use a generalization of an algorithm of Voronoi [16] to find a perfect form A such that $\text{Min}(B) \subsetneq \text{Min}(A)$. Let $H = A - B$. Then

$$\langle H, q(v) \rangle = 0 \quad \text{if } v \in M \quad \text{and} \quad \langle H, q(v) \rangle > 0 \quad \text{if } v \in \text{Min}(A) \setminus M.$$

Thus the hyperplane $\{\theta \mid \langle H, \theta \rangle = 0\}$ is a supporting hyperplane for the subset of $S_A = \pi^{-1}(\sigma(A))$ spanned by $\{q(v) \mid v \in M\}$. Therefore M defines a face of S_A in $\tilde{\Sigma}$ as desired.   □

Definition 2.8. Let Σ_n^* denote a set of representatives, modulo the action of $\text{GL}_N(\mathcal{O})$, of n-dimensional cells of X_N^* that meet X_N. Let $\Sigma^* = \cup_n \Sigma_n^*$. The cells that are quotients of cones not meeting C_N are said to be *at infinity*.

We note that $\pi(\tilde{\Sigma}) = \text{GL}_N(\mathcal{O}) \cdot \Sigma^*$. In the following corollary we explain why we can work with Σ^* and its stabilizer subgroups as if we were working with the proper, contractible $\text{GL}_N(\mathcal{O})$-complex W.

Corollary 2.9. *The map $W \to \tilde{\Sigma}$ of Theorem 2.7 is a $\text{GL}_N(\mathcal{O})$-equivariant bijection.*

Proof. To see this, let $\mu(M)$ be a cell in W. Then the forms $B \in \mu(M)$ have $\text{Min}(B) = M$ and $\min(B) = 1$. Under the identification in Theorem 2.7, $\mu(M)$ corresponds to the cone C_σ' with spanning vectors $\{q(v) \mid v \in M\}$. Let σ' be the corresponding cell in X_N^*. There is a cell $\sigma \in \Sigma^*$ that is $\text{GL}_N(\mathcal{O})$-equivalent to σ'. If $\gamma \in \text{GL}_N(\mathcal{O})$, then $\gamma \cdot \mu(M) = \mu(\{(\gamma^*)^{-1} v \mid v \in M\})$. Thus if $\gamma \in \text{Stab}(\mu(M))$, then $(\gamma^*)^{-1} \in \text{Stab}(C_\sigma')$. It is therefore clear that $\text{Stab}(C_\sigma') = \text{Stab}(\sigma') \simeq \text{Stab}(\sigma) \simeq \text{Stab}(\mu(M))$.   □

The map $g \mapsto (g^*)^{-1}$ is an isomorphism of groups, so the duality between W mod $\Gamma = \text{GL}_N(\mathcal{O})$ and Σ^* in Theorem 2.7 allows us to work with Σ^* and its stabilizer subgroups as if we were working with a proper, contractible $\text{GL}_N(\mathcal{O})$-complex. This fact will be used later to compute the mass formula (§4).

2.3. **An example.** We consider the Eisenstein integers $\mathcal{O} = \mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{-3}}{2}$, and give the details of the Σ^* computation for $\Gamma = \text{GL}_3(\mathcal{O})$. In particular, we give representatives of the $\text{GL}_3(\mathcal{O})$-equivalence classes of cells in X_3^* and describe the facets of each cell. This includes not only $\text{GL}_3(\mathcal{O})$-representatives for $\pi(\tilde{\Sigma})$, but also representatives for the cells at infinity.

Let A denote the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ -\omega-1 & -\omega-1 & -\omega & -1 & -\omega & -1 & 1 & 1 & 0 & -\omega & -1 & 0 & 1 \\ 1 & \omega & 0 & 0 & \omega & 1 & -1 & 0 & 1 & \omega^2 & \omega & 0 & -\omega \end{bmatrix}.$$

We describe Σ^* in terms of the columns of A. In what follows, we denote by a list of integers $[m_1, \ldots, m_k]$ the image in X_3^* of the cone generated by the columns of A indexed by $m_1, \ldots, m_k$.

There are two equivalence classes of perfect Hermitian forms modulo the Γ-action, so the 8-dimensional cells in Σ^* fall into two orbits, with representatives

$$a_1 = [1, 2, 3, 4, 5, 6, 7, 8, 9] \quad \text{and} \quad a_2 = [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].$$

Recall that the f-vector of a d-dimensional polytope σ is the vector $(f_{-1}, f_0, f_1, \ldots, f_d)$, where f_k is the number of k-dimensional faces of σ and $f_{-1} := f_d$. The f-vector of a_1 is

$$f(a_1) = (1, 9, 36, 84, 126, 126, 84, 36, 9, 1).$$

Since a_1 has nine vertices in an 8-dimensional space, we see that a_1 is a simplex, and this is indeed compatible with its f-vector. As such, the faces of a_1 are easily described as all of the subsets of $\{1, 2, \ldots, 9\}$. The cone a_2 is not a simplex, but it is a simplicial polytope since all of its 81 facets are simplices. The f-vector of a_2 is

$$f(a_2) = (1, 12, 66, 216, 459, 648, 594, 324, 81, 1).$$

For simplicial polytopes, one often prefers to use the (simpler) h-vector, which is related to the f-vector via the transformation

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}.$$

Here, t is a formal variable that just serves to access the binomial coefficients. Using this, one finds that the h-vector of a_2 is

$$h(a_2) = (1, 4, 10, 16, 19, 16, 10, 4, 1).$$

The 7-dimensional cells in Σ^* fall into two equivalence classes under the action of Γ, namely

$$b_1 = [1, 2, 3, 4, 5, 6, 7, 8] \quad \text{and} \quad b_2 = [4, 6, 7, 8, 9, 11, 12, 13];$$

these are both simplicial. All nine of the facets of a_1 are of type b_1. Of the 81 facets of a_2, nine are of type b_1 and 72 are of type b_2.

The 6-dimensional cells in Σ^* fall into three equivalence classes

$$c_1 = [1, 2, 3, 4, 5, 6, 7], \quad c_2 = [4, 6, 8, 9, 11, 12, 13], \quad \text{and} \quad c_3 = [4, 6, 7, 8, 9, 11, 13].$$

The eight facets of b_1 are type c_1. There are six facets of b_2 of type c_2 and one each of types c_1 and c_3.

The 5-dimensional cells of Σ^* fall into four equivalence classes

$$d_1 = [1, 2, 3, 4, 5, 6], \qquad\qquad d_2 = [1, 2, 3, 4, 5, 7],$$
$$d_3 = [4, 6, 8, 9, 11, 12], \qquad\qquad d_4 = [4, 8, 9, 11, 12, 13].$$

The cell c_1 has one facet of type d_1 and six facets of type d_2. The cell c_2 has three facets of type d_2, three facets of type d_3, and one facet of type d_4. The cell c_3 has one facet of type d_1 and three facets of type d_3.

The 4-dimensional cells of Σ^* fall into three equivalence classes

$$e_1 = [1, 2, 3, 4, 5], \quad e_2 = [1, 2, 3, 4, 7], \quad \text{and} \quad e_3 = [4, 6, 8, 9, 11].$$

All six of the facets of d_1 are type e_1. The cell d_2 has three facets of type e_1 and e_2 each. The cell d_3 has one facet of type e_1, three facets of type e_2, and two facets of type e_3. All six of the facets of d_4 are type e_2.

The 3-dimensional cells of Σ^* fall into two equivalence classes

$$f_1 = [1, 2, 3, 4] \quad \text{and} \quad f_2 = [1, 2, 4, 5].$$

The cell e_1 has three facets of type f_1 and two facets of type f_2. The cell e_2 has one facet of type e_1 and four facets of type f_2. The cell e_3 has one facet at infinity and four facets of type f_2.

The 2-dimensional cells of Σ^* fall into a single equivalence class

$$g_1 = [1, 2, 3].$$

All of the facets of f_1 are type g_1. The cell f_2 has one facet at infinity and three facets of type g_1. The boundary of g_1 consists of three 1-cells. Note that since we are considering an $N = 3$ example, every cone generated by two minimal vectors is necessarily at infinity. In particular, g_1 has all of its boundary edges at infinity, but is not itself at infinity. An analogous situation occurs in the Voronoi tessellation of the upper half-plane shown in Figure 1 where we have a 1-cell whose boundary vertices are at infinity, but is not itself at infinity.

We now consider the cells in Σ^* and their relationship with boundary components in the Satake compactification X_3^*. The symmetric space X_3 has dimension 8, and the compactification X_3^* has boundary components of dimensions 3 and 0. The former are isomorphic to hyperbolic 3-space $\mathfrak{H}_3$, and the latter are of course points.

The cells that lie at infinity in the 3-dimensional boundary components are certain facets in the boundaries of three equivalence classes above, represented by the 4-cell e_3, the 3-cell f_2, and the 2-cell g_1. The cell e_3 has one boundary facet at infinity, f_2 has one boundary facet at infinity, and g_1 has all its boundary facets at infinity. These boundary faces have dimensions 3, 2, 1 respectively, and all cells of a given dimension are $\mathrm{GL}_3(\mathcal{O})$-equivalent. Taken together they form ideal polyhedra that tessellate the hyperbolic 3-spaces.

More precisely, fix a boundary component X_2 of dimension 3 and identify it with $\mathfrak{H}_3$. Then the cells at infinity meeting X_2 tessellate $X_2 \simeq \mathfrak{H}_3$ exactly as in Cremona's study of modular symbols for imaginary quadratic fields in the case $\mathbb{Q}(\sqrt{-3})$ [13]. To construct this hyperbolic tessellation, one takes the ideal tetrahedron in $\mathfrak{H}_3$ with vertices at 0, 1, ω, and ∞, and translates it by $\mathrm{GL}_2(\mathcal{O})$ (cf. [13, Fig. 2.3.3]). In particular all the dimension 3 boundary components inherit a tessellation by ideal tetrahedra isomorphic to this one. Indeed, one can check that the cell appearing in the boundary of e_3 indexed by $[4, 6, 9, 11]$ is the one at infinity, and that mod $\mathrm{GL}_3(\mathcal{O})$ this set of minimal vectors can be taken to the columns of

$$\begin{bmatrix} \omega & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which give the ideal vertices of the tessellation of the standard boundary component of dimension 3 corresponding to the standard maximal parabolic subgroup of $\mathrm{SL}_3(\mathbb{C})$ of type $(2, 1)$. The $\mathrm{GL}_3(\mathcal{O})$-action transports this tessellation to the other boundary components.

This almost accounts for all the cells at infinity. The remaining cells at infinity are points appearing in the boundaries of the 1-dimensional cells. As one might expect, these points are nothing more than the boundary components of dimension 0; they lie in a single $\mathrm{GL}_3(\mathcal{O})$ orbit. In terms of the hyperbolic 3-space picture, these points are exactly the ideal vertices of the ideal polyhedra tessellating the $\mathfrak{H}_3$'s. This reflects the inductive nature of Satake's construction: inside X_3^*, any 3-dimensional boundary component X_2 inherits a Satake compactification X_2^* by adjoining the 0-dimensional boundary components in its closure.

3. The cohomology and homology

The material in this section follows [19, §3], [39, §2], and [3], all of which rely on [12]. Recall that $\Gamma = \mathrm{GL}_N(\mathcal{O})$. In this section, we introduce a complex $\mathrm{Vor}_{N,D} = (V_*(\Gamma), d_*)$ of $\mathbb{Z}[\Gamma]$-modules whose homology is isomorphic to the group cohomology of Γ modulo small primes. More precisely, for any positive integer n let $\mathcal{S}_n$ be the Serre class of finite abelian groups with orders only divisible by primes less than or equal to n [35]. Then the main result of this section (Theorem 3.7) is that, if $n = n(N, D)$ is larger than all the primes dividing the orders of finite subgroups of Γ, then modulo $\mathcal{S}_n$ the homology of $\mathrm{Vor}_{N,D}$ is isomorphic to the group cohomology of Γ.

As above let $\Sigma_n^* = \Sigma_n^*(\Gamma)$ denote a finite set of representatives, modulo the action of Γ, of n-dimensional cells of X_N^* that meet X_N. A cell σ is called *orientable* if every element in $\mathrm{Stab}(\sigma)$ preserves the orientation of σ. By $\Sigma_n = \Sigma_n(\Gamma)$ we denote the set of orientable cells in $\Sigma_n^*(\Gamma)$. Let $\Sigma^* = \cup_n \Sigma_n^*$ and let $\Sigma = \cup_n \Sigma_n$.

3.1. Steinberg homology.

The arithmetic group Γ—like any arithmetic group—is a virtual duality group [6, Theorem 11.4.4]. This means that there is a $\mathbb{Z}[\Gamma]$-module I that plays the role of an "orientation module" for an analogue of Poincaré duality: for all coefficient modules M there is an isomorphism between the cohomology of Γ' with coefficients in M and the homology of Γ' with coefficients in $M \otimes I$, for any torsion-free finite index subgroup $\Gamma' \subset \Gamma$. For more details see [6, §11]. The module I is called the *dualizing module* for Γ. Borel and Serre prove that I is isomorphic to $H^\nu(\Gamma, \mathbb{Z}[\Gamma])$, where $\nu = \mathrm{vcd}\,\Gamma$ is the *virtual cohomological dimension* of Γ (cf. §4); note that for $\Gamma = \mathrm{GL}_N(\mathcal{O})$, we have $\nu = N^2 - N$.

One can show that I is isomorphic to $\mathrm{St}_\Gamma \otimes \Omega$, where St_Γ is the *Steinberg module* (the top non-vanishing reduced homology group of the Tits building attached to GL_N / F), and Ω is the *orientation module* (which records whether or not an element changes the orientation of the symmetric space). Since the action of $\mathrm{GL}_N(\mathcal{O})$ preserves the orientation of $\mathcal{H}^N(\mathbb{C})$, (cf. Remark 2.5), the module Ω is trivial, and thus $H^\nu(\Gamma, \mathbb{Z}[\Gamma]) = \mathrm{St}_\Gamma$.

For any Γ-module M, the *Steinberg homology* [12, p. 279], denoted $H_*^{\mathrm{St}}(\Gamma, M)$, is defined as $H_*^{\mathrm{St}}(\Gamma, M) = H_*(\Gamma, \mathrm{St}_\Gamma \otimes M)$. Let $\hat{H}^*$ denote the *Farrell cohomology* $\hat{H}^*(\Gamma, M)$ of Γ with coefficients in M (cf. [12, X.3]). There is a long exact sequence

$$(3) \qquad \cdots \to H_{\nu-i}^{\mathrm{St}} \to H^i \to \hat{H}^i \to H_{\nu-(i+1)}^{\mathrm{St}} \to H^{i+1} \to \hat{H}^{i+1} \to \cdots$$

that implies that we can understand the group cohomology by understanding the Steinberg homology and the Farrell cohomology. In particular, if the Farrell cohomology vanishes (which is the case when the torsion primes in Γ are invertible in M, see [12, IX.9 et seq.]), then the Steinberg homology is exactly the group cohomology $H_{\nu-k}^{\mathrm{St}}(\Gamma, M) \simeq H^k(\Gamma, M)$.

We now specialize to the case $M = \mathbb{Z}$ with trivial Γ-action, and in the remainder of this section omit the coefficients from homology and cohomology groups.

Proposition 3.1. *Let b be an upper bound on the torsion primes for $\Gamma = \mathrm{GL}_N(\mathcal{O})$. Then, modulo the Serre class $\mathcal{S}_b$, we have*

$$H_{\nu-k}^{\mathrm{St}}(\Gamma) \simeq H^k(\Gamma).$$

3.2. The Voronoi complex.

Let $V_n(\Gamma)$ denote the free abelian group generated by $\Sigma_n(\Gamma)$. Let $d_n \colon V_n(\Gamma) \to V_{n-1}(\Gamma)$ be the map defined in [19, §3.1] (see also Section 6.2), and denote the complex $(V_*(\Gamma), d_*)$ by $\mathrm{Vor}_{N,D}$.

Let ∂X_N^* denote the cells in X_N^* that do not meet X. Then ∂X_N^* is a Γ-invariant subcomplex of X_N^*. Let $H_*^\Gamma(X_N^*, \partial X_N^*)$ denote the relative equivariant homology of the pair $(X_N^*, \partial X_N^*)$ with integral coefficients (cf. [12, VII.7]).

Proposition 3.2. *Let b be an upper bound on the torsion primes for $\Gamma = \mathrm{GL}_N(\mathcal{O})$. Modulo the Serre class $\mathcal{S}_b$, we have*

$$H_n(\mathrm{Vor}_{N,D}) \simeq H_n^\Gamma(X_N^*, \partial X_N^*).$$

Proof. This result follows from [39, Proposition 2]. The argument is explained in detail for $F = \mathbb{Q}$ in [19, §3.2] and can be extended to imaginary quadratic F. For the convenience of the reader we recall the argument.

There is a spectral sequence E_{pq}^r converging to the equivariant homology groups

$$H_{p+q}^\Gamma(X_N^*, \partial X_N^*)$$

of the homology pair $(X_N^*, \partial X_N^*)$ such that

(4) $$E_{pq}^1 = \bigoplus_{\sigma \in \Sigma_p^*} H_q(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma) \Rightarrow H_{p+q}^\Gamma(X_N^*, \partial X_N^*),$$

where $\mathbb{Z}_\sigma$ is the orientation module for σ.

When σ is not orientable the homology $H_0(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma)$ is killed by 2. Otherwise, $H_0(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma) \simeq \mathbb{Z}_\sigma$. Therefore modulo $\mathcal{S}_2$ we have

$$E_{n,0}^1 \simeq \bigoplus_{\sigma \in \Sigma_n} \mathbb{Z}_\sigma.$$

Furthermore, when $q > 0$, $H_q(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma)$ lies in the Serre class $\mathcal{S}_b$, where b is the upper bound on the torsion primes for Γ. In other words, modulo $\mathcal{S}_b$, the E^1 page of the spectral sequence (4) is concentrated in the bottom row, and there is an identification of the bottom row groups with the groups $V_n(\Gamma)$ defined above. Finally, [39, Proposition 2] shows that $d_n^1 = d_n$. □

Remark 3.3. If we do not work modulo $\mathcal{S}_b$, then there are other entries in the spectral sequence to consider before we get $H_n^\Gamma(X_N^*, \partial X_N^*)$. In particular, the small torsion in $H_*(\mathrm{Vor}_{N,D})$ may not agree in general with the small torsion in $H_{p+q}^\Gamma(X_N^*, \partial X_N^*)$.

3.3. **Equivariant relative homology to Steinberg homology.** For $V \subset F^N$ a proper subspace, let $C(V)$ be the set of matrices $A \in C_N^*$ such that the kernel of A is $V \otimes_\mathbb{Q} \mathbb{R}$, and let $X(V) = \pi(C(V))$. The closure $\overline{C(V)}$ of $C(V)$ in the usual topology induced from $\mathcal{H}^N(\mathbb{C})$ consists of matrices whose kernel contains V. Let $\overline{X(V)} = \pi(\overline{C(V)})$.

The following lemma shows that $\overline{X(V)}$ is contractible. Soulé proves this for $F = \mathbb{Q}$ in [39, Lemma 1] and [38, Lemma 2], and the same proof with simple modifications applies to our setting as follows.

Lemma 3.4. *For any proper subspace $V \subset F^N$, the CW-complex $\overline{X(V)}$ is contractible.*

Proof. Let A be a perfect Hermitian form over F. Then $\sigma(A) \cap \overline{X(V)}$ is the perfect cone of the intersection $\mathrm{Min}(A) \cap V^\perp$, where $V^\perp$ is the orthogonal complement to V in F^N. It follows that $\overline{X(V)}$ is a sub-CW-complex of X_N^*.

Now we show that $\overline{X(V)}$ is contractible. Since $\overline{X(V)}$ has the same homotopy type as $X(V)$, it is enough to prove that $X(V)$ is contractible. For this it suffices to prove that the CW-topology on $X(V)$ coincides with its usual topology, since $X(V)$ in this topology is clearly contractible (because it is convex). To prove this, we argue that the covering of $X(V)$ by the closed sets of the form $\sigma(A) \cap X(V)$, where A is perfect, is locally finite.

Given any positive definite Hermitian form A, let $A^\perp$ be its restriction to the real space $V_\mathbb{R}^\perp = V^\perp \otimes_\mathbb{Q} \mathbb{R}$. Then $X(V)$ is isomorphic to the symmetric space for the group $\mathrm{Aut}(V_\mathbb{R}^\perp)$

via the map $A \mapsto A^\perp$. If $\gamma \in \Gamma$ satisfies $\gamma \cdot X(V) \cap X(V) \neq \emptyset$, then in fact γ stabilizes V. Let $P \subset \Gamma$ be the stabilizer of V and let $\alpha \colon P \to \Gamma' = \mathrm{Aut}(V_\mathbb{R}^\perp \cap L)$ be the projection map. Then the set of cells $\sigma(A) \cap X(V)$, as A ranges over the perfect Hermitian forms, is finite modulo Γ': if $[\sigma(A) \cap X(V)] \cap [(\gamma \cdot \sigma(A)) \cap X(V)] \neq \emptyset$, then $\gamma \in P$, and thus $(\gamma \cdot \sigma(A) \cap X(V)) = \alpha(\gamma)(\sigma(A) \cap X(V))$.

To conclude the argument one uses Siegel sets; we refer to [7] for their definition and to [4, Ch. II] for their properties in our setting. Given any point $x \in X(V)$, one can find an open set $U \ni x$ and a Siegel set $\mathfrak{S}$ such that $\mathfrak{S} \supset U$. Furthermore, any cell of the form $\sigma(A) \cap X(V)$ is itself contained in another Siegel set $\mathfrak{S}'$. Thus if U meets $\gamma(\sigma(A) \cap X(V))$ for $\gamma \in \Gamma'$, we must have that $\gamma \cdot \mathfrak{S}'$ meets $\mathfrak{S}$. But this is only possible for finitely many γ by the standard properties of Siegel sets. Thus the covering of $X(V)$ by the closed sets of the form $\sigma(A) \cap X(V)$ (A perfect) is locally finite, which completes the proof. $\qquad\square$

Proposition 3.5. *For every $n \geq 0$, there are canonical isomorphisms of Γ-modules*

$$H_n(X_N^*, \partial X_N^*) \simeq \begin{cases} \mathrm{St}_\Gamma & \text{if } n = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since X_N^* is contractible, the long exact sequence for the pair $(X_N^*, \partial X_N^*)$

$$\cdots \to H_n(\partial X_N^*) \to H_n(X_N^*) \to H_n(X_N^*, \partial X_N^*) \to H_{n-1}(\partial X_N^*) \to H_{n-1}(X_N^*) \to \cdots$$

implies $H_n(X_N^*, \partial X_N^*) \simeq \tilde{H}_{n-1}(\partial X_N^*)$ for $n \geq 1$, where $\tilde{H}$ denotes reduced homology.

From properties of Hermitian forms it is clear that $\overline{X(V)} \cap \overline{X(W)} = \overline{X(V \cap W)}$ for each pair of proper, non-zero subspaces $V, W \subset F^N$. Thus the nerve of the covering $\mathcal{U}$ of ∂X_N^* by $\overline{X(V)}$ as V ranges over non-zero, proper subspaces of F^N is the spherical Tits building $T_{F,N}$. Since each $\overline{X(V)}$ in $\mathcal{U}$ is contractible by Lemma 3.4, the cover $\mathcal{U}$ is a good cover (i.e., non-empty finite intersections are diffeomorphic to $\mathbb{R}^d$ for some d). It follows that the relative homology groups are isomorphic, i.e. $\tilde{H}_{n-1}(\partial X_N^*) \simeq \tilde{H}_{n-1}(T_{F,N})$. By the Solomon-Tits theorem [37], the latter is isomorphic to St_Γ if $n-1 = N-2$ and is trivial otherwise. $\qquad\square$

Proposition 3.6. *For all n, we have*

$$H_n^\Gamma(X_N^*, \partial X_N^*) = H_{n-(N-1)}^{\mathrm{St}}(\Gamma).$$

Proof. There is a spectral sequence [39, equation (2)] computing the relative equivariant homology

$$E_{pq}^2 = H_p(\Gamma, H_q(X_N^*, \partial X_N^*)) \Rightarrow H_{p+q}^\Gamma(X_N^*, \partial X_N^*).$$

Proposition 3.5 implies the E^2 page of the spectral sequence is concentrated in the $q = N-1$ column. Then

$$H_{p+(N-1)}^\Gamma(X_N^*, \partial X_N^*) \simeq H_p(\Gamma, H_q(X_N^*, \partial X_n^*)) = H_p(\Gamma, \mathrm{St}_\Gamma) = H_p^{\mathrm{St}}(\Gamma),$$

and the result follows. $\qquad\square$

Theorem 3.7. *Let b be an upper bound on the torsion primes for $\mathrm{GL}_N(\mathcal{O})$. Modulo the Serre class $\mathcal{S}_b$, one has*

$$H_n(\mathrm{Vor}_{N,D}) \simeq H^{N^2-1-n}(\mathrm{GL}_N(\mathcal{O})).$$

Proof. Let $\Gamma = \mathrm{GL}_N(\mathcal{O})$. Modulo $\mathcal{S}_b$, Propositions 3.2, 3.6, and 3.1 imply

$$H_n(\mathrm{Vor}_{N,D}) \simeq H_n^\Gamma(X_N^*, \partial X_N^*) \simeq H_{n-(N-1)}^{\mathrm{St}}(\Gamma) \simeq H^{N^2-1-n}(\Gamma).$$

$\qquad\square$

3.4. **Torsion elements in Γ.** To finish this section we discuss the possible torsion that can arise in the stabilizer subgroups of cells in our complexes. This allows us to make the bound in Theorem 3.7 effective.

Lemma 3.8. *Let p be an odd prime, and let $F/\mathbb{Q}$ be a quadratic field. Let $\Phi_p = x^{p-1} + x^{p-2} + \cdots + x + 1$ be the pth cyclotomic polynomial. Then Φ_p factors over F as a product of irreducible polynomials of degree $(p-1)/2$ if $F = \mathbb{Q}(\sqrt{p^*})$, where $p^* = (-1)^{(p-1)/2}p$, and is irreducible otherwise.*

Proof. Let ζ_p denote a primitive p^{th} root of unity. Consider the diagram of Galois extensions below:

$$
\begin{array}{ccc}
 & F(\zeta_p) & \\
\diagup & & \diagdown \\
F & & \mathbb{Q}(\zeta_p) \\
\diagdown & & \diagup \\
 & F \cap \mathbb{Q}(\zeta_p) & \\
 & | & \\
 & \mathbb{Q} &
\end{array}
$$

If $F = \mathbb{Q}(\sqrt{p^*})$, then F is the unique quadratic subfield of $\mathbb{Q}(\zeta_p)$. It follows that Φ_p factors over F as a product of irreducible polynomials of degree $(p-1)/2$. If $F \neq \mathbb{Q}(\sqrt{p^*})$, then $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. It follows that $[F(\zeta_p) : \mathbb{Q}(\zeta_p)] = [F : \mathbb{Q}] = 2$ and so $[F(\zeta_p) : F] = p - 1$. Thus Φ_p is irreducible over F if $F \neq \mathbb{Q}(\sqrt{p^*})$. $\qquad\square$

Lemma 3.9. *Let p be an odd prime, and let $F/\mathbb{Q}$ be a imaginary quadratic field. If $g \in \mathrm{GL}_N(F)$ has order p, then*

$$
p \leq \begin{cases} N + 1 & \textit{if } p \equiv 1 \bmod 4, \\ 2N + 1 & \textit{otherwise.} \end{cases}
$$

Proof. If $p \equiv 1 \bmod 4$, then $p^* > 0$. In particular, $F \neq \mathbb{Q}(\sqrt{p^*})$ and so by Lemma 3.8, Φ_p is irreducible over F. Then the minimal polynomial of g is Φ_p. By the Cayley-Hamilton Theorem, Φ_p divides the characteristic polynomial of g. Therefore $p - 1 \leq N$. Similarly, if $p \not\equiv 1 \bmod 4$, then $(p-1)/2 \leq N$. $\qquad\square$

Lemmas 3.8 and 3.9 immediately imply the following:

Proposition 3.10. *If $g \in \mathrm{GL}_3(\mathcal{O}_F)$ has prime order q, then $q \in \{2, 3, 7\}$ for $F = \mathbb{Q}(\sqrt{-7})$ and $q \in \{2, 3\}$ otherwise. If $g \in \mathrm{GL}_4(\mathcal{O}_F)$ has prime order q, then $q \in \{2, 3, 5, 7\}$ for $F = \mathbb{Q}(\sqrt{-7})$ and $q \in \{2, 3, 5\}$ otherwise.*

In Tables 1–12 we give the factorizations of the orders of the stabilizers of the cells in Σ^*.

4. A MASS FORMULA FOR THE VORONOI COMPLEX

The computation of the cell complex is a relatively difficult task and there are many ways that small mistakes in the computation could cause the final answer to be incorrect. Hence it is very important to have checks that allow us to give strong evidence for the correctness of our computations. One is that the complexes we construct actually are chain complexes, namely that their differentials square to zero. Another is the *mass formula*, stated in Theorem 4.6. According to this formula, the alternating sum over the cells of

Σ^* of the inverse orders of the stabilizer subgroups must vanish. A good reference for this section is [12, Ch. IX, §§6–7], and we follow it closely. The main theorem underlying this computation is due to Harder [21].

4.1. **Euler characteristics.** We begin by recalling some definitions. The *cohomological dimension* cd Γ of a group Γ is the largest $n \in \mathbb{Z} \cup \{\infty\}$ such that there exists a $\mathbb{Z}\Gamma$-module M with $H^n(\Gamma; M) \neq 0$. The *virtual cohomological dimension* vcd Γ of Γ is defined to be the cohomological dimension of any torsion-free finite index subgroup of Γ (one can show that this is well-defined). We recall that Γ is said to be of *finite homological type* if (i) vcd $\Gamma < \infty$ and (ii) for every $\mathbb{Z}\Gamma$-module M that is finitely generated as an abelian group, the homology group $H_i(\Gamma; M)$ is finitely generated for all i.

Definition 4.1. Let Γ be a torsion-free group of finite homological type, and let $H_*(\Gamma)$ denote the homology of Γ with (trivial) $\mathbb{Z}$-coefficients. The *Euler characteristic* $\chi(\Gamma)$ is

$$\chi(\Gamma) = \sum_i (-1)^i \operatorname{rank}_{\mathbb{Z}}(H_i(\Gamma)).$$

Proposition 4.2. [12, Theorem 6.3] *If Γ is torsion-free and of finite homological type and $\Gamma' \subseteq \Gamma$ is a subgroup of finite index, then*

$$\chi(\Gamma') = [\Gamma : \Gamma'] \cdot \chi(\Gamma).$$

One can use Proposition 4.2 to extend the notion of Euler characteristic to groups with torsion. Namely, if Γ is an arbitrary group of finite homological type with a torsion-free subgroup Γ' of finite index, one sets

$$\tag{5} \chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma : \Gamma']}.$$

By Proposition 4.2, this is independent of the choice of Γ'.

Proposition 4.3. *If Γ is a finite group, then $\chi(\Gamma) = 1/|\Gamma|$.*

Proof. Take Γ' to be the trivial subgroup in (5). $\qquad\square$

Theorem 4.4. [21] *Let F be a number field with ring of integers $\mathcal{O}$, and let $\zeta_F(s)$ be the Dedekind zeta function of F. Then*

$$\chi(\mathrm{SL}_N(\mathcal{O})) = \prod_{k=2}^{N} \zeta_F(1-k).$$

In particular, since for imaginary quadratic F the Dedekind zeta function $\zeta_F(m)$ vanishes for all $m \in \mathbb{Z} < 0$, we have that $\chi(\mathrm{SL}_N(\mathcal{O})) = 0$ when $N \geq 2$.

Corollary 4.5. *Let F be an imaginary quadratic field with ring of integers $\mathcal{O}$. Then $\chi(\mathrm{GL}_N(\mathcal{O})) = 0$ when $N \geq 2$.*

Proof. This follows immediately from Theorem 4.4 and the definition of the Euler characteristic since $\mathrm{SL}_N(\mathcal{O})$ is of finite index in $\mathrm{GL}_N(\mathcal{O})$. $\qquad\square$

4.2. **The mass formula.** Now we turn to a different concept, the *equivariant Euler characteristic* $\chi_\Gamma(X)$. Let X be any cell complex with Γ action such that (i) X has finitely many cells mod Γ, and (ii) for each $\sigma \in X$, the stabilizer subgroup $\mathrm{Stab}_\Gamma(\sigma)$ is finite. One defines

$$\chi_\Gamma(X) = \sum_{\sigma \in S} (-1)^{\dim \sigma} \chi(\mathrm{Stab}_\Gamma(\sigma)),$$

where S is a set of representatives of cells of X mod Γ. One knows that if Γ has a torsion-free subgroup of finite index, and the cell complex X is contractible, then in fact $\chi_\Gamma(X) = \chi(\Gamma)$ [12, Proposition 7.3 (e')].

The well-rounded retract W defined in §2.2 is a proper, contractible Γ-complex and so its equivariant Euler characteristic is defined. We compute its equivariant Euler characteristic, phrased in terms of cells in Σ^* using Corollary 2.9, to get a mass formula.

Theorem 4.6 (Mass Formula). *We have*

(6)
$$\sum_{\sigma \in \Sigma_k^*} (-1)^k \frac{1}{|\mathrm{Stab}_\Gamma(\sigma)|} = 0.$$

Proof. Let $\Gamma = \mathrm{GL}_N(\mathcal{O})$. By Proposition 4.3 we have $\chi(\mathrm{Stab}_\Gamma(\sigma)) = 1/|\mathrm{Stab}_\Gamma(\sigma)|$. Furthermore, by Theorem 2.7 and Corollary 2.9, we can identify cells of W mod Γ with cells in Σ^*. Thus (6) represents the equivariant Euler characteristic $\chi_\Gamma(W)$. Since W is contractible [1,2] and Γ has torsion-free subgroups of finite index (any sufficiently small congruence subgroup is torsion-free), the remark at the end of §4.1 implies $\chi_\Gamma(W) = \chi(\Gamma)$. Finally, by Corollary 4.5, we have $\chi(\Gamma) = 0$, which completes the proof. □

Using the stabilizer information in Tables 1–12 one can easily verify that $\chi(\mathrm{GL}_n(\mathcal{O})) = 0$ for each of our examples. The most spectacular example is $\mathrm{GL}_4(\mathcal{O}_{-4})$. If we add together the terms $1/|\mathrm{Stab}_\Gamma(\sigma)|$ for cells σ of the same dimension to a single term for $\mathrm{GL}_4(\mathcal{O}_{-4})$, we find (ordering the terms by increasing dimension in Σ^*)

$$-\frac{11}{3072} + \frac{127}{960} - \frac{4187}{2304} + \frac{28375}{2304} - \frac{868465}{18432} + \frac{126127}{1152} - \frac{81945}{512}$$
$$+ \frac{340955}{2304} - \frac{48655}{576} + \frac{16075}{576} - \frac{21337}{4608} + \frac{101}{384} - \frac{17}{92160} = 0.$$

The other groups have been checked similarly.

5. Explicit homology classes

By Theorem 3.7 we have $H_{N^2-1}(\mathrm{Vor}_{N,D} \otimes \mathbb{Q}) \simeq H^0(\mathrm{GL}_N(\mathcal{O}), \mathbb{Q})$, which in turn is isomorphic to $\mathbb{Q}$. This suggests that there should be a canonical generator for this homology group, a fact already explored in [19, Section 5]. An obvious choice is the analogue of the chain presented there, namely

$$\xi := \xi_{N,D} := \sum_\sigma \frac{1}{|\mathrm{Stab}(\sigma)|} [\sigma],$$

where σ runs through the cells in $\Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$. In this section we verify that this is true for our examples. We should point out that all the cells in $\Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$ are orientable. The reason is that the group $\mathrm{GL}_n(\mathbb{C})$ is connected and so the determinant of its action on $\mathcal{H}^N(\mathbb{C})$ is positive. Since the stabilizers are included in $\mathrm{GL}_N(\mathcal{O})$ and the faces in $\Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$ are full-dimensional, the orientation has to be preserved.

Theorem 5.1. *When $N = 3$ and $D \geq -24$ or $N = 4$ and $D = -3, -4$, the chain ξ is a cycle and thus generates $H_{N^2-1}(\mathrm{Vor}_{N,D} \otimes \mathbb{Q})$.*

Proof. The proof is an explicit computation with differential matrix A representing the map $V_{N^2-1}(\Gamma) \to V_{N^2-2}(\Gamma)$ (cf. §3.2). Note that the signs of the entries of A depend on a choice of orientation for each of the cells in Σ_{N^2-1} and Σ_{N^2-2}. In each non-zero row of A there are exactly two non-zero entries. Each non-zero entry $A_{i,j}$ has absolute value $|\mathrm{Stab}(\sigma_j)|/|\mathrm{Stab}(\tau_i)|$, where $\sigma_j \in \Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$ and $\tau_i \in \Sigma_{N^2-2}(\mathrm{GL}_N(\mathcal{O}))$. One then checks that there is a choice of orientations such that the non-zero entries in a given row have opposite signs. $\square$

For example, consider $N = 4$ and $D = -4$. The relevant differential matrix is

$$d_{15} = \begin{bmatrix} 0 & 0 \\ 1920 & -256 \end{bmatrix},$$

with kernel generated by $(2, 15) = 92160(1/46080, 1/6144)$. The orders of the two stabilizer groups for the cells in $\Sigma_{15}(\mathrm{GL}_N(\mathcal{O}_{-4}))$ are 46080 and 6144, respectively, and thus $\xi_{4,-4}$ is a cycle.

Remark 5.2. It seems likely that Theorem 5.1 holds for all $\mathrm{GL}_N(\mathcal{O}_D)$, although we do not presently have a proof.

6. Algorithms

In this section we give more details about the specific algorithms we used in this project. There are several stages needed to compute the Voronoi homology:

(1) Determination of the perfect forms.
(2) Computation of the Voronoi complex and its differentials.
(3) Computation of the homology.

We discuss each one of these below in turn.

We remark that all our computations were performed with two different implementations, one in `MAGMA` [9] and one in `GAP` [43]. In addition the determination of the perfect forms was also done using a `C++` [42] implementation currently under development that we plan to use as our main codebase in the future. The first two implementations produced identical results, and the third implementation successfully recovered the list of perfect forms. Complete source code for the `GAP` implementation, including data for all the Voronoi complexes, can be found at [14]. Another version of the source code, which does only the cohomology of perfect forms, can be found at [17].

6.1. **Determination of the perfect forms.** Recall that $\mathcal{H}^N(\mathbb{C})$ is the vector space of Hermitian forms in N variables, and that each field F endows this space with a rational structure. We begin by describing a model for $\mathcal{H}^N(\mathbb{C})$ that allows us to perform all computations with exact arithmetic. To do this we first represent the space $\mathcal{H}^N(\mathbb{C})$ as a vector space of symmetric $2N \times 2N$ real matrices.

Let $\mathcal{S}^N$ the space of $N \times N$ real symmetric matrices and $\mathcal{AS}^N$ the space of $N \times N$ real antisymmetric matrices. Considering the action of complex conjugation, we write a matrix $A \in \mathcal{H}^N(\mathbb{C})$ as $A = U + V$ with $U \in \mathcal{S}^N$ and $V \in \mathcal{AS}^N$. Then for any $x, y \in \mathbb{Z}^N$, we have

$$\begin{aligned} A[x + \omega y] &= (x + \omega y)^* A(x + \omega y) \\ &= x^T U x + P {}^t y U y + x^t S U y + x^T (\omega - \omega^*) V y, \end{aligned}$$

where $P = \alpha^* \alpha$ and $S = \alpha + \alpha^*$, and $\omega \in F$ is defined at the beginning of Section 2. Putting $W = (\omega - \omega^*)V/2$, we can define the space

$$\mathcal{SP}(\mathbb{Z}[\omega]) := \left\{ \begin{pmatrix} U & (S/2)U + W \\ (S/2)U + W & PU \end{pmatrix} \in \mathcal{S}^{2N} \text{ for } U \in \mathcal{S}^N \text{ and } W \in \mathcal{AS}^N \right\}.$$

The space $\mathcal{SP}(\mathbb{Z}[\omega])$ has a natural basis of $\mathbb{Q}$-rational matrices, and using this basis, all our computations can be carried out over the integers. With this model for $\mathcal{H}^N(\mathbb{C})$, the positive definite elements of $\mathcal{SP}(\mathbb{Z}[\omega])$ correspond to the positive definite Hermitian forms, and the group $\Gamma = \mathrm{GL}_N(\mathcal{O})$ is a subgroup of $\mathrm{GL}_{2N}(\mathbb{Z})$. We remark that this "restriction of scalars" construction can be carried out for general number fields F, but that one obtains rational spaces only when F is totally real or CM.

Thus we are reduced to performing computations in a space of real symmetric matrices and its subcone of positive definite matrices. By now there is an extensive literature giving tools to work with such spaces. For instance, specific algorithms have been developed in [32] for enumerating the set of perfect forms when it is finite; we will only briefly restate them here and refer to [32] for detailed definitions and algorithms. Another good source is [36].

As in Definition 2.1, let $\mathrm{Min}(A)$ be the set of minimal vectors of a positive definite form $A \in \mathcal{S}^N$. The key algorithmic functions available for a given space $\mathcal{SP}$ of symmetric matrices are as follows:

(1) *Enlarging sets of miminal vectors.* Let A be a positive definite quadratic form and let $v \in \mathrm{Min}(A)$ be a minimal vector. Suppose $B \neq 0$ is a quadratic form with $B[v] = 0$. Then we can find $\lambda > 0$ such that $\mathrm{Min}(A)$ is properly contained in $\mathrm{Min}(A + \lambda B)$. [32, Algorithm 2].

(2) *Finding perfect forms.* Starting with a positive definite form A and its set of minimal vectors $\mathrm{Min}(A)$, we can iterate the enlarging algorithm [32, Algorithm 2] until we obtain a perfect form [32, Page 371].

(3) *Finding the cone of a perfect form.* Given one perfect form A we can compute the cone $\sigma(A)$ it determines and then the facets of $\sigma(A)$ using the Recursive Adjacency Decomposition Method [36, Section 3.2], [31].

(4) *Finding neighbors of perfect forms.* Suppose τ is a facet of a perfect cone $\sigma(A)$. Let $\mathrm{Min}(\tau) \subset \mathrm{Min}(A)$ be the set of corresponding shortest vectors. Up to a positive multiple there is a unique matrix $B \in \mathcal{SP}$ such that (i) $B[v] = 0$ for $v \in \mathrm{Min}(\tau)$ and (ii) $B[v] > 0$ for $v \in \mathrm{Min}(A) \smallsetminus \mathrm{Min}(\tau)$. In other words, B determines a linear form that cuts τ out from $\sigma(A)$ and is positive on $\sigma(A) \setminus \tau$. If B is not positive definite then there is a perfect form A' such that $\sigma(A') \cap \sigma(A) = \tau$; we say A' is a *neighbor* of A. Any neighboring form of A can be easily determined using the enlarging algorithm [32, Algorithm 2]. On the other hand, if B is positive definite, then one can show that there is no perfect form neighboring A along τ. In this case the facet τ is called a *dead end* [32]. In all our cases considered here, there were no dead ends.

(5) *Equivalence and automorphisms.* In order to compute with perfect forms, it is indispensable to be able to determine when two forms lie in the same Γ-orbit and to determine a form's stabilizer in Γ. In the classical setting of positive definite forms over $\mathbb{Q}$, the best tools are the algorithms `ISOM` / `AUTO` due to Plesken–Souvignier [27]. Applying these algorithms in our setting using the model $\mathcal{SP}(\mathbb{Z}[\omega])$ produces stabilizers that are too large and false equivalences, since in our case we only want the matrices $P \in \mathrm{GL}_{2N}(\mathbb{Z})$ that belong to the embedding of Γ into $\mathrm{GL}_{2N}(\mathbb{Z})$. Adaptations of `ISOM` / `AUTO` allowing this are described in in [11, Example 5].

By combining the preceding algorithms, one obtains the generalization of Voronoi's classical algorithm that allows one to find all the perfect forms in the cone of positive definite Hermitian forms up to Γ-equivalence. Details are given in [31, 32, 36].

6.2. **Computing the Voronoi complex and the differentials.** After the steps in Section 6.1 are finished, we have obtained a list of representatives of the perfect forms and their stabilizer groups. Once one has this list of perfect forms, one essentially has the set of top-dimensional cells of $\Sigma^*(\Gamma)$.

One then needs to determine the full polyhedral structure of all these cells. This is done in an obvious way using well known algorithms in polyhedral geometry: one finds all the facets of a given cell, determines their facets, and so on. The subtlety in the process lies in testing equivalence of cells. Ultimately one is only interested in those faces that do not lie at infinity (i.e. those that meet X_N). Given two cells $\tau, \tau' \in \Sigma_k^*(\Gamma)$, one first computes two barycenter matrices $\mathrm{Cent}(\tau)$, $\mathrm{Cent}(\tau')$ by taking the sums of their corresponding extreme rays. Since the cells τ, τ' meet X_N, the matrices $\mathrm{Cent}(\tau)$ and $\mathrm{Cent}(\tau')$ are positive definite. Thus we can check whether or not they are equivalent using the same methods involved for enumerating perfect cones.

After carrying this out, the result is the full set $\Sigma^*(\Gamma)$, along with information about inclusion relations between cells. This information is exactly what one needs to compute the differential in the Voronoi complex, although there are subtleties in determining signs. Recall from Section 3 that $\Sigma_k(\Gamma) \subset \Sigma_k^*(\Gamma)$ is the subset of all k-cells whose stabilizers are orientation-preserving. The groups in the Voronoi complex are then the free abelian groups $V_k(\Gamma)$ generated by the elements of $\Sigma_k(\Gamma)$, and we must determine the differential $d_k \colon V_k(\Gamma) \to V_{k-1}(\Gamma)$. This is done as follows.

For each closed cell σ in X_N^*, we first fix an orientation of σ, i.e. an orientation of the real vector space $\mathbb{R}(\sigma)$ of symmetric matrices spanned by the forms $q(v)$ with $v \in \mathrm{Min}(\sigma)$. Let $\sigma \in \Sigma_k$ and let τ' be a face of σ that is equivalent under Γ to an element in Σ_{k-1} (i.e. τ' neither lies on the boundary nor has elements in its stabilizer reversing the orientation). Given a positive basis B' of $\mathbb{R}(\tau')$ we get a basis B of $\mathbb{R}(\sigma) \supset \mathbb{R}(\tau')$ by appending to B' the vector $\mathrm{Cent}(\sigma)$. We let $\varepsilon(\tau', \sigma) = \pm 1$ be the sign of the orientation of B in the oriented vector space $\mathbb{R}(\sigma)$.

Next, let $\tau \in \Sigma_{k-1}$ be the (unique) cell equivalent to τ' and let $\gamma \in \Gamma$ be such that $\tau' = \tau \cdot \gamma$. We define $\eta(\tau, \tau') = 1$ (resp. $\eta(\tau, \tau') = -1$) when γ is compatible (resp. incompatible) with the chosen orientations of $\mathbb{R}(\tau)$ and $\mathbb{R}(\tau')$.

Finally we define, as in [19], the differential by

$$(7) \qquad d_k(\sigma) = \sum_{\tau \in \Sigma_{n-1}} \sum_{\tau'} \eta(\tau, \tau')\, \varepsilon(\tau', \sigma)\, \tau.$$

6.3. **Computing the homology.** After one has the differentials, computing the cohomology is a standard problem in exact linear algebra. Any cohomology group H is obtained in the form $H = \mathrm{Ker}(A)/\mathrm{Im}(B)$. However, it is important to note that one does not need actually to find the kernel explicitly and compute the quotient. For the rational part of the cohomology, one simply needs the ranks of the matrices A and B. For the torsion of H, one simply needs the invariant factors of the Smith normal form of B (see the discussion after Theorem 7.3 below). For both computations, the `linbox` library [23] for exact linear algebra is adequate and is able to take advantage of the natural sparsity of the matrices of the differentials.

7. RESULTS AND TABLES

We conclude by presenting the results of our computations. First we summarize the most interesting numerical results from the tables, namely those involving the homology computations for $\mathrm{GL}_*(\mathcal{O}_D)$.

Theorem 7.1. *Let $0 > D \geq -24$ be a fundamental discriminant.*

(1) *The Voronoi homology $H_n(\mathrm{Vor}_{3,D})$ can have non-trivial rank only for $n = 2, 4, 5,$ and 8.*

(2) *In each case for $n = 8$ we have $H_8(\mathrm{Vor}_{3,D}) \cong \mathbb{Z}$.*

(3) *In homology degree 2, 4 and 5, we have the following table of ranks (empty slots being zero)*

D	-3	-4	-7	-8	-11	-15	-19	-20	-23	-24
#{*perf. forms*}	2	1	2	2	12	90	157	212	870	596
rank H_2								1	4	1
rank H_4	1	1	1	1	1	3	2	4	5	5
rank H_5	1	1	2	2	2	5	3	6	1	7

(4) *The p-Sylow subgroups for the primes occurring in (the orders of) the respective homology groups are given in the following table, where we use the shorthand $n(G)$ to denote that the p-Sylow subgroup in $H_n(\mathrm{Vor}_{3,D})$ is equal to G. We also use the "topologist's convention" that $\mathbb{Z}_n$ denotes the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$.*

D	-3	-4	-7	-8	-11	-15	-19	-20
Syl_2		$5(\mathbb{Z}_2)$		$4(\mathbb{Z}_2), 5(\mathbb{Z}_2)$	$5(\mathbb{Z}_4)$	$4(\mathbb{Z}_4)$	$5(\mathbb{Z}_2 \times \mathbb{Z}_8)$	$3(\mathbb{Z}_2), 4(\mathbb{Z}_2^4), 7(\mathbb{Z}_2^2 \times \mathbb{Z}_4)$
Syl_3	$7(\mathbb{Z}_9)$		$7(\mathbb{Z}_3)$		$7(\mathbb{Z}_3^2)$	$7(\mathbb{Z}_3^5)$	$7(\mathbb{Z}_3^4)$	$5(\mathbb{Z}_3), 7(\mathbb{Z}_3^5)$
Syl_5						$7(\mathbb{Z}_5)$		
Syl_7			$5(\mathbb{Z}_7)$					$7(\mathbb{Z}_7)$

D	-23	-24
Syl_2	$3(\mathbb{Z}_2), 4(\mathbb{Z}_2^{13}), 5(\mathbb{Z}_2^5), 7(\mathbb{Z}_2^3 \times \mathbb{Z}_4^2)$	$3(\mathbb{Z}_2^7), 4(\mathbb{Z}_2^5), 5(\mathbb{Z}_2^8 \times \mathbb{Z}_4^2), 6(\mathbb{Z}_2), 7(\mathbb{Z}_2^4 \times \mathbb{Z}_4^2)$
Syl_3	$4(\mathbb{Z}_3^3), 7(\mathbb{Z}_3^7)$	$7(\mathbb{Z}_3^3)$

Theorem 7.2. *Let V_4 be either the Voronoi complex for $\mathrm{GL}_4(\mathcal{O}_{-3})$ or $\mathrm{GL}_4(\mathcal{O}_{-4})$. Then $H_n(V_4)$ has non-trivial rank only for $n = 6, 7, 9, 10, 12,$ and 15. More precisely,*

(1) *For $n = 3k$ we have rank $H_{3k}(V_4) = \begin{cases} 1 & \text{if } k > 1, \\ 0 & \text{if } k = 1. \end{cases}$*

(2) *For $n = 7$ we have rank $H_n(V_4) = 2$.*

(3) *For $n = 10$ we have rank $H_n(V_4) = 1$.*

Moreover, the only torsion primes appearing are 2, 3 and 5.

Invoking Theorem 3.7, we also get information on the cohomology of $\mathrm{GL}_4(\mathcal{O}_D)$:

Theorem 7.3. *Let $D = -3$ or -4. Then $H^n(\mathrm{GL}_4(\mathcal{O}_D))$ has non-trivial rank only for $n = 0, 3, 5, 6, 8,$ and 9. More precisely,*

(1) *For $n = 3k$ we have rank $H^n(\mathrm{GL}_4(\mathcal{O}_D)) = 1$ if $0 \leq k \leq 3$.*

(2) *For $n = 5$ we have* $\operatorname{rank} H^n(\operatorname{GL}_4(\mathcal{O}_D)) = 1$.
(3) *For $n = 8$ we have* $\operatorname{rank} H^n(\operatorname{GL}_4(\mathcal{O}_D)) = 2$.

The remainder of the text is devoted to presenting details about the Voronoi complexes and full information about their homology. The notation is as follows:

- The first three columns concern the cell decomposition of X_N^* mod Γ:
  - n is dimension of the cells in the (partially) compactified symmetric space X_N^*.
  - $|\Sigma_n^*|$ is the number of Γ-orbits in the cells that meet $X_N \subset X_N^*$.
  - $|\operatorname{Stab}|$ gives the sizes of the stabilizer subgroups in factored form. The notation $A(k)$ means that, of the $|\Sigma_n^*|$ cells of dimension n, k of them have a stabilizer subgroup of order A.
- The next four columns concern the differentials d_n of the Voronoi complex $\operatorname{Vor}_{N,D}$:
  - $|\Sigma_n|$ is the number of orientable Γ-orbits in Σ_n^*.
  - nnz is the number of non-zero entries in the differential $d_n \colon V_n(\Gamma) \to V_{n-1}(\Gamma)$.
  - rank is the rank of d_n.
  - elem. div. gives the elementary divisors of d_n. As in the stabilizer column, the notation $d(k)$ means that the elementary divisor d occurs with multiplicity k. If the rank of d_n is zero then this column is left empty.
- Finally, the last column gives the homology of the Voronoi complex. One can easily check that $H_n \simeq \mathbb{Z}^r \oplus \bigoplus (\mathbb{Z}/d\mathbb{Z})^k$, where $r = |\Sigma_n| - \operatorname{rank}(d_n) - \operatorname{rank}(d_{n+1})$ and the sum is taken over the elementary divisors $d(k)$ from row $n + 1$. As above, to save space we denote the finite cyclic group $\mathbb{Z}/d\mathbb{Z}$ by $\mathbb{Z}_d$. By Theorem 3.7 we have $H_n(\operatorname{Vor}_{N,D}) \simeq H^{N^2-1-n}(\operatorname{GL}_N(\mathcal{O}))$ modulo the torsion primes in $\operatorname{GL}_N(\mathcal{O})$. These primes are visible in the third column of each table.

TABLE 1. The cell complex, differentials, and homology for $\operatorname{GL}_3(\mathcal{O}_{-3})$.

n	$|\Sigma_n^*|$	$|\operatorname{Stab}|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	1	$2^4 3^4(1)$	0	0	0		0
3	2	$2^4 3^2(1),\ 2^3 3^3(1)$	0	0	0		0
4	3	$2^2 3^2(1),\ 2^4 3^3(1),\ 2^4 3^1(1)$	1	0	0		$\mathbb{Z}$
5	4	$2^4 3^2(1),\ 2^2 3^2(1),\ 2^1 3^2(1),$ $2^2 3^3(1)$	2	0	0		$\mathbb{Z}$
6	3	$2^2 3^2(1),\ 2^1 3^2(1),\ 2^2 3^3(1)$	1	2	1	$1(1)$	0
7	2	$2^4 3^2(1),\ 2^2 3^2(1)$	1	0	0		$\mathbb{Z}_9$
8	2	$2^4 3^4(2)$	2	2	1	$9(1)$	$\mathbb{Z}$

TABLE 2. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-4})$.

n	$|\Sigma_n^*|$	$|\mathrm{Stab}|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	2	$2^7 3^1(2)$	0	0	0		0
3	3	$2^3 3^1(1), 2^5 3^1(2)$	0	0	0		0
4	4	$2^4(1), 2^3(1), 2^5(1), 2^7 3^1(1)$	1	0	0		$\mathbb{Z}$
5	5	$2^2 3^1(2), 2^3(1), 2^5 3^1(1), 2^5(1)$	4	0	0		$\mathbb{Z} \oplus \mathbb{Z}_2$
6	3	$2^2 3^1(2), 2^4(1)$	3	10	3	$1(2), 2(1)$	0
7	1	$2^4(1)$	0	0	0		0
8	1	$2^7 3^1(1)$	1	0	0		$\mathbb{Z}$

TABLE 3. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-7})$.

n	$|\Sigma_n^*|$	$|\mathrm{Stab}|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	3	$2^4 3^1(3)$	0	0	0		0
3	6	$2^2 3^1(2), 2^3 3^1(1), 2^2(1), 2^4 3^1(1),$ $2^4(1)$	0	0	0		0
4	9	$2^2 3^1(1), 2^1(1), 2^3 3^1(1), 2^2(2),$ $2^4(1), 2^3(3)$	3	0	0		$\mathbb{Z}$
5	11	$2^2 3^1(1), 2^1(2), 2^2(1), 2^4 3^1(3),$ $2^1 3^1(3), 2^3(1)$	10	8	2	$1(2)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_7$
6	8	$2^2(2), 2^1 3^1(2), 2^3(3), 2^1 3^1 7^1(1)$	6	19	6	$1(5), 7(1)$	0
7	2	$2^1 7^1(1), 2^2(1)$	1	0	0		$\mathbb{Z}_3$
8	2	$2^4 3^1 7^1(1), 2^1 3^1 7^1(1)$	2	2	1	$3(1)$	$\mathbb{Z}$

TABLE 4. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-8})$.

n	$|\Sigma_n^*|$	$|\mathrm{Stab}|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	5	$2^2 3^1(2), 2^4 3^1(2), 2^4(1)$	0	0	0		0
3	16	$2^1(2), 2^2(5), 2^3 3^1(1), 2^2 3^1(5),$ $2^4 3^1(1), 2^5(2)$	2	0	0		0
4	26	$2^1(14), 2^2(9), 2^3(1), 2^5 3^1(1),$ $2^4(1)$	16	12	2	$1(2)$	$\mathbb{Z} \oplus \mathbb{Z}_2$
5	37	$2^1(25), 2^2(4), 2^1 3^1(4), 2^3(2),$ $2^4 3^1(2)$	36	104	13	$1(12), 2(1)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$
6	28	$2^1(18), 2^2(2), 2^1 3^1(4), 2^3(2),$ $2^2 3^1(2)$	26	166	21	$1(20), 2(1)$	0
7	7	$2^1(6), 2^2(1)$	6	45	5	$1(5)$	0
8	2	$2^1 3^1(1), 2^5(1)$	2	6	1	$1(1)$	$\mathbb{Z}$

TABLE 5. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-11})$.

n	$|\Sigma_n^*|$	$|\mathrm{Stab}|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	8	$2^23^1(2), 2^33^1(1), 2^2(2), 2^43^1(1),$ $2^4(2)$	1	0	0		0
3	34	$2^1(13), 2^2(11), 2^3(2), 2^43^1(1),$ $2^23^1(4), 2^33^1(3)$	15	6	1	1(1)	0
4	91	$2^1(67), 2^2(15), 2^13^1(1), 2^3(5),$ $2^23^1(1), 2^4(1), 2^43^1(1)$	75	193	14	1(14)	$\mathbb{Z}$
5	150	$2^1(124), 2^2(13), 2^13^1(7), 2^3(2),$ $2^23^1(3), 2^43^1(1)$	147	700	60	1(60)	$\mathbb{Z}^2 \oplus \mathbb{Z}_4$
6	125	$2^1(110), 2^2(6), 2^13^1(6), 2^3(3)$	122	859	85	1(84), 4(1)	0
7	51	$2^1(44), 2^2(2), 2^13^1(3), 2^23^1(1),$ $2^4(1)$	48	404	37	1(37)	$\mathbb{Z}_3^2$
8	12	$2^1(4), 2^13^1(6), 2^4(2)$	12	88	11	1(9), 3(2)	$\mathbb{Z}$

TABLE 6. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-15})$.

n	$|\Sigma_n^*|$	$|\mathrm{Stab}|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	34	$2^1(4), 2^2(8), 2^13^1(1), 2^3(8),$ $2^43^1(3), 2^23^1(3), 2^4(7)$	10	0	0		$\mathbb{Z}$
3	217	$2^1(102), 2^2(77), 2^13^1(2), 2^3(23),$ $2^23^1(5), 2^4(2), 2^33^1(3), 2^43^1(3)$	128	175	9	1(9)	0
4	689	$2^1(546), 2^2(114), 2^13^1(1),$ $2^3(20), 2^23^1(2), 2^4(5), 2^33^1(1)$	604	2112	119	1(119)	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^4$
5	1224	$2^1(1109), 2^2(84), 2^13^1(7),$ $2^3(13), 2^23^1(5), 2^4(3), 2^43^1(3)$	1185	6373	482	1(478), 2(4)	$\mathbb{Z}^5$
6	1139	$2^1(1081), 2^2(47), 2^13^1(7), 2^3(4)$	1102	7771	698	1(698)	0
7	522	$2^1(489), 2^2(30), 2^13^1(1), 2^3(1),$ $2^23^1(1)$	493	4162	404	1(404)	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_{12}$
8	90	$2^1(78), 2^2(3), 2^13^1(5), 2^3(2),$ $2^23^1(2)$	90	972	89	1(84), 3(2), 6(2), 12(1)	$\mathbb{Z}$

TABLE 7. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-19})$.

n	$|\Sigma_n^*|$	$|\,$Stab$\,|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	43	$2^1(23)$, $2^2(10)$, $2^43^1(1)$, $2^13^1(3)$, $2^3(1)$, $2^23^1(2)$, $2^4(2)$, $2^33^1(1)$	29	0	0		0
3	359	$2^1(304)$, $2^2(38)$, $2^13^1(3)$, $2^3(7)$, $2^23^1(4)$, $2^33^1(2)$, $2^43^1(1)$	314	664	29	1(29)	0
4	1293	$2^1(1234)$, $2^2(52)$, $2^3(4)$, $2^4(1)$, $2^33^1(1)$, $2^43^1(1)$	1255	5410	285	1(285)	$\mathbb{Z}^2$
5	2347	$2^1(2287)$, $2^2(37)$, $2^13^1(13)$, $2^3(7)$, $2^23^1(2)$, $2^43^1(1)$	2339	13596	968	1(968)	$\mathbb{Z}^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$
6	2169	$2^1(2137)$, $2^2(15)$, $2^13^1(10)$, $2^3(7)$	2164	15404	1368	1(1366), 2(1), 8(1)	0
7	958	$2^1(950)$, $2^2(6)$, $2^13^1(1)$, $2^4(1)$	952	8181	796	1(796)	$\mathbb{Z}_3^4$
8	157	$2^1(149)$, $2^13^1(6)$, $2^4(2)$	157	1884	156	1(152), 3(4)	$\mathbb{Z}$

TABLE 8. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-20})$.

n	$|\Sigma_n^*|$	$|\,$Stab$\,|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
2	69	$2^1(21)$, $2^2(26)$, $2^13^1(4)$, $2^3(6)$, $2^23^1(3)$, $2^4(7)$, $2^43^1(2)$	31	0	0		$\mathbb{Z}$
3	538	$2^1(398)$, $2^2(98)$, $2^13^1(4)$, $2^3(22)$, $2^23^1(7)$, $2^4(3)$, $2^33^1(4)$, $2^43^1(2)$	425	772	30	1(30)	$\mathbb{Z}_2$
4	1895	$2^1(1721)$, $2^2(153)$, $2^3(15)$, $2^23^1(1)$, $2^4(4)$, $2^43^1(1)$	1804	7464	395	1(394), 2(1)	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^4$
5	3382	$2^1(3223)$, $2^2(117)$, $2^13^1(15)$, $2^3(15)$, $2^23^1(7)$, $2^4(4)$, $2^43^1(1)$	3345	19167	1405	1(1401), 2(4)	$\mathbb{Z}^6 \oplus \mathbb{Z}_3$
6	3061	$2^1(2976)$, $2^2(61)$, $2^13^1(15)$, $2^3(8)$, $2^23^1(1)$	3017	21502	1934	1(1933), 3(1)	0
7	1330	$2^1(1293)$, $2^2(37)$	1294	11127	1083	1(1083)	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_{12}$
8	212	$2^1(202)$, $2^2(2)$, $2^13^1(5)$, $2^3(2)$, $2^23^1(1)$	212	2532	211	1(206), 3(2), 6(2), 12(1)	$\mathbb{Z}$

TABLE 9. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-23})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	nnz	rank	elem. div.	H_n
2	204	$2^1(89)$, $2^2(65)$, $2^13^1(4)$, $2^3(27)$, $2^23^1(6)$, $2^4(10)$, $2^43^1(3)$	126	0	0		$\mathbb{Z}^4$
3	1777	$2^1(1402)$, $2^2(295)$, $2^13^1(2)$, $2^3(56)$, $2^23^1(11)$, $2^4(3)$, $2^33^1(5)$, $2^43^1(3)$	1477	3272	122	$1(122)$	$\mathbb{Z}_2$
4	6589	$2^1(6112)$, $2^2(434)$, $2^3(35)$, $2^23^1(2)$, $2^4(5)$, $2^33^1(1)$	6285	26837	1355	$1(1354)$, $2(1)$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^{10} \oplus \mathbb{Z}_6^3$
5	12214	$2^1(11866)$, $2^2(291)$, $2^13^1(19)$, $2^3(23)$, $2^23^1(6)$, $2^4(6)$, $2^43^1(3)$	12119	69891	4925	$1(4912)$, $2(10)$, $6(3)$	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^5$
6	11627	$2^1(11461)$, $2^2(138)$, $2^13^1(16)$, $2^3(10)$, $2^23^1(2)$	11568	81720	7184	$1(7179)$, $2(5)$	0
7	5303	$2^1(5250)$, $2^2(48)$, $2^13^1(2)$, $2^23^1(2)$, $2^4(1)$	5253	44741	4384	$1(4384)$	$\mathbb{Z}_{12}^2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_6^3$
8	870	$2^1(853)$, $2^2(3)$, $2^13^1(10)$, $2^3(2)$, $2^4(2)$	870	10464	869	$1(862)$, $12(2)$, $3(2)$, $6(3)$	$\mathbb{Z}$

TABLE 10. The cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-24})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	nnz	rank	elem. div.	H_n
2	158	$2^1(90)$, $2^2(41)$, $2^13^1(3)$, $2^3(13)$, $2^23^1(1)$, $2^4(8)$, $2^43^1(2)$	104	0	0		$\mathbb{Z}$
3	1396	$2^1(1214)$, $2^2(142)$, $2^13^1(4)$, $2^3(24)$, $2^23^1(6)$, $2^4(1)$, $2^33^1(3)$, $2^43^1(2)$	1247	2967	103	$1(103)$	$\mathbb{Z}_2^7$
4	5090	$2^1(4859)$, $2^2(199)$, $2^13^1(1)$, $2^3(23)$, $2^23^1(1)$, $2^4(5)$, $2^33^1(1)$, $2^43^1(1)$	4957	22280	1144	$1(1137)$, $2(7)$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^5$
5	9091	$2^1(8889)$, $2^2(161)$, $2^13^1(12)$, $2^3(20)$, $2^23^1(5)$, $2^4(3)$, $2^43^1(1)$	9043	53385	3808	$1(3803)$, $2(5)$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_4^2$
6	8319	$2^1(8187)$, $2^2(102)$, $2^13^1(13)$, $2^3(15)$, $2^23^1(2)$	8263	58948	5228	$1(5218)$, $2(8)$, $4(2)$	$\mathbb{Z}_2$
7	3662	$2^1(3617)$, $2^2(42)$, $2^3(2)$, $2^4(1)$	3630	31020	3035	$1(3034)$, $2(1)$	$\mathbb{Z}_{12}^2 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_6$
8	596	$2^1(578)$, $2^2(8)$, $2^13^1(2)$, $2^3(4)$, $2^23^1(2)$, $2^4(2)$	596	7188	595	$1(589)$, $12(2)$, $2(3)$, $6(1)$	$\mathbb{Z}$

TABLE 11. The cell complex, differentials, and homology for $\mathrm{GL}_4(\mathcal{O}_{-3})$.

n	$|\Sigma_n^*|$	$|\,\mathrm{Stab}\,|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
3	2	$2^4 3^5(1)$, $2^7 3^5(1)$	0	0	0		0
4	5	$2^4 3^2(1)$, $2^4 3^2 5^1(1)$, $2^5 3^3(1)$, $2^5 3^4(1)$, $2^3 3^4(1)$	0	0	0		0
5	12	$2^2 3^2(2)$, $2^3 3^2(1)$, $2^2 3^3(1)$, $2^3 3^1(1)$, $2^5 3^4(1)$, $2^3 3^3(1)$, $2^4 3^1(2)$, $2^5 3^2(2)$, $2^6 3^4(1)$	0	0	0		0
6	34	$2^5 3^4(1)$, $2^1 3^1(1)$, $2^4 3^1(2)$, $2^2 3^1(8)$, $2^5 3^3(1)$, $2^1 3^2(2)$, $2^3 3^4(1)$, $2^3 3^1(4)$, $2^5 3^2(1)$, $2^3 3^3(1)$, $2^4 3^2(2)$, $2^2 3^3(4)$, $2^2 3^2(6)$	8	0	0		$\mathbb{Z}$
7	82	$2^1 3^1(21)$, $2^5 3^1(1)$, $2^2 3^1(23)$, $2^1 3^2(7)$, $2^2 3^3(3)$, $2^3 3^1(5)$, $2^3 3^4(1)$, $2^2 3^2(8)$, $2^3 3^3(2)$, $2^7 3^4(1)$, $2^4 3^1(4)$, $2^1 3^3(2)$, $2^4 3^2(1)$, $2^3 3^2(3)$	50	58	7	$1(7)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$
8	166	$2^3 3^3(2)$, $2^1 3^1(88)$, $2^4 3^3(1)$, $2^2 3^1(36)$, $2^1 3^2(13)$, $2^5 3^3(1)$, $2^3 3^1(5)$, $2^2 3^2(7)$, $2^4 3^1(2)$, $2^1 3^3(2)$, $2^3 3^2(5)$, $2^2 3^3(3)$, $2^2 3^5(1)$	129	604	41	$1(40)$, $2(1)$	$\mathbb{Z}_9$
9	277	$2^1 3^1(191)$, $2^5 3^2(1)$, $2^2 3^1(34)$, $2^1 3^2(17)$, $2^5 3^5(2)$, $2^3 3^1(6)$, $2^1 3^1 5^1(1)$, $2^2 3^2(9)$, $2^1 3^3(2)$, $2^3 3^2(3)$, $2^5 3^1(1)$, $2^2 3^3(7)$, $2^7 3^3(1)$, $2^2 3^5(1)$, $2^4 3^2 5^1(1)$	228	1616	88	$1(87)$, $9(1)$	$\mathbb{Z} \oplus \mathbb{Z}_{24}$
10	324	$2^1 3^1(246)$, $2^2 3^1(35)$, $2^1 3^2(16)$, $2^3 3^1(7)$, $2^2 3^2(9)$, $2^1 3^3(2)$, $2^4 3^3(1)$, $2^2 3^3(5)$, $2^3 3^3(2)$, $2^5 3^2(1)$	286	2531	139	$1(138)$, $24(1)$	$\mathbb{Z} \oplus \mathbb{Z}_2^4$
11	259	$2^1 3^1(200)$, $2^2 3^1(24)$, $2^1 3^2(11)$, $2^3 3^1(6)$, $2^2 3^2(9)$, $2^2 3^4(1)$, $2^4 3^1(1)$, $2^1 3^3(1)$, $2^2 3^3(2)$, $2^4 3^2(2)$, $2^4 3^3(2)$	237	2283	146	$1(142)$, $2(4)$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_6$
12	142	$2^1 3^1(91)$, $2^2 3^1(20)$, $2^1 3^2(9)$, $2^3 3^1(5)$, $2^2 3^2(11)$, $2^1 3^3(4)$, $2^2 3^3(1)$, $2^4 3^2(1)$	122	1252	91	$1(88)$, $2(2)$, $6(1)$	$\mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$
13	48	$2^1 3^3(1)$, $2^3 3^3(1)$, $2^1 3^1(22)$, $2^2 3^1(6)$, $2^1 3^2(7)$, $2^3 3^1(2)$, $2^2 3^2(8)$, $2^4 3^1(1)$	36	369	30	$1(28)$, $3(1)$, $12(1)$	$\mathbb{Z}_{15}$
14	15	$2^1 3^2(2)$, $2^2 3^2(2)$, $2^1 3^3(1)$, $2^2 3^3(3)$, $2^1 3^1(1)$, $2^4 3^2 5^1(1)$, $2^3 3^1(1)$, $2^2 3^1(2)$, $2^1 3^1 5^1(1)$, $2^6 3^2(1)$	10	51	6	$1(5)$, $15(1)$	$\mathbb{Z}_{12} \oplus \mathbb{Z}_{288}$
15	5	$2^6 3^3(1)$, $2^7 3^5 5^1(1)$, $2^4 3^5(1)$, $2^2 3^3(2)$	5	16	4	$1(2)$, $12(1)$, $288(1)$	$\mathbb{Z}$

TABLE 12. The cell complex, differentials, and homology for $\mathrm{GL}_4(\mathcal{O}_{-4})$.

n	$|\Sigma_n^*|$	$|\mathrm{Stab}|$	$|\Sigma_n|$	nnz	rank	elem. div.	H_n
3	4	$2^{11}3^1(2)$, $2^9 3^1(1)$, $2^7 3^1(1)$	0	0	0		0
4	10	$2^3 3^1(1)$, $2^8(1)$, $2^4 3^1(1)$, $2^5 3^1 5^1(1)$, $2^7(1)$, $2^5 3^1(2)$, $2^8 3^1(1)$, $2^7 3^1(1)$, $2^5(1)$	0	0	0		0
5	33	$2^{10}3^1(1)$, $2^2(1)$, $2^7 3^2(1)$, $2^6 3^1(1)$, $2^3(6)$, $2^4 3^1(2)$, $2^4(7)$, $2^7(1)$, $2^5 3^1(2)$, $2^3 3^1(2)$, $2^{10}(1)$, $2^6(2)$, $2^5(6)$	5	0	0		0
6	98	$2^7(1)$, $2^7 3^1(2)$, $2^2(26)$, $2^3(37)$, $2^2 3^1(1)$, $2^4(10)$, $2^8 3^2(1)$, $2^3 3^1(1)$, $2^5(9)$, $2^4 3^1(4)$, $2^6(1)$, $2^6 3^1(1)$, $2^5 3^1(4)$	48	35	5	$1(5)$	$\mathbb{Z} \oplus \mathbb{Z}_2^4$
7	258	$2^2(147)$, $2^3(69)$, $2^2 3^1(1)$, $2^4(18)$, $2^3 3^1(2)$, $2^5(10)$, $2^4 3^1(3)$, $2^6(2)$, $2^5 3^1(2)$, $2^7(3)$, $2^{11}3^2(1)$	189	682	42	$1(38)$, $2(4)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^3$
8	501	$2^2(397)$, $2^3(67)$, $2^2 3^1(4)$, $2^4(18)$, $2^3 3^1(4)$, $2^5(5)$, $2^4 3^1(1)$, $2^6(3)$, $2^7(1)$, $2^5 3^2(1)$	435	2972	145	$1(142)$, $2(3)$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$
9	704	$2^2(603)$, $2^3(58)$, $2^2 3^1(6)$, $2^4(15)$, $2^2 5^1(1)$, $2^3 3^1(7)$, $2^5(6)$, $2^4 3^1(2)$, $2^6(1)$, $2^3 3^2(1)$, $2^7(1)$, $2^5 3^1 5^1(1)$, $2^8 3^2(1)$, $2^9 3^1(1)$	639	5928	290	$1(287)$, $2(2)$, $4(1)$	$\mathbb{Z} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4$
10	628	$2^2(571)$, $2^3(31)$, $2^2 3^1(4)$, $2^4(13)$, $2^3 3^1(3)$, $2^5(1)$, $2^6(2)$, $2^3 3^2(1)$, $2^7(1)$, $2^8(1)$	597	6701	348	$1(343)$, $2(4)$, $4(1)$	$\mathbb{Z} \oplus \mathbb{Z}_2^7 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{24}$
11	369	$2^2(320)$, $2^3(25)$, $2^2 3^1(4)$, $2^4(12)$, $2^3 3^1(4)$, $2^5(3)$, $2^6 3^2(1)$	346	4544	248	$1(237)$, $2(7)$, $4(2)$, $8(1)$, $24(1)$	0
12	130	$2^2(103)$, $2^3(9)$, $2^2 3^1(8)$, $2^4(3)$, $2^3 3^1(1)$, $2^5(2)$, $2^2 3^2(1)$, $2^4 3^1(1)$, $2^6(1)$, $2^5 3^1(1)$	120	1787	98	$1(98)$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$
13	31	$2^2(13)$, $2^6 3^1(1)$, $2^3(7)$, $2^2 3^1(4)$, $2^4(1)$, $2^3 3^1(2)$, $2^6(1)$, $2^5 3^2(1)$, $2^9(1)$	22	337	21	$1(19)$, $2(1)$, $8(1)$	$\mathbb{Z}_5$
14	7	$2^3 3^1(2)$, $2^4(2)$, $2^5 3^1 5^1(1)$, $2^2 5^1(1)$, $2^7 3^1(1)$	2	3	1	$5(1)$	$\mathbb{Z}_{128}$
15	2	$2^{10}3^2 5^1(1)$, $2^{11}3^1(1)$	2	2	1	$128(1)$	$\mathbb{Z}$

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