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An explicit spine for the Picard modular group over the Gaussian integers

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Abstract

Let $\Gamma\backslash D$ be an arithmetic quotient of a symmetric space of non-compact type. A spine D_0 is a Γ-equivariant deformation retraction of D with dimension equal to the virtual cohomological dimension of Γ. We explicitly construct a spine for the case of $\Gamma = \mathrm{SU}(2,1;\mathbb{Z}[i])$. The spine is then used to compute the cohomology of $\Gamma\backslash D$ with various local coefficients.

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1. Introduction

Let G be the real points of the $\mathbb{Q}$-rank 1 linear algebraic group $\mathrm{SU}(2,1)$, and let D be the associated non-compact symmetric space. Let Γ be an arithmetic subgroup of the rational points $G(\mathbb{Q})$. Let (E,ρ) be a Γ-module over R. If Γ is torsion-free, the locally symmetric space $\Gamma\backslash D$ is a $K(\Gamma,1)$ since D is contractible, and the group cohomology of Γ is isomorphic to the cohomology of the locally symmetric space, i.e., $H^*(\Gamma,E)\cong H^*(\Gamma\backslash D;\mathbb{E})$, where $\mathbb{E}$ denotes the local system defined by (E,ρ) on $\Gamma\backslash D$. When Γ has torsion, the correct treatment involves the language of orbifolds, but the isomorphism of cohomology is still valid by using a suitable sheaf $\mathbb{E}$ as long as the orders of the torsion elements of Γ are invertible in R.

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The *virtual cohomological dimension* (vcd) of G is the smallest integer p such that cohomology of $\Gamma\backslash D$ vanishes in degrees above p, where $\Gamma \subset G(\mathbb{Q})$ is any torsion-free arithmetic subgroup. Borel and Serre [9] show that the discrepancy between the dimension of D and the $\operatorname{vcd}(G)$ is given by the $\mathbb{Q}$-rank of G, the dimension of a maximal $\mathbb{Q}$-split torus in G. Thus in our case, D is 4-dimensional, and the virtual cohomological dimension of Γ is 3. There is in fact a 3-dimensional deformation retract $D_0 \subset D$ that is invariant under the action of Γ and compact modulo subgroups of Γ [21]. Such spaces are known as *spines*.

Spines have been constructed for many groups [1,8,11,13–15,18,20]. In [3], Ash describes the *well-rounded retract*, a method for constructing a spine for all linear symmetric spaces. Ash and McConnell [7] extend [3] to the Borel–Serre compactification and relate the retraction to a combination of geodesic actions. The well-rounded retract has been used in the computation of cohomology [2,4–6,13,15,17–20].

The well-rounded retract proves the existence and gives a method of explicitly defining spines in linear symmetric spaces. There were no non-linear examples until MacPherson and McConnell [14] constructed a spine in the Siegel upper half-space for the group $\mathrm{Sp}_4(\mathbb{Z})$.

In this paper, we provide another non-linear example by using the method of [21] to compute a spine for $\mathrm{SU}(2,1;\mathbb{Z}[i])$. Sections 2 and 3 set notation and define the exhaustion functions that are used to describe the pieces of the spine. In Section 4, we classify certain of configurations of isotropic line in $\mathbb{C}^3$. We show the spine has the structure of a cell complex with cells related to these configurations in Section 6. Explicit Γ-representatives of cells are fixed, and their stabilizers are computed in Section 8. After subdivision, we obtain a regular cell complex for D_0 on which Γ acts cellularly. In Section 10, we recall some facts about orbifolds and develop machinery to investigate the cohomology of Γ. The results of Section 10 hold in more generality, and may be of independent interest. We apply these methods in Section 11 to $\Gamma = \mathrm{SU}(2,1;\mathbb{Z}[i])$ to compute the cohomology of Γ with coefficients in various Γ-modules.

2. Preliminaries

Let G be the identity component of the real points of the algebraic group $\mathbf{G} = \mathrm{SU}(2,1)$, realized explicitly as

$$\mathrm{SU}(2,1) = \left\{ g \in \mathrm{SL}(3,\mathbb{C}) \;\middle|\; g^* \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix} \right\}.$$

Alternatively, let $\mathcal{Q}$ be the $(2,1)$-quadratic form on $\mathbb{C}^3$ defined by

$$\mathcal{Q}(u,v) = u^* \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix} v.$$

Then G is the group of determinant 1 complex linear transformations of $\mathbb{C}^3$ that preserve $\mathcal{Q}$. Let Γ be the arithmetic subgroup $\Gamma = \mathrm{SU}(2,1) \cap \mathrm{SL}_3(\mathbb{Z}[i])$.

Let θ denote the Cartan involution given by inverse conjugate transpose and let K be the fixed points under θ. Let $D = G/K$ be the associated Riemannian symmetric space of non-compact type. Let $\mathcal{P}$ denote the set of (proper) rational parabolic subgroups of $\mathbf{G}$.

Let $P_0 \subset G$ be the rational parabolic subgroup of upper triangular matrices,

$$P_0 = \left\{ \begin{pmatrix} y\zeta & \beta\zeta^{-2} & \zeta(r + i|\beta|^2/2)/y \\ 0 & \zeta^{-2} & i\bar{\beta}\zeta/y \\ 0 & 0 & \zeta/y \end{pmatrix} \middle| \zeta, \beta \in \mathbb{C},\ |\zeta| = 1,\ r \in \mathbb{R},\ y \in \mathbb{R}_{>0} \right\},$$

and fix subgroups N_0, A_0, and M_0:

$$N_0 = \left\{ \begin{pmatrix} 1 & \beta & r + i|\beta|^2/2 \\ 0 & 1 & i\bar{\beta} \\ 0 & 0 & 1 \end{pmatrix} \middle| \beta \in \mathbb{C},\ r \in \mathbb{R} \right\},$$

$$A_0 = \left\{ \begin{pmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/y \end{pmatrix} \middle| y \in \mathbb{R}_{>0} \right\},$$

$$M_0 = \left\{ \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^{-2} & 0 \\ 0 & 0 & \zeta \end{pmatrix} \middle| \zeta \in \mathbb{C},\ |\zeta| = 1 \right\}.$$

P_0 acts transitively on D, and every point $z \in D$ can be written as $p \cdot x_0$ for some $p \in P_0$. Using Langlands decomposition, there exists $u \in N_0, a \in A_0$, and $m \in M_0$ such that $p = uam$. Since $M_0 \subset K$, z can be written as $ua \cdot x_0$. Denote such a point $z = (y, \beta, r)$.

Zink showed that Γ has class number 1 [22]. Thus $\Gamma\backslash G(\mathbb{Q})/P_0(\mathbb{Q})$ consists of a single point, and all the parabolic subgroups of G are Γ-conjugate. The rational parabolic subgroups of G are parametrized by the maximal isotropic subspaces of $\mathbb{C}^3$ which they stabilize. These are 1-dimensional, and so to each $P \in \mathcal{P}$, there is an associated reduced, isotropic vector $v_P \in \mathbb{Z}[i]^3$. (A vector $(n, p, q)^t \in \mathbb{Z}[i]^3$ is *reduced* if (n, p, q) generate $\mathbb{Z}[i]$ as an ideal.) Similarly, given a reduced, isotropic vector v in $\mathbb{Z}[i]^3$, there is an associated rational parabolic subgroup P_v. Notice, however, that v_P is only well-defined up to scaling by $\mathbb{Z}[i]^* = \{\pm 1, \pm i\}$. Thus, the vectors v and εv will be treated interchangeably for $\varepsilon \in \mathbb{Z}[i]^*$. If $P = {}^{\gamma}Q$ for some $\gamma \in \Gamma$, then $v_P = \gamma v_Q$.

Unless explicitly mentioned otherwise, the vector v_P will be written as $v_P = (n, p, q)^t$. The isotropic condition $\mathcal{Q}(v_P, v_P) = 0$ implies that

$$|p|^2 = 2\,\mathrm{Im}(n\bar{q}). \tag{1}$$

In particular, $q \neq 0$ for $P \neq P_0$. Furthermore, since there are no isotropic 2-planes in $\mathbb{C}^3$,

$$\mathcal{Q}(v_P, v_Q) \neq 0 \quad \text{for } P \neq Q. \tag{2}$$

Because these elements of Γ will be used frequently, set once and for all

$$w = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 1+i & i \\ 0 & 1 & 1+i \\ 0 & 0 & 1 \end{pmatrix}, \quad \check{\sigma} = \begin{pmatrix} 1 & i(1+i) & i \\ 0 & 1 & -i(1+i) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \text{and } \xi = \tau w \tau \sigma w \epsilon^3.$$

Note that $\check{\sigma}$ is contained in the group generated by $\{\epsilon, w, \sigma\}$. In particular, $\check{\sigma} = w\sigma\epsilon w\sigma^{-1}w$.

3. Construction of the spine

In this section we briefly describe the construction of a Γ-invariant, 3-dimensional cell complex which is a deformation retract of D. This construction is described for the general $\mathbb{Q}$-rank 1 case in [21].

We first define an exhaustion function f_P for every rational parabolic subgroup $P \subseteq G$. These exhaustion functions are then used to define a decomposition of D into sets $D(\mathcal{I})$ for $\mathcal{I} \subset \mathcal{P}$.

3.1. Exhaustion functions

Let $z = (y, \beta, r) \in D$ and P a rational parabolic subgroup of G with associated isotropic vector $(n, p, q)^t$. Then the exhaustion function f_P can be written as

$$f_0(z) \equiv f_{P_0}(z) = y, \tag{3}$$

$$f_P(z) = \frac{y}{(|n - \beta p + (i|\beta|^2/2 - r)q|^2 + y^2|p - i\bar{\beta}q|^2 + y^4|q|^2)^{1/2}}. \tag{4}$$

The family of exhaustion functions defined above is Γ-invariant in the sense that

$$f_{\gamma P}(z) = f_P\bigl(\gamma^{-1} \cdot z\bigr) \quad \text{for } \gamma \in \Gamma. \tag{5}$$

3.2. Admissible sets

For a parabolic P, define $D(P) \subset D$ to be the set of $z \in D$ such that $f_P(z) \geqslant f_Q(z)$ for every $Q \in \mathcal{P} \setminus \{P\}$. More generally, for a subset $\mathcal{I} \subseteq \mathcal{P}$,

$$E(\mathcal{I}) = \bigl\{z \in D \mid f_P(z) = f_Q(z) \text{ for every pair } P, Q \in \mathcal{I}\bigr\}, \tag{6}$$

$$D(\mathcal{I}) = \bigcap_{P \in \mathcal{I}} D(P), \tag{7}$$

$$D'(\mathcal{I}) = D(\mathcal{I}) \setminus \bigcup_{\mathcal{I}' \supsetneq \mathcal{I}} D\bigl(\mathcal{I}'\bigr). \tag{8}$$

It follows that $D'(\mathcal{I}) \subseteq D(\mathcal{I}) \subset E(\mathcal{I})$ and $D(\mathcal{I}) = \coprod_{\tilde{\mathcal{I}} \supseteq \mathcal{I}} D'(\tilde{\mathcal{I}})$. Let $f_{\mathcal{I}}$ denote the restriction to $E(\mathcal{I})$ of f_P for $P \in \mathcal{I}$.

Definition 3.1. Let $\mathcal{I} \subseteq \mathcal{P}$, $P \in \mathcal{I}$, and $z \in E(\mathcal{I})$. Then z is called a *first contact for* $\mathcal{I}$ if $f_{\mathcal{I}}(z)$ is a global maximum of $f_{\mathcal{I}}$ on $E(\mathcal{I})$.

Definition 3.2. A subset $\mathcal{I} \subset \mathcal{P}$ is called *admissible* if $D(\mathcal{I})$ is non-empty and *strongly admissible* if $D'(\mathcal{I})$ is non-empty.

Let $D_0 \subset D$ denote the subset

$$D_0 = \coprod_{|\mathcal{I}|>1} D'(\mathcal{I}). \tag{9}$$

The deformation is defined separately on each $D(P)$ for $P \in \mathcal{P}$. For $z \in D(P)$, we use the (negative) gradient flow of f_P to flow z to a point on D_0. This corresponds to using the geodesic action [9] of A_P on z [21].

Let f_{D_0} denote the function on D_0 given by

$$f_{D_0}(z) = f_{\mathcal{I}}(z) \quad \text{for } z \in D(\mathcal{I}). \tag{10}$$

3.3. First contact points

Given the explicit description of the exhaustion functions in coordinates, one can readily describe the set $E(\{P_0, P\})$. Writing $z = (y, \beta, r)$ and using (3),

$$\left(\frac{f_0(z)}{f_P(z)}\right)^2 = \left|n - \beta p + \left(i|\beta|^2/2 - r\right)q\right|^2 + y^2\left|p - i\overline{\beta}q\right|^2 + y^4|q|^2. \tag{11}$$

Proposition 3.3. *Let P be a rational parabolic subgroup of G with associated isotropic vector $(n, p, q)^t$. Every $z = (y, \beta, r) \in E(\{P_0, P\})$ satisfies*

$$y^2 = -\frac{1}{2}\left|\frac{p}{q} - i\bar{\beta}\right|^2 + \sqrt{\frac{1}{|q|^2} - \left(\mathrm{Re}\left(\frac{n - \beta p}{q} - r\right)\right)^2}.$$

Proof. Note that for $z \in E(\{P_0, P\})$, $(\frac{f_0(z)}{f_P(z)})^2 = 1$. Then the result follows from (11) using the quadratic formula to solve for y^2, and simplifying the result using the isotropic condition (1). □

Proposition 3.3 allows us to easily calculate the first contact point for $\{P_0, P\}$. The Γ-invariance of the exhaustion functions allows us to translate this for general $\{P, Q\}$.

Proposition 3.4. *Let $\mathcal{I} = \{P, Q\} \subset \mathcal{P}$. Let z be a first contact point for $\mathcal{I}$. Then*

$$f_P(z) = f_Q(z) = \frac{1}{\sqrt{|\mathcal{Q}(v_P, v_Q)|}}.$$

In particular, the first contact for $\{P_0, P\}$ is $z = (1/\sqrt{|q|}, i(\overline{p/q}), \mathrm{Re}(n/q))$.

4. Configurations of vectors

Definition 4.1. Let $\mathcal{J}$ be a subset of vectors in $\mathbb{C}^3$. Then $\mathcal{J}$ is said to be *c-bounded* if

$$\left|\mathcal{Q}(u, v)\right|^2 \leqslant c \quad \text{for every } u \text{ and } v \text{ in } \mathcal{J}.$$

Set once and for all the following sets $\mathcal{J}_j^i$ of isotropic vectors in $\mathbb{C}^3$.

$$\mathcal{J}_1^2 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \qquad \mathcal{J}_2^2 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}i\\1+i\\1+i\end{pmatrix}\right\},$$

$$\mathcal{J}_1^3 = \mathcal{J}_1^2 \cup \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{J}_2^3 = \mathcal{J}_1^2 \cup \left\{ \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{J}_3^3 = \mathcal{J}_1^2 \cup \left\{ \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{J}_1^4 = \mathcal{J}_1^3 \cup \left\{ \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{J}_2^4 = \mathcal{J}_3^3 \cup \left\{ \begin{pmatrix} -1 \\ -1+i \\ 1+i \end{pmatrix} \right\},$$

$$\mathcal{J}^5 = \mathcal{J}_1^4 \cup \left\{ \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{J}^8 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1+i \\ 1+i \end{pmatrix}, \begin{pmatrix} -1+i \\ 1+i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1-i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1+i \\ 1+i \end{pmatrix}, \begin{pmatrix} 2i \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 2 \\ 2 \end{pmatrix} \right\}.$$

One easily checks that these sets are not Γ-conjugate, and $\mathcal{J}^8$ is 4-bounded, while the other sets are 2-bounded.

Proposition 4.2. *A* 2*-bounded set of reduced, integral, isotropic vectors is Γ-conjugate to exactly one of $\mathcal{J}_1^2$, $\mathcal{J}_2^2$, $\mathcal{J}_1^3$, $\mathcal{J}_2^3$, $\mathcal{J}_3^3$, $\mathcal{J}_1^4$, $\mathcal{J}_2^4$, or $\mathcal{J}^5$.*

Proof. Let $\mathcal{J}$ denote such a subset of order 2. Since G has class number 1, we can assume that one of the vectors of $\mathcal{J}$ is $(1,0,0)^t$. Let $v = (n, p, q)^t$ be the other vector of $\mathcal{J}$. Let σ, $\check{\sigma}$, τ, and ϵ be defined as in Section 2. These elements preserve $(1,0,0)^t$ (up to scaling by $\mathbb{Z}[i]^*$). By applying powers of σ and $\check{\sigma}$ to v, we can add any $\mathbb{Z}$-linear combination of $(1+i)q$ and $(1+i)iq$ to p to force it into the square in the complex plane with vertices q, iq, $-q$, and $-iq$. By applying powers of $-i\epsilon$ to v, one can force p to lie in the triangle with q, iq, and 0 as vertices while leaving q fixed. Then by applying powers of τ to v, one can add a $\mathbb{Z}$-scalar multiple of q to n so that n now has the form $dq + ciq$, where $-\frac{1}{2} < d \leqslant \frac{1}{2}$ for some $c \in \mathbb{R}$. Since v is isotropic, $|p|^2 = 2\,\mathrm{Im}(n\bar{q}) = 2c|q|^2$. Since $\mathcal{J}$ is 2-bounded, $|q|^2 \leqslant 2$, so that in particular, $\mathcal{J}$ is Γ-equivalent to $\mathcal{J}_1^2$ or $\mathcal{J}_2^2$.

For $|\mathcal{J}| > 2$, we can arrange that $\mathcal{J} \supset \mathcal{J}_1^2$ or $\mathcal{J} \supset \mathcal{J}_2^2$. The 2-bounded condition in each of these two cases has only finitely many solutions. The proposition then follows from listing the solutions and identifying Γ-conjugate sets. $\square$

5. Reduction theory

Proposition 5.1. *Every point $z \in D$ is conjugate under Γ_{P_0} to a point (y, β, r), where $-\frac{1}{2} < r \leqslant \frac{1}{2}$ and β lies in the square in the complex plane with vertices* 0, $\frac{1+i}{2}$, i, *and* $\frac{-1+i}{2}$.

Proof. Consider the elements $\{\sigma,\ \check{\sigma},\ \tau,\ \epsilon\} \subset \Gamma_{P_0}$ defined in Section 2. The action of $\{\sigma,\ \check{\sigma},\ \tau,\ \epsilon\}$ leaves $D(P_0) \cap D_0$ stable, and is given explicitly by

$$\begin{aligned}
\sigma \cdot (y, \beta, r) &= \big(y, \beta + (1+i), r - \mathrm{Re}(\beta) + \mathrm{Im}(\beta)\big),\\
\check{\sigma} \cdot (y, \beta, r) &= \big(y, \beta + i(1+i), r - \mathrm{Re}(\beta) - \mathrm{Im}(\beta)\big),\\
\tau \cdot (y, \beta, r) &= (y, \beta, r+1), \quad \text{and}\\
\epsilon \cdot (y, \beta, r) &= (y, -i\beta, r).
\end{aligned} \tag{12}$$

Thus, by applying powers of σ and $\check{\sigma}$, β can be put in the square in the complex plane with vertices 1, i, -1, and $-i$. By applying a power of ϵ, β can be put in the square in the complex plane with vertices 0, $\frac{1+i}{2}$, i, and $\frac{-1+i}{2}$. Then by applying powers of τ, it can be arranged that $-\frac{1}{2} < r \leqslant \frac{1}{2}$ without changing the value of β. $\square$

Proposition 5.2. *For each $z \in D_0$,*

$$\frac{1}{\sqrt[4]{5}} < f_{D_0}(z) \leqslant 1.$$

Proof. Since $f_{D_0}(z) = f_{D_0}(\gamma \cdot z)$ for $z \in D_0$ and $\gamma \in \Gamma$, and every point in D_0 is Γ-conjugate to a point of $D(P_0)$, it suffices to determine the range of f_{D_0} on $D(P_0) \cap D_0$. In fact, it suffices to determine the range on a subset $F \subseteq D(P_0) \cap D_0$, provided the Γ-translates of F cover $D(P_0) \cap D_0$. The action of $\{\sigma, \check{\sigma}, \tau, \epsilon\}$ leaves $D(P_0) \cap D_0$ stable, so it suffices to determine the range on $F = D(P_0) \cap D_0 \cap T$, where T is the strip in D defined in Proposition 5.1. By construction, $f_{D_0}(z) = \max_{P \in \mathcal{P}}\{f_P(z)\}$ and Proposition 3.4 shows that $f_{D_0}(z) \leqslant 1$. Thus it suffices to show that

$$\min_{z \in F}\{f_0(z)\} > \frac{1}{\sqrt[4]{5}}. \tag{13}$$

Note that on F, $f_0(z) \geqslant f_P(z)$ for all $P \in \mathcal{P}$. By Proposition 3.3, this is equivalent to the condition that

$$f_0(z)^2 = y^2 \geqslant -\frac{1}{2}\left|\frac{p}{q} - i\overline{\beta}\right|^2 + \sqrt{\frac{1}{|q|^2} - \left(\mathrm{Re}\left(\frac{n - \beta p}{q} - r\right)\right)^2}. \tag{14}$$

Consider the rational parabolic subgroups ${}^{w}P_0$, P, and Q corresponding to the vectors $v_w = (0,0,1)^t$, $v_P = (i, 1+i, 1+i)^t$ and $v_Q = (-1, 1+i, 1+i)^t$. Divide F into the following regions:

$$\begin{aligned}
F_P &= \left\{x = (y, \beta, r) \in F \;\middle|\; |\beta| > \frac{9}{10},\ \frac{3}{10} < r \leqslant \frac{1}{2}\right\},\\
F_Q &= \left\{x = (y, \beta, r) \in F \;\middle|\; |\beta| > \frac{9}{10},\ -\frac{1}{2} \leqslant r < -\frac{3}{10}\right\},\\
F_w &= F \setminus \{F_1 \cup F_2\}.
\end{aligned}$$

Since β is constrained to lie in the square indicated in Proposition 5.1, one may calculate that on F_P and F_Q

$$-\frac{(10 - \sqrt{62})}{20} < \mathrm{Re}(\beta) < \frac{10 - \sqrt{62}}{20}, \tag{15}$$

$$|\beta - i|^2 < \frac{81 - 10\sqrt{62}}{100}. \tag{16}$$

Comparing f_0 to f_P on F_P, f_0 to f_Q on F_Q, and f_0 to f_w on F_w, (14) gives the desired bound on each piece. Since $F = F_P \cup F_Q \cup F_w$, this proves the result. $\square$

6. Admissible sets

Since v_P and v_Q are integral vectors, $\mathcal{Q}(v_P, v_Q) \in \mathbb{Z}[i]$. Propositions 3.4 and 5.2 imply the following.

Proposition 6.1. *Let $\mathcal{I}$ be an admissible set. Then*

$$\left|\mathcal{Q}(v_P, v_Q)\right|^2 \leqslant 4 \quad \textit{for every } P \textit{ and } Q \textit{ in } \mathcal{I}.$$

In particular, the set of vectors associated to an admissible set is 4*-bounded.*

Proposition 6.2. *Let $\mathcal{I} = \{P, Q\}$ be an admissible subset of $\mathcal{P}$. If $|\mathcal{Q}(v_P, v_Q)|^2 = 4$, then there exists a strongly admissible set $\tilde{\mathcal{I}} \subset \mathcal{P}$ of order* 8 *such that $D(\mathcal{I}) = D(\tilde{\mathcal{I}}) = \{z\}$, where z is the first contact for $\mathcal{I}$. Let $\tilde{\mathcal{J}}$ be the set of isotropic vectors associated to $\tilde{\mathcal{I}}$. Then $\tilde{\mathcal{J}}$ is* 4*-bounded and is Γ-equivalent to $\mathcal{J}^8$. A set of isotropic vectors that is not* 2*-bounded, but is associated to a strongly admissible set, is Γ-equivalent to $\mathcal{J}^8$.*

Proof. Suppose $\mathcal{I} = \{P, Q\}$ is admissible and $|\mathcal{Q}(v_P, v_Q)|^2 = 4$. Since G has class number 1, $\mathcal{I}$ is Γ-conjugate to a set of the form $\{P_0, P\}$. By Γ-action, one only needs to consider the cases when $v_P = (i, 2, 2)^t$ and $v_P = (1, 0, 2)^t$.

In the case that $v = (1, 0, 2)^t$ we claim that $\{P_0, P_v\}$ is not admissible, and hence $D(\{P_0, P_v\})$ is empty. To see this, it suffices to show that $f_Q(x) > f_0(x)$ on $E(\{P_0, P_v\})$ for some Q. Proposition 3.3 implies that $E(\{P_0, P_v\})$ is the set where

$$y^2 = -\frac{|\beta|^2}{2} + \sqrt{r - r^2}. \tag{17}$$

In particular, $0 < r < 1$. Explicit computation with (3) and (17) shows that on $E(\{P_0, P_v\})$, for $Q = {}^{w}P_0$, $f_Q(z) = \frac{f_0(z)}{\sqrt{r}} > f_0(z)$.

Next, consider the case $v = (i, 2, 2)^t$. Proposition 3.3 implies that $E(\{P_0, P_v\})$ is the set where

$$y^2 = -\frac{|\beta - i|^2}{2} + \frac{1}{2}\sqrt{1 - 4\big(\mathrm{Re}(\beta) + r\big)^2}. \tag{18}$$

Consider the rational parabolic subgroups P_1 and P_2 with associated isotropic vectors $(i, 1+i, 1+i)^t$ and $(-1, 1+i, 1+i)^t$ respectively. Explicit computation with (3) and (18) shows that on $E(\{P_0, P_v\})$,

$$f_{P_1}(x) = \frac{f_0(x)}{1 - 2r - \mathrm{Re}(\beta)} \quad \text{and} \quad f_{P_2}(x) = \frac{f_0(x)}{1 + 2r + \mathrm{Re}(\beta)}.$$

Thus on $D(\{P_0, P_v\})$, where $f_{P_i}(z) \leqslant f_0(z)$, we see that $2r = -\mathrm{Re}(\beta)$, and so

$$D(\mathcal{I}) = D\big(\{P_0, P_v, P_1, P_2\}\big).$$

In particular, note that $|\beta - i|^2 < \sqrt{1 - 4r^2} \leqslant 1$.

Consider the rational parabolic subgroups Q_1, Q_2, Q_3, and Q_4 with associated isotropic vectors $(0,0,1)^t$, $(1+i,1-i,1)^t$, $(2i,2,1)^t$, and $(-1+i,1+i,1)^t$ respectively. Explicit computation shows that on S,

$$f_{Q_1}(x)=\frac{f_0(x)}{|\beta|^2},\qquad f_{Q_2}(x)=\frac{f_0(x)}{|\beta-(-1+i)|^2},$$
$$f_{Q_3}(x)=\frac{f_0(x)}{|\beta-2i|^2},\qquad f_{Q_4}(x)=\frac{f_0(x)}{|\beta-(1+i)|^2}.$$

Since $|\beta-i|^2<1$ and $f_{Q_j}(x)\leqslant f_0(x)$ on $D(\mathcal{I})$, we see that β is forced to equal i. Thus

$$D(\mathcal{I})=D\big(\{P_0,P_v,P_1,P_2,Q_1,Q_2,Q_3,Q_4\}\big)=\left\{\left(\frac{1}{\sqrt{2}},i,0\right)\right\}.$$

One checks that $\mathcal{J}^8$ is not properly contained in any 4-bounded set to complete the proof. □

Corollary 6.3. *The set of isotropic vectors associated to a strongly admissible set of order less than eight is* 2*-bounded. Furthermore, there are no strongly admissible sets of order greater than eight.*

Proof. Propositions 6.1 and 6.2 imply the first statement. The second follows from explicit calculation that shows there are no 4-bounded sets that properly contain $\mathcal{J}^8$. □

Proposition 6.4. *Let $\mathcal{I}^i_j$ denote the set of rational parabolic subgroups associated $\mathcal{J}^i_j$ as defined in Section* 4*. Then*

$$\mathcal{I}^2_1=\{P_0,{}^{w}P_0\},\qquad \mathcal{I}^2_2=\{P_0,{}^{\xi}P_0\},$$
$$\mathcal{I}^3_1=\mathcal{I}^2_1\cup\{{}^{\tau w}P_0\},\qquad \mathcal{I}^3_2=\mathcal{I}^2_1\cup\{{}^{\sigma w}P_0\},\qquad \mathcal{I}^3_3=\mathcal{I}^2_1\cup\{{}^{\tau\sigma w}P_0\},$$
$$\mathcal{I}^4_1=\mathcal{I}^3_1\cup\{{}^{\sigma w}P_0\},\qquad \mathcal{I}^4_2=\mathcal{I}^3_3\cup\{{}^{w^{-1}\tau\check{\sigma}w}P_0\},$$
$$\mathcal{I}^5=\mathcal{I}^4_1\cup\{{}^{\tau\sigma w}P_0\},$$
$$\mathcal{I}^8=\{P_0,{}^{w}P_0,{}^{w\tau\sigma w}P_0,{}^{\tau^{-1}\sigma w}P_0,{}^{\tau\check{\sigma}w}P_0,{}^{\tau w\tau\sigma w}P_0,{}^{\tau^2\check{\sigma}\sigma w}P_0,{}^{\epsilon w\xi^4 w}P_0\}.$$

Using (3) and Propositions 3.4 and 6.4, one calculates the following first contact points $z(\mathcal{I})$.

Proposition 6.5. *Let $\phi=\frac{1+\sqrt{5}}{2}$. Then*

$$z(\mathcal{I}^2_1)=(1,0,0),\qquad z(\mathcal{I}^2_2)=\left(\frac{1}{\sqrt[4]{2}},i,\frac{1}{2}\right),$$
$$z(\mathcal{I}^3_1)=\left(\sqrt[4]{\frac{3}{4}},0,\frac{1}{2}\right),\qquad z(\mathcal{I}^3_2)=\left(\frac{\sqrt{3}}{2},\frac{1+i}{2},0\right),$$
$$z(\mathcal{I}^3_3)=\left(\frac{1}{\sqrt{2}}\sqrt{\phi^2\sqrt{\phi}-2},\frac{1}{2}\big(\phi^2-\sqrt{\phi}+i(1-\phi+\sqrt{\phi})\big),\frac{1}{2}(1-\phi+\sqrt{\phi})\right),$$

$$z(\mathcal{I}_1^4)=\left(\frac{\sqrt{-3+\sqrt{3}+\sqrt{2}+\sqrt{6}}}{2},\frac{1+\sqrt{3}-\sqrt{2}}{4}(1+\sqrt{3}i),\frac{1}{2}\right),$$
$$z(\mathcal{I}_2^4)=\left(\sqrt{\phi-1},\frac{3+i}{2}(1-\sqrt{2}),0\right),$$
$$z(\mathcal{I}^5)=\left(\frac{\sqrt{-1+2\sqrt{3}}}{2},\frac{1+i}{2},\frac{1}{2}\right),\qquad z(\mathcal{I}^8)=\left(\frac{1}{\sqrt{2}},i,0\right).$$

Theorem 6.6. *Up to Γ-conjugacy, the strongly admissible sets are exactly*

$$\mathcal{I}_1^2,\ \mathcal{I}_2^2,\ \mathcal{I}_1^3,\ \mathcal{I}_2^3,\ \mathcal{I}_3^3,\ \mathcal{I}_1^4,\ \mathcal{I}_2^4,\ \mathcal{I}^5,\ \mathcal{I}^8.$$

Proof. Propositions 4.2 and 6.2 and Corollary 6.3 imply that every strongly admissible set must be one of the ones listed. Thus, it suffices to show that for each set $\mathcal{I}$ listed, $D'(\mathcal{I})$ is non-empty. In particular, it suffices to show that $D'(\mathcal{I})$ contains its first contact point. One uses (3) and Proposition 6.5 to show that for $P\in\mathcal{P}\setminus\mathcal{I}$, $f_P(z(\mathcal{I}))<f_{\mathcal{I}}(z(\mathcal{I}))$. For example, $z(\mathcal{I}_1^2)=(1,0,0)$ and

$$f_P(1,0,0)=\frac{1}{\sqrt{|n|^2+|p|^2+|q|^2}}<1\quad\text{for all } P\in\mathcal{P}\setminus\mathcal{I}_1^2.$$

The other cases follow similarly. □

7. Pieces of the spine

Theorem 7.1. *Let (y,β,r) denote a point in D. Then*

(i) $E(\mathcal{I}_1^2)=\{y^2=-\frac{|\beta|^2}{2}+\sqrt{1-r^2}\}$,
(ii) $E(\mathcal{I}_2^2)=\{y^2=-\frac{1}{2}|1-i\bar\beta|^2+\sqrt{\frac{1}{2}-(\frac{1}{2}-\mathrm{Re}(\beta)-r)^2}\}$,
(iii) $D(\mathcal{I}_1^2)\subseteq E(\mathcal{I}_1^2)\cap\{\frac{1}{\sqrt5}<y^2\leqslant 1,\ -\frac12<r\leqslant\frac12,\ |\beta|^2<-\frac{2}{\sqrt5}+2\}$,
(iv) $D(\mathcal{I}_2^2)\subseteq E(\mathcal{I}_2^2)\cap\{0\leqslant \mathrm{Re}(\beta)+r<\sqrt{\frac{3}{10}}+\frac12\}$,
(v) $D(\mathcal{I}_1^3)=D(\mathcal{I}_1^2)\cap\{r=\frac12\}$,
(vi) $D(\mathcal{I}_2^3)=D(\mathcal{I}_1^2)\cap\{y^2=-\frac12|1+i-i\bar\beta|^2+\sqrt{1-(\mathrm{Re}(\beta(1+i))+r)^2}\}$,
(vii) $D(\mathcal{I}_3^3)=D(\mathcal{I}_1^2)\cap\{y^2=-\frac12|1+i-i\bar\beta|^2+\sqrt{1-(1-r-\mathrm{Re}(\beta(1+i)))^2}\}$,
(viii) $D(\mathcal{I}_1^4)=D(\mathcal{I}_1^2)\cap\{r=\frac12,\ |\beta-\frac{1+\sqrt3}{4}(1+\sqrt3 i)|^2=\frac12\}$,
(ix) $D(\mathcal{I}_2^4)=D(\mathcal{I}_1^2)\cap\{r=0,\ |\beta-\frac{3+i}{2}|^2=\frac12\}$,
(x) $D(\mathcal{I}^5)=\{(\frac{\sqrt{-1+2\sqrt3}}{2},\frac{1+i}{2},\frac12)\}$,
(xi) $D(\mathcal{I}^8)=\{(\frac{1}{\sqrt2},i,0)\}$.

Proof. Proposition 3.3 implies (i) and (ii). Similarly, (v)–(xi) follow from computer calculations and repeated uses of Proposition 3.3.

To show the bounds in (iii), let $P={}^{w}P_0$ and consider the rational parabolic subgroups $Q={}^{\tau}P_0$ and $R={}^{\tau^{-1}}P_0$ with associated isotropic vector $v_Q=(1,0,1)^t$ and $v_R=(-1,0,1)^t$ respectively. Then (3) implies that

$$f_P(z) = \frac{y}{(|i|\beta|^2/2 - r|^2 + y^2|\beta|^2 + y^4)^{1/2}}, \tag{19}$$

$$f_Q(z) = \frac{y}{(|1 + i|\beta|^2/2 - r|^2 + y^2|\beta|^2 + y^4)^{1/2}}, \tag{20}$$

$$f_R(z) = \frac{y}{(|-1 + i|\beta|^2/2 - r|^2 + y^2|\beta|^2 + y^4)^{1/2}}. \tag{21}$$

For $z \in D(\mathcal{I}_1^2)$, $f_P(z) \geqslant f_Q(z)$ and $f_{{}^w P_0}(z) \geqslant f_Q(z)$ and hence (19), (20), and (21) imply that $-\frac{1}{2} \leqslant r \leqslant \frac{1}{2}$. Since $f_0(z) = y$, Proposition 5.2 implies that $\frac{1}{\sqrt{5}} < y^2 \leqslant 1$. It follows that $|\beta|^2 < -\frac{2}{\sqrt{5}} + 2$ on $D(\mathcal{I}_1^2)$.

To show (iv), note that $y^2 > \frac{1}{\sqrt{5}}$ on $D(\mathcal{I}_2^2)$ by Proposition 5.2, and hence

$$\operatorname{Re}(\beta) + r < \sqrt{\frac{3}{10}} + \frac{1}{2}.$$

Let $P = {}^{\xi} P_0$ and consider the rational parabolic subgroup Q associated to the isotropic vector $v_Q = (-1, 1+i, 1+i)^t$. For $z \in D(\mathcal{I}_2^2)$, $f_P(z) = y$ and $f_P(z) \geqslant f_Q(z)$ and hence (3) implies that

$$\bigl|i - \beta(1+i) + \bigl(i|\beta|^2/2 - r\bigr)(1+i)\bigr|^2 \leqslant \bigl|-1 - \beta(1+i) + \bigl(i|\beta|^2/2 - r\bigr)(1+i)\bigr|^2. \tag{22}$$

Thus (22) and (3) imply $0 \leqslant \operatorname{Re}(\beta) + r$. $\square$

From the explicit description of representatives of Γ-conjugacy classes of admissible sets given above and the exhaustion functions given in (3), one can calculate a strict lower bound of f_{D_0}.

Proposition 7.2. *For each $z \in D_0$,*

$$\frac{1}{\sqrt{2}} \leqslant f_{D_0}(z) \leqslant 1.$$

8. Stabilizers

Proposition 8.1. *The stabilizer in Γ of $D'(\mathcal{I}_1^2)$ is $\{e, \epsilon, \epsilon^2, \epsilon^3, w, \epsilon w, \epsilon^2 w, \epsilon^3 w\}$ and is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ via the morphism which sends ϵ to the generator of $\mathbb{Z}/4\mathbb{Z}$ and ϵw to the generator of $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Note that $\operatorname{Stab}_\Gamma(D'(\mathcal{I}_1^2)) = \operatorname{Stab}_\Gamma(\mathcal{I}_1^2)$. One can calculate that $w^2 \in \Gamma_{P_0}$ and hence $w \in \operatorname{Stab}_\Gamma(\mathcal{I}_1^2)$ and acts non-trivially on the ordered pair $(P_0, {}^w P_0)$. Thus, $\operatorname{Stab}_\Gamma(\mathcal{I}_1^2) = L \cup L \cdot w$, where $\gamma \in L$ if and only if ${}^\gamma P_0 = P_0$ and ${}^{\gamma w} P_0 = {}^w P_0$. Since a parabolic subgroup is its own normalizer, one computes that

$$L = \Gamma \cap P_0 \cap {}^w P_0 = \bigl\{e, \epsilon, \epsilon^2, \epsilon^3\bigr\},$$

and hence,

$$\mathrm{Stab}_\Gamma\big(D'\big(\mathcal{I}_1^2\big)\big) = \big\{e, \epsilon, \epsilon^2, \epsilon^3, w, \epsilon w, \epsilon^2 w, \epsilon^3 w\big\}.$$

It is easily checked that the map given in the proposition is an isomorphism. □

Proposition 8.2. *The stabilizer in Γ of $D'(\mathcal{I}_2^2)$ is the cyclic group of order eight generated by ξ.*

Proof. Note that $\mathrm{Stab}_\Gamma(D'(\mathcal{I}_2^2)) = \mathrm{Stab}_\Gamma(\mathcal{I}_2^2)$. Recall that $\mathcal{I}_2^2 = \{P_0, {}^\xi P_0\}$. One can calculate that $\xi^2 = \epsilon\tau\sigma^{-1} \in \Gamma_{P_0}$ and hence ξ is in $\mathrm{Stab}_\Gamma(\mathcal{I}_2^2)$ and acts non-trivially on the ordered pair $(P_0, {}^\xi P_0)$. Thus, $\mathrm{Stab}_\Gamma(\mathcal{I}_2^2) = L \cup L \cdot \xi$, where $\gamma \in L$ if and only if ${}^\gamma P_0 = P_0$ and ${}^{\gamma\xi} P_0 = {}^\xi P_0$. Since a parabolic subgroup is its own normalizer, one computes that

$$L = \Gamma \cap P_0 \cap {}^\xi P_0 = \big\{e, \xi^2, \xi^4, \xi^6\big\},$$

and hence $\mathrm{Stab}_\Gamma(D'(\mathcal{I}_2^2))$ is the cyclic group of order eight generated by ξ. □

Proposition 8.3. *The stabilizer in Γ of $D'(\mathcal{I}_1^3)$ is the cyclic group of order twelve generated by $\tau\epsilon w$.*

Proof. Note that $\mathrm{Stab}_\Gamma(D'(\mathcal{I}_1^3)) = \mathrm{Stab}_\Gamma(\mathcal{I}_1^3)$. One can easily check that

$${}^\tau\big\{P_0, {}^w P_0\big\} = \big\{P_0, {}^{\tau w} P_0\big\} \quad \text{and} \quad {}^{w\tau^{-1}w}\big\{P_0, {}^w P_0\big\} = \big\{{}^w P_0, {}^{\tau w} P_0\big\}.$$

Thus $\mathrm{Stab}_\Gamma(\mathcal{I}_1^3) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned}
\Gamma_1 &= \big\{\gamma \in \mathrm{Stab}_\Gamma\big(\mathcal{I}_1^2\big) \bigm| {}^{\gamma\tau w} P_0 = {}^{\tau w} P_0\big\},\\
\Gamma_2 &= \big\{\gamma \in \tau \cdot \mathrm{Stab}_\Gamma\big(\mathcal{I}_1^2\big) \bigm| {}^{\gamma\tau w} P_0 = {}^{w} P_0\big\}, \quad \text{and}\\
\Gamma_3 &= \big\{\gamma \in w\tau^{-1}w \cdot \mathrm{Stab}_\Gamma\big(\mathcal{I}_1^2\big) \bigm| {}^{\gamma\tau w} P_0 = P_0\big\}.
\end{aligned}$$

One calculates using Proposition 8.1 that

$$\begin{aligned}
\Gamma_1 &= \big\{e, \epsilon, \epsilon^2, \epsilon^3\big\},\\
\Gamma_2 &= \big\{\tau w, \tau\epsilon w, \tau\epsilon^2 w, \tau\epsilon^3 w\big\}, \quad \text{and}\\
\Gamma_3 &= \big\{w\tau^{-1}, w\tau^{-1}\epsilon, w\tau^{-1}\epsilon^2, w\tau^{-1}\epsilon^3\big\}.
\end{aligned}$$

Explicit matrix multiplication shows that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is the cyclic group of order twelve generated by $\tau\epsilon w$. □

Proposition 8.4. *The stabilizer in Γ of $D'(\mathcal{I}_2^3)$ is*

$$\mathrm{Stab}_\Gamma\big(D'\big(\mathcal{I}_2^3\big)\big) = \big\{e, \epsilon w, w\check{\sigma}^{-1}w, w\check{\sigma}^{-1}\epsilon^3, \sigma\epsilon^3 w, \sigma\epsilon^2\big\}$$

and is isomorphic to $\mathfrak{S}_3$.

Proof. Note that $\mathrm{Stab}_\Gamma(D'(\mathcal{I}_2^3)) = \mathrm{Stab}_\Gamma(\mathcal{I}_2^3)$. Fix an ordering of $\mathcal{I}_2^3$. Since a parabolic subgroup is its own normalizer, the group which preserves the ordering is $\Gamma \cap P_0 \cap {}^w P_0 \cap {}^{\sigma w} P_0$. The proof of Proposition 8.1 shows that $\Gamma \cap P_0 \cap {}^w P_0 = \langle \epsilon \rangle$ and one can easily check that $\langle \epsilon \rangle \cap {}^{\sigma w} P_0 = \{e\}$. Thus $\mathrm{Stab}_\Gamma(\mathcal{I}_2^3)$ is isomorphic to a subgroup of $\mathfrak{S}_3$. One checks that $\{e, \epsilon w, w\check{\sigma}^{-1}w, w\check{\sigma}^{-1}\epsilon^3, \sigma\epsilon^3 w, \sigma\epsilon^2\}$ is a set of six distinct elements in $\mathrm{Stab}_\Gamma(\mathcal{I}_2^3)$. Thus this set is exactly $\mathrm{Stab}_\Gamma(\mathcal{I}_2^3)$ and is isomorphic to $\mathfrak{S}_3$. $\square$

Proposition 8.5. *The stabilizer in Γ of $D'(\mathcal{I}_3^3)$ is trivial.*

Proof. Since $|\mathcal{Q}(v_0, v_w)|^2 = 1$, $|\mathcal{Q}(v_0, v_{\tau\sigma w})|^2 = 1$, and $|\mathcal{Q}(v_w, v_{\tau\sigma w})|^2 = 2$, if $\gamma \in \mathrm{Stab}_\Gamma(\mathcal{I}_3^3)$, then ${}^\gamma\mathcal{I}_1^2 = \mathcal{I}_1^2$ or ${}^\gamma\mathcal{I}_1^2 = \{P_0, {}^{\tau\sigma w}P_0\}$. Thus $\mathrm{Stab}_\Gamma(\mathcal{I}_3^3) = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \left\{\gamma \in \mathrm{Stab}_\Gamma\left(\mathcal{I}_1^2\right) \,\middle|\, {}^{\gamma\tau\sigma w}P_0 = {}^{\tau\sigma w}P_0\right\} \quad \text{and}$$
$$\Gamma_2 = \left\{\gamma \in \tau\sigma \cdot \mathrm{Stab}_\Gamma\left(\mathcal{I}_1^2\right) \,\middle|\, {}^{\gamma\tau\sigma w}P_0 = {}^{w}P_0\right\}.$$

One checks using Proposition 8.1 that $\Gamma_1 = \{e\}$ and $\Gamma_2 = \emptyset$. $\square$

Proposition 8.6. *The stabilizer in Γ of $D'(\mathcal{I}_1^4)$ is trivial.*

Proof. Note that $\mathrm{Stab}_\Gamma(D'(\mathcal{I}_1^4)) = \mathrm{Stab}_\Gamma(\mathcal{I}_1^4)$. The set of isotropic vectors associated to $\mathcal{I}_1^3 \subset \mathcal{I}_1^4$ span a two-dimensional subspace of $\mathbb{C}^3$ while the isotropic vectors associated to any other order three subset of $\mathcal{I}_1^4$ span all of $\mathbb{C}^3$. Thus if $\gamma \in \mathrm{Stab}_\Gamma(\mathcal{I}_1^4)$ then ${}^\gamma\mathcal{I}_1^3 = \mathcal{I}_1^3$ and ${}^{\gamma\sigma w}P_0 = {}^{\sigma w}P_0$ Therefore one calculates from Proposition 8.3 that $\mathrm{Stab}_\Gamma(\mathcal{I}_1^4) = \mathrm{Stab}_\Gamma(\mathcal{I}_1^3) \cap {}^{\sigma w}P_0 = \{e\}$. $\square$

Proposition 8.7. *The stabilizer in Γ of $D'(\mathcal{I}_2^4)$ is cyclic of order two generated by ϵw.*

Proof. Explicit computation show that

$$ {}^{\epsilon w}\mathcal{I}_3^3 = \left\{P_0, {}^{w}P_0, {}^{w^{-1}\tau\check{\sigma}w}P_0\right\},$$
$$ {}^{\sigma\tau w\tau\sigma^{-1}}\mathcal{I}_3^3 = \left\{P_0, {}^{\tau\sigma w}P_0, {}^{w^{-1}\tau\check{\sigma}w}P_0\right\}, \quad \text{and}$$
$$ {}^{\sigma\tau w\tau\sigma^{-1}w}\mathcal{I}_3^3 = \left\{{}^{w}P_0, {}^{\tau\sigma w}P_0, {}^{w^{-1}\tau\check{\sigma}w}P_0\right\}.$$

Then $\mathrm{Stab}_\Gamma(\mathcal{I}_2^4) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\Gamma_1 = \left\{\gamma \in \mathrm{Stab}_\Gamma\left(\mathcal{I}_3^3\right) \,\middle|\, {}^{\gamma w^{-1}\tau\check{\sigma}w}P_0 = {}^{w^{-1}\tau\check{\sigma}w}P_0\right\},$$
$$\Gamma_2 = \left\{\gamma \in \epsilon w \cdot \mathrm{Stab}_\Gamma\left(\mathcal{I}_3^3\right) \,\middle|\, {}^{\gamma w^{-1}\tau\check{\sigma}w}P_0 = {}^{\tau\sigma w}P_0\right\},$$
$$\Gamma_3 = \left\{\gamma \in \sigma\tau w\tau\sigma^{-1} \cdot \mathrm{Stab}_\Gamma\left(\mathcal{I}_3^3\right) \,\middle|\, {}^{\gamma w^{-1}\tau\check{\sigma}w}P_0 = {}^{w}P_0\right\}, \quad \text{and}$$
$$\Gamma_4 = \left\{\gamma \in \sigma\tau w\tau\sigma^{-1}w \cdot \mathrm{Stab}_\Gamma\left(\mathcal{I}_3^3\right) \,\middle|\, {}^{\gamma w^{-1}\tau\check{\sigma}w}P_0 = P_0\right\}.$$

By Proposition 8.5, $\mathrm{Stab}_\Gamma(\mathcal{I}_3^3)$ is trivial. It is easy to check that $\Gamma_1 = \{e\}$, $\Gamma_2 = \{\epsilon w\}$, and $\Gamma_3 = \Gamma_4 = \emptyset$. $\square$

Proposition 8.8. *The stabilizer group of $D'(\mathcal{I}^5)$ is the cyclic group of order two generated by $\sigma\epsilon^2$.*

Proof. Note that $\mathrm{Stab}_\Gamma(D'(\mathcal{I}^5)) = \mathrm{Stab}_\Gamma(\mathcal{I}^5)$. With the exception of P_0, for every $P \in \mathcal{I}^5$, there exists a $Q \in \mathcal{I}^5$ such that $|\mathcal{Q}(v_P, v_Q)|^2 = 2$. Therefore if $\gamma \in \mathrm{Stab}_\Gamma(\mathcal{I}^5)$, then ${}^\gamma P_0 = P_0$. This implies that

$$\begin{aligned}
\mathrm{Stab}_\Gamma(\mathcal{I}^5) &= \Gamma_{P_0} \cap \mathrm{Stab}_\Gamma\big(\big\{{}^w P_0, {}^{\tau w} P_0, {}^{\sigma w} P_0, {}^{\tau\sigma w} P_0\big\}\big) \\
&= \Gamma_{P_0} \cap {}^{w\tau^{-1}\epsilon w\sigma^{-1}\tau^{-1}}\mathrm{Stab}_\Gamma\big(\mathcal{I}_2^4\big) \\
&= \Gamma_{P_0} \cap {}^{w\tau^{-1}\epsilon w\sigma^{-1}\tau^{-1}}\{e, \epsilon w\} \quad \text{from Proposition 8.7.}
\end{aligned}$$

One easily checks that this intersection is $\{e, \sigma\epsilon^2\}$. □

Proposition 8.9. *The stabilizer in Γ of $D'(\mathcal{I}^8)$ is the group of order* 32 *generated by ξ^2 and ϵw given below*:

$$\begin{aligned}
\mathrm{Stab}_\Gamma\big(D'(\mathcal{I}^8)\big) = \big\{&e, \xi^2, \xi^4, \xi^6, \epsilon w, \epsilon w\xi^2, \epsilon w\xi^4, \epsilon w\xi^6, \xi^2\epsilon w, \xi^2\epsilon w\xi^2, \xi^2\epsilon w\xi^4, \\
&\xi^2\epsilon w\xi^6, \xi^4\epsilon w, \xi^4\epsilon w\xi^2, \xi^4\epsilon w\xi^4, \xi^4\epsilon w\xi^6, \xi^6\epsilon w, \xi^6\epsilon w\xi^2, \\
&\xi^6\epsilon w\xi^4, \xi^6\epsilon w\xi^6, \epsilon w\xi^2\epsilon w, \epsilon w\xi^4\epsilon w, \epsilon w\xi^6\epsilon w, \xi^2\epsilon w\xi^2\epsilon w, \\
&\xi^2\epsilon w\xi^4\epsilon w, \xi^2\epsilon w\xi^6\epsilon w, \xi^4\epsilon w\xi^2\epsilon w, \xi^4\epsilon w\xi^4\epsilon w, \xi^4\epsilon w\xi^6\epsilon w, \\
&\xi^6\epsilon w\xi^2\epsilon w, \xi^6\epsilon w\xi^4\epsilon w, \xi^6\epsilon w\xi^6\epsilon w\big\}.
\end{aligned}$$

It is isomorphic to the group of order 32 *with Hall–Senior number* 31 [12].

Proof. Note that $\mathrm{Stab}_\Gamma(D'(\mathcal{I}^8)) = \mathrm{Stab}_\Gamma(\mathcal{I}^8)$. Consider $\mathcal{I}_1^2 \subset \mathcal{I}^8$. Since the set of isotropic vectors associated to $\mathcal{I}_1^2$ is 1-bounded, if $\gamma \in \mathrm{Stab}_\Gamma(\mathcal{I}^8)$, then the set of isotropic vectors associated to ${}^\gamma\mathcal{I}_1^2$ is 1-bounded. There are 16 such subsets $\mathcal{I} = {}^{\gamma_\mathcal{I}}\mathcal{I}_1^2 \subset \mathcal{I}^8$ with this property. Then

$$\mathrm{Stab}_\Gamma\big(\mathcal{I}^8\big) = \coprod_{\substack{\mathcal{I}\subset\mathcal{I}^8 \\ \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}_1^2}} \Gamma_\mathcal{I},$$

where

$$\Gamma_\mathcal{I} = \big\{\gamma \in \gamma_\mathcal{I} \cdot \mathrm{Stab}_\Gamma\big(\mathcal{I}_1^2\big) \bigm| {}^\gamma\big\{\mathcal{I}^8 \setminus \mathcal{I}_1^2\big\} = \big\{\mathcal{I}^8 \setminus \mathcal{I}\big\}\big\}.$$

One can compute that each $\Gamma_\mathcal{I}$ has exactly two elements. Explicit computation of each $\Gamma_\mathcal{I}$ gives the 32 elements listed.

Fix an ordering $[\mathcal{I}^8]$. This induces a homomorphism $\psi : \mathrm{Stab}_\Gamma(\mathcal{I}^8) \to \mathfrak{S}_8$. The subgroup of $\mathrm{Stab}_\Gamma(\mathcal{I}^8)$ which fixes the ordering is $\ker(\psi) = \Gamma \cap (\bigcap_{P\in\mathcal{I}^8} P)$. One calculates that this intersection is trivial. Thus $\mathrm{Stab}_\Gamma(\mathcal{I}^8)$ is isomorphic to its image $\psi(\mathrm{Stab}_\Gamma(\mathcal{I}^8))$ in $\mathfrak{S}_8$. In particular, non-trivial elements of $\psi(\mathrm{Stab}_\Gamma(\mathcal{I}^8))$ written in disjoint cycle notation consist of 2, 4, or 8-cycles. Thus the exponent of $\mathrm{Stab}_\Gamma(\mathcal{I}^8)$ is 8. One computes that the center of $\mathrm{Stab}_\Gamma(\mathcal{I}^8)$ is cyclic of order four. There are only three groups of order 32 with exponent 8 and center which is cyclic of order four, namely $\mathcal{G}_{26}$, $\mathcal{G}_{31}$, and $\mathcal{G}_{32}$ in the lists given in [12]. These are distinguished by the number of conjugacy classes of maximal elementary abelian subgroups, that is, subgroups which are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some r. $\mathcal{G}_{26}$, $\mathcal{G}_{31}$, and $\mathcal{G}_{32}$ have one, two, and three conjugacy classes of maximal elementary abelian subgroups, respectively. One calculates that $\mathrm{Stab}_\Gamma(\mathcal{I}^8)$ has two conjugacy classes of maximal elementary abelian subgroups. $\square$

9. Structure of the spine

9.1. Cell structure and action of the stabilizers

From the previous computations, one can get a very explicit description of the cellular structure of the spine. For example, to find the strongly admissible sets of order three that are on the boundary of $\mathcal{I}_1^2$, one needs to find all rational parabolic subgroups Q such that $\mathcal{I} = \mathcal{I}_1^2 \cup \{Q\}$ is Γ-conjugate to either $\mathcal{I}_1^3$, $\mathcal{I}_2^3$, or $\mathcal{I}_3^3$ and hence reduces to a simple calculation of the $\mathcal{Q}$ inner-products of v_Q with $(1,0,0)^t$ and $(0,0,1)^t$. More precisely, define a map $\mathfrak{Q}: \{\text{ordered subsets of } \mathcal{P}\} \to \mathrm{Mat}(\mathbb{Z})$ as follows: Given an ordered subset $[\mathcal{I}] = (P_1, \ldots, P_n) \subset \mathcal{P}$, let $\mathfrak{Q}([\mathcal{I}])$ be the $n \times n$ matrix whose (i, j)-component is $|\mathcal{Q}(v_{P_i}, v_{P_j})|^2$.

Proposition 9.1. *Let $\mathcal{I}$ be strongly admissible set. Then*

$$\mathfrak{Q}([\mathcal{I}]) = \begin{cases} \begin{pmatrix} 0&1\\1&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}_1^2, \\ \begin{pmatrix} 0&2\\2&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}_2^2, \\ \begin{pmatrix} 0&1&1\\1&0&1\\1&1&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}_1^3 \text{ or } \mathcal{I}_2^3, \\ \begin{pmatrix} 0&1&2\\1&0&1\\2&1&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}_3^3, \\ \begin{pmatrix} 0&1&1&2\\1&0&1&1\\1&1&0&1\\2&1&1&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}_1^4, \\ \begin{pmatrix} 0&1&1&2\\1&0&2&1\\1&2&0&1\\2&1&1&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}_2^4, \\ \begin{pmatrix} 0&1&1&1&2\\1&0&1&2&1\\1&1&0&1&1\\1&2&1&0&1\\2&1&1&1&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}^5, \\ \begin{pmatrix} 0&1&2&1&1&2&1&4\\1&0&1&2&2&1&4&1\\2&1&0&1&1&4&1&2\\1&2&1&0&4&1&2&1\\1&2&1&4&0&1&2&1\\2&1&4&1&1&0&1&2\\1&4&1&2&2&1&0&1\\4&1&2&1&1&2&1&0 \end{pmatrix} & \text{if } \mathcal{I} \text{ is } \Gamma\text{-conjugate to } \mathcal{I}^8. \end{cases}$$

Table 1 summarizes the strongly admissible sets up to Γ-conjugacy and gives their stabilizers.

Table 1
Stabilizer groups of strongly admissible sets

S.A.S.	Stabilizer	Generators
$\mathcal{I}_1^2$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$\langle \epsilon w, \epsilon \rangle$
$\mathcal{I}_2^2$	$\mathbb{Z}/8\mathbb{Z}$	$\langle \xi \rangle$
$\mathcal{I}_1^3$	$\mathbb{Z}/12\mathbb{Z}$	$\langle \tau \epsilon w \rangle$
$\mathcal{I}_2^3$	$\mathfrak{S}_3$	$\langle \epsilon w, \sigma \epsilon^2 \rangle$
$\mathcal{I}_3^3$	trivial	$\langle e \rangle$
$\mathcal{I}_1^4$	trivial	$\langle e \rangle$
$\mathcal{I}_2^4$	$\mathbb{Z}/2\mathbb{Z}$	$\langle \epsilon w \rangle$
$\mathcal{I}^5$	$\mathbb{Z}/2\mathbb{Z}$	$\langle \sigma \epsilon^2 \rangle$
$\mathcal{I}^8$	$\langle \epsilon w, \xi^2 \rangle$	$\mathcal{G}_{31}$

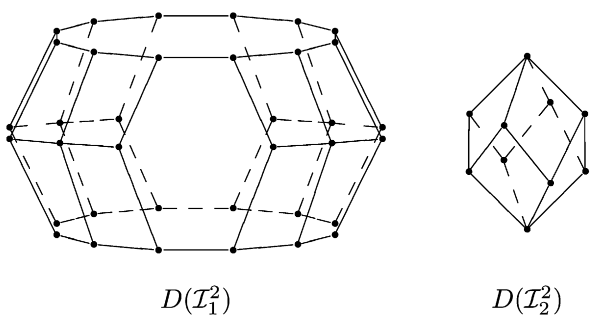

Fig. 1. The sets associated to the two Γ-conjugacy classes of strongly admissible sets of order two are shown here. $D(\mathcal{I}_1^2)$ is homeomorphic to a polytope with dodecagon, hexagon and quadrilateral faces, while $D(\mathcal{I}_2^2)$ has only quadrilateral faces.

In Fig. 1, the dodecagon faces are type $\mathcal{I}_1^3$, the hexagon faces are type $\mathcal{I}_2^3$, and the quadrilateral faces are type $\mathcal{I}_3^3$. The edges that bound a hexagon or dodecagon are type $\mathcal{I}_1^4$. The edges where two quadrilaterals meet are type $\mathcal{I}_2^4$. The vertices of edges of type $\mathcal{I}_1^4$ are type $\mathcal{I}^5$, and the others are type $\mathcal{I}^8$. The incidence table is given in Table 2, where the entry below the diagonal means that each column cell has that many row cells in its boundary, and the entry above the diagonal means the column cell appears in the boundary of this many row cells. The entries below the diagonal can be read off from Figs. 2–4, while the entries above the diagonal can be easily computed from Proposition 9.1, since the Γ-conjugacy class of a strongly admissible can be distinguished by the pairwise $\mathcal{Q}$-inner products of its associated isotropic vectors, except to distinguish $\mathcal{I}_1^3$ and $\mathcal{I}_2^3$, we must also compute the dimension of the span of their isotropic vectors.

To understand how the cells in Fig. 1 are glued together to form the spine, one must remember that the spine is in a four-dimensional symmetric space. Each 2-cell is a boundary face of exactly three 3-cells that only meet along this face. In particular, there are two other 3-cells of type $\mathcal{I}_1^2$ that are glued to each dodecagon face of $D(\mathcal{I}_1^2)$. There are two other 3-cells of type $\mathcal{I}_1^2$ that are glued to each hexagon face of $D(\mathcal{I}_1^2)$. There is one other 3-cells of type $\mathcal{I}_1^2$ and one 3-cell of

Table 2
Incidence types

	$\mathcal{I}^2_1$	$\mathcal{I}^2_2$	$\mathcal{I}^3_1$	$\mathcal{I}^3_2$	$\mathcal{I}^3_3$	$\mathcal{I}^4_1$	$\mathcal{I}^4_2$	$\mathcal{I}^5$	$\mathcal{I}^8$
$\mathcal{I}^2_1$	$*$	$*$	3	3	2	5	4	8	16
$\mathcal{I}^2_2$	$*$	$*$	0	0	1	1	2	2	8
$\mathcal{I}^3_1$	2	0	$*$	$*$	$*$	1	0	2	0
$\mathcal{I}^3_2$	4	0	$*$	$*$	$*$	1	0	2	0
$\mathcal{I}^3_3$	12	8	$*$	$*$	$*$	2	4	6	32
$\mathcal{I}^4_1$	40	8	12	6	2	$*$	$*$	4	0
$\mathcal{I}^4_2$	16	8	0	0	2	$*$	$*$	1	16
$\mathcal{I}^5$	32	8	12	6	3	2	1	$*$	$*$
$\mathcal{I}^8$	4	2	0	0	1	0	1	$*$	$*$

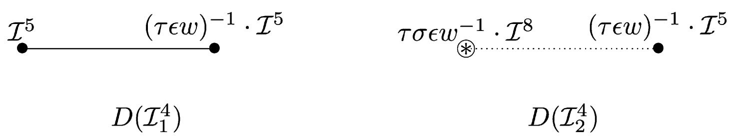

Fig. 2. Strongly admissible sets of order 4.

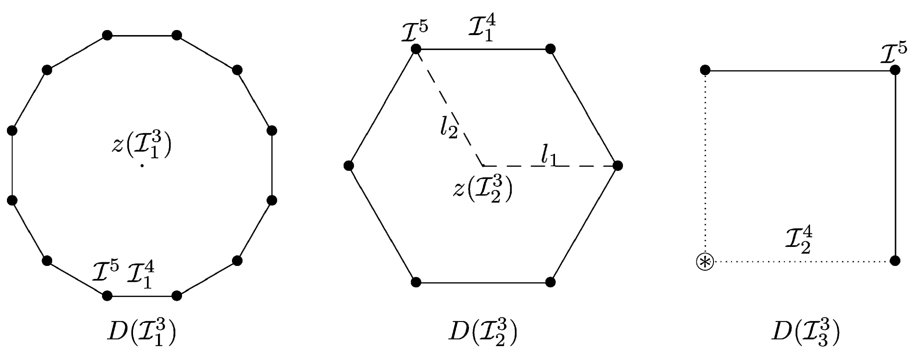

Fig. 3. Strongly admissible sets of order 3.

type $\mathcal{I}^2_2$ that is glued to each quadrilateral face of $D(\mathcal{I}^2_1)$. In particular, two 3-cells of type $\mathcal{I}^2_2$ are never glued to each other along a 2-cell face.

Looking at the incidence table we see that there are two quadrilaterals, one hexagon, and one dodecagon glued to each edge of type $\mathcal{I}^4_1$. Thus, given a dodecagon face and a hexagon face of $D(\mathcal{I}^2_1)$, the four other 3-cells emanating from these two faces can be paired so that each pair is glued to each other along a quadrilateral face. Given a dodecagon face and an adjacent quadrilateral face of $D(\mathcal{I}^2_1)$, the four other 3-cells emanating from these two faces can be paired so that one pair is glued to each other along a quadrilateral face and the other pair is glued to each other along a hexagon. Similarly, given a hexagon face and an adjacent quadrilateral face of $D(\mathcal{I}^2_1)$, the four other 3-cells emanating from these two faces can be paired so that one pair

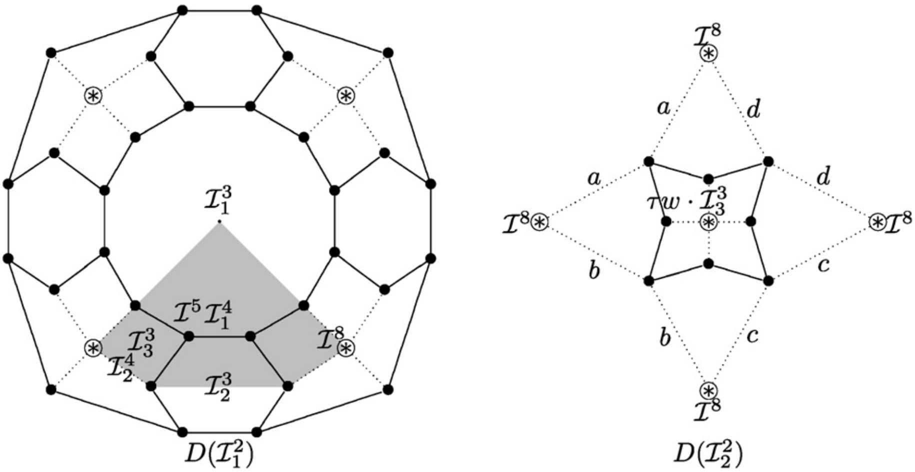

Fig. 4. Strongly admissible sets of order 2.

is glued to each other along a quadrilateral face and the other pair is glued to each other along a dodecagon.

Looking at the incidence table we see that there are four quadrilateral faces glued to each edge of type $\mathcal{I}_2^4$. Thus, given two adjacent quadrilateral faces of $D(\mathcal{I}_1^2)$, the four other 3-cells emanating from these two faces can be paired so that each pair is glued to each other along a quadrilateral face.

In Figs. 2–4, a 0-cell that is Γ-conjugate to $\mathcal{I}^5$ will be denoted by $\bullet$, and a 0-cell that is Γ-conjugate to $\mathcal{I}^8$ will be denoted by $\circledast$. A 1-cell that is Γ-conjugate to $\mathcal{I}_1^4$ will be denoted with a solid line, and a 1-cell that is Γ-conjugate to $\mathcal{I}_2^4$ will be denoted with a dotted line.

The 1-cell $D(\mathcal{I}_1^4)$ is shown in Fig. 2. The boundary consists of two 0-cells that are both conjugate to $D(\mathcal{I}^5)$. By Proposition 8.6, the stabilizer of $D(\mathcal{I}_1^4)$ is trivial.

The 1-cell $D(\mathcal{I}_2^4)$ is shown in Fig. 2. The boundary consists of two 0-cells (one that is conjugate to $D(\mathcal{I}^5)$ and one that is conjugate to $D(\mathcal{I}^8)$). By Proposition 8.7, the stabilizer of $D(\mathcal{I}_2^4)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by ϵw. This acts on the cell by fixing it pointwise.

The 2-cell $D(\mathcal{I}_1^3)$ is shown in Fig. 3. Its boundary consists of twelve 1-cells (all of which are conjugate to $D(\mathcal{I}_1^4)$) and twelve 0-cells (all of which are conjugate to $D(\mathcal{I}^5)$). By Proposition 8.3 the stabilizer of $D(\mathcal{I}_1^3)$ is isomorphic to $\mathbb{Z}/12\mathbb{Z}$ and is generated by $\tau\epsilon w$. It is easily checked that $\tau\epsilon w$ acts on the figure by rotation by $\frac{\pi}{6}$ about the first contact $z(\mathcal{I}_1^3)$ for $D(\mathcal{I}_1^3)$.

The 2-cell $D(\mathcal{I}_2^3)$ is shown in Fig. 3. The boundary consists of six 1-cells (all of which are conjugate to $D(\mathcal{I}_1^4)$) and six 0-cells (all of which are conjugate to $D(\mathcal{I}^5)$). The point $z(\mathcal{I}_2^3)$ is the first contact for $\mathcal{I}_2^3$. The lines l_j represent the gradient flows of the f_0 function restricted to $D(\mathcal{I}_2^3)$ from the first contact to two of the vertices of $D(\mathcal{I}_2^3)$. By Proposition 8.4, the stabilizer of $D(\mathcal{I}_2^3)$ is isomorphic to $\mathfrak{S}_3$ generated by ϵw and $\sigma^2\epsilon^2$. One can check that ϵw acts as reflection about l_1 and $\sigma\epsilon^2$ acts as reflection about l_2.

The 2-cell $D(\mathcal{I}_3^3)$ is shown in Fig. 3. The boundary consists of four 1-cells (two that are conjugate to $D(\mathcal{I}_1^4)$ and two that are conjugate to $D(\mathcal{I}_2^4)$) and four 0-cells (three that are conjugate to $D(\mathcal{I}^5)$ and one that is conjugate to $D(\mathcal{I}^8)$). By Proposition 8.5, the stabilizer of $D(\mathcal{I}_3^3)$ is trivial.

See Fig. 4 for a picture of the face relations for $D(\mathcal{I}_1^2)$. The outside of the figure is the dodecagon bottom face seen in Fig. 1. The boundary of $D(\mathcal{I}_1^2)$ consists of twenty-two 2-cells (two that are conjugate to $D(\mathcal{I}_1^3)$, four that are conjugate to $D(\mathcal{I}_2^3)$, and sixteen that are conjugate to $D(\mathcal{I}_3^3)$), fifty-six 1-cells (forty that are conjugate to $D(\mathcal{I}_1^4)$ and sixteen that are conjugate to $D(\mathcal{I}_2^4)$), and thirty-six 0-cells (thirty-two that are conjugate to $D(\mathcal{I}^5)$ and four that are conjugate to $D(\mathcal{I}^8)$). By Proposition 8.1 the stabilizer of $D(\mathcal{I}_1^2)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generated by ϵ and ϵw. It is easily calculated that ϵ acts by rotating the figure by $-\frac{\pi}{2}$ and ϵw acts by an inversion, sending the interior 12-gon to the exterior one.

The other 3-cell $D(\mathcal{I}_2^2)$ is shown in Fig. 4. The labels $\{a, b, c, d\}$ are added to show the identifications that need to be made. The boundary of $D(\mathcal{I}_2^2)$ consists of eight 2-cells (all of which are conjugate to $D(\mathcal{I}_3^3)$), sixteen 1-cells (eight that are conjugate to $D(\mathcal{I}_1^4)$ and eight that are conjugate to $D(\mathcal{I}_2^4)$) and ten 0-cells (eight that are conjugate to $D(\mathcal{I}^5)$ and two that are conjugate to $D(\mathcal{I}^8)$). By Proposition 8.2, the stabilizer of $D(\mathcal{I}_2^2)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$ generated by ξ. It is easily checked that ξ acts on the figure by the composition of an inversion sending the exterior point to the interior point and a rotation of $\frac{\pi}{4}$.

9.2. The subdivision

From the explicit description of the cells and Γ-action, it is straightforward to subdivide the spine so that the stabilizer of each cell fixes the cell pointwise.

First consider $D(\mathcal{I}_1^2)$. It must be divided into eight 3-cells that are conjugate via $\mathrm{Stab}_\Gamma(\mathcal{I}_1^2)$. One of the 3-cells X is represented by the shaded region in Fig. 4. It is easy to see that the boundary of X consists of two 2-cells, C_1 and C_2, that are Γ-conjugate to $D(\mathcal{I}_3^3)$, half of the 2-cell $D(\mathcal{I}_2^3)$ B, one-fourth of the 2-cell $D(\mathcal{I}_1^3)$, A, and three new faces, E_1, E_2, and D that lie inside $D(\mathcal{I}_1^2)$ and such that E_1 and E_2 are Γ-conjugate. Next we turn to subdividing the boundary faces of X. From the description of the action of the $\mathrm{Stab}_\Gamma(\mathcal{I}_1^3)$ on $D(\mathcal{I}_1^3)$, it follows that A must be subdivided into three Γ-conjugate 2-cells A_1, A_2, and A_3. Similarly, because of the action of $\mathrm{Stab}_\Gamma(\mathcal{I}_2^3)$ on $D(\mathcal{I}_2^3)$, B must be subdivided into three Γ-conjugate 2-cells B_1, B_2, and B_3. One can compute that the stabilizers of E_i are trivial, and that the stabilizer of D fixes D pointwise. The 1-cells of X, both the ones that are Γ-conjugate to $D(\mathcal{I}_j^4)$ $(j = 1, 2)$ and the new ones that are introduced for the subdivision, do not need to be subdivided because the stabilizers are either trivial, or the stabilizer acts on the 1-cell by fixing it pointwise. This yields X in Fig. 5.

Similarly, $D(\mathcal{I}_2^2)$ must be subdivided into eight 3-cells that are conjugate via $\mathrm{Stab}_\Gamma(\mathcal{I}_2^2)$. One of the 3-cells Y can be viewed in Fig. 4 by taking a face C_1 that is Γ-conjugate to $D(\mathcal{I}_3^3)$ and looking at the set of points of $D(\mathcal{I}_2^2)$ that would flow to C_1 under the gradient flow for $-f_{\mathcal{I}_2^2}$ and adding the first contact point of $\mathcal{I}_2^2$. Then the boundary of Y consists of five 2-cells, C_1, F_1, F_2, G_1, and G_2 such that F_1 is Γ-conjugate to F_2 and G_1 is Γ-conjugate to G_2. The calculations above show that the stabilizer of C_1 is trivial, and it is easy to check that the stabilizers of F_1, F_2, G_1, and G_2 are trivial as well. Hence the 2-cells do not need to be subdivided. Similarly, one checks that the stabilizers of the 1-cells are either trivial or fix the 1-cell pointwise. This yields Y in Fig. 5.

In the figures, the labels that only differ by a subscript are conjugate under Γ. For each Γ-conjugacy class, we fix a representative and compute the stabilizers. The results are given in Table 3.

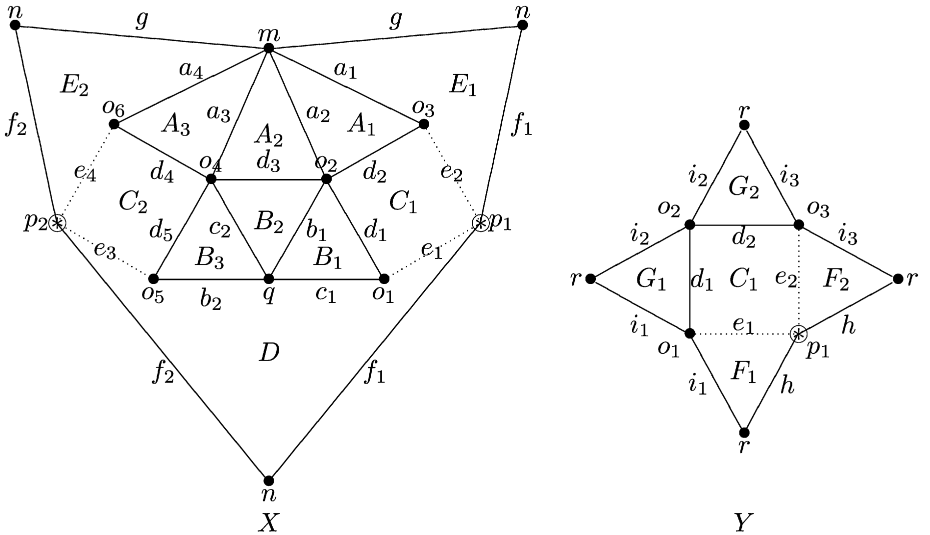

Fig. 5. Subdivided cells.

Table 3
Representative cells and their stabilizers

Cell	Dimension	Stabilizer	Generators
X	3	trivial	$\langle e\rangle$
Y	3	trivial	$\langle e\rangle$
A_1	2	trivial	$\langle e\rangle$
B_1	2	trivial	$\langle e\rangle$
C_1	2	trivial	$\langle e\rangle$
D	2	$\mathbb{Z}/2\mathbb{Z}$	$\langle \epsilon w\rangle$
E_1	2	trivial	$\langle e\rangle$
F_1	2	trivial	$\langle e\rangle$
G_1	2	trivial	$\langle e\rangle$
a_1	1	trivial	$\langle e\rangle$
b_1	1	$\mathbb{Z}/2\mathbb{Z}$	$\langle \sigma\epsilon w\sigma^{-1}\rangle$
c_1	1	$\mathbb{Z}/2\mathbb{Z}$	$\langle \epsilon w\rangle$
d_1	1	trivial	$\langle e\rangle$
e_1	1	$\mathbb{Z}/2\mathbb{Z}$	$\langle \epsilon w\rangle$
f_1	1	$\mathbb{Z}/2\mathbb{Z}$	$\langle \epsilon w\rangle$
g	1	$\mathbb{Z}/4\mathbb{Z}$	$\langle \epsilon\rangle$
h	1	$\mathbb{Z}/4\mathbb{Z}$	$\langle \xi^2\rangle$
i_1	1	trivial	$\langle e\rangle$
m	0	$\mathbb{Z}/12\mathbb{Z}$	$\langle \tau\epsilon w\rangle$
n	0	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$\langle \epsilon w, \epsilon\rangle$
o_1	0	$\mathbb{Z}/2\mathbb{Z}$	$\langle \epsilon w\rangle$
p_1	0	$\mathcal{G}_{31}$	$\langle \epsilon w, \xi^2\rangle$
q	0	$\mathfrak{S}_3$	$\langle \epsilon w, \sigma\epsilon^2\rangle$
r	0	$\mathbb{Z}/8$	$\langle \xi\rangle$

10. Cohomology of $\Gamma\backslash D_0$ with local coefficients

In this section only, we generalize to the case where $D = G/K$ is a non-compact symmetric space, where G is the group of real points of a semisimple linear algebraic group G defined over $\mathbb{Q}$. Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. Let $D_0 \subset D$ denote a spine with CW-structure such that the Γ-stabilizer of a cell fixes the cell pointwise.

In order to set notation, we recall the definition of orbifold and the sheaf associated to the local system. We then prove that the cohomology of $\Gamma\backslash D$ with local coefficients is isomorphic the cohomology of $\Gamma\backslash D_0$ with local coefficients.

10.1. Orbifolds

The notion of an orbifold was first introduced by Satake in [16]. He called them V-manifolds. Let M be a Hausdorff topological space. A *local uniformizing system* (*l.u.s.*) $\{U, \Gamma_U, \phi\}$ for an open set $V \subset M$ is a collection of the following objects:

(i) U: a connected open subset of $\mathbb{R}^n$.
(ii) G: a finite group of linear transformations of U to itself such that the set of fixed points of G is either all of U or at least codimension 2.
(iii) ϕ: a continuous Γ_U-invariant map $U \to V$ such that the induced map $\Gamma_U\backslash U \to V$ is a homeomorphism.

Let $V \subset V' \subset M$ be two open sets and let $\{U, \Gamma_U, \phi\}$ and $\{U', \Gamma_{U'}, \phi'\}$ be local uniformizing systems for V and V' respectively. An *injection* $\lambda : \{U, \Gamma_U, \phi\} \to \{U', \Gamma_{U'}, \phi'\}$ is a smooth injection $\lambda : U \to U'$ such that for any $\gamma \in \Gamma_U$, there exists a $\gamma' \in \Gamma_{U'}$ such that $\lambda \circ \gamma = \gamma' \circ \lambda$ and $\phi = \phi' \circ \lambda$.

Let $\mathfrak{L}$ be a family of local uniformizing systems for open sets in M. Then an open set $V \subset M$ is said to be $\mathfrak{L}$-*uniformized* if there exists a local uniformizing system for V in $\mathfrak{L}$.

Definition 10.1. An *orbifold* is a Hausdorff space M and a family $\mathfrak{L}$ of local uniformizing systems for open sets in M satisfying the following conditions.

(i) Let $V \subset V' \subset M$ be two open sets and let $\{U, \Gamma_U, \phi\}$ and $\{U', \Gamma_{U'}, \phi'\}$ be local uniformizing systems for V and V' respectively. Then there exists an injection $\{U, \Gamma_U, \phi\} \to \{U', \Gamma_{U'}, \phi'\}$.
(ii) The $\mathfrak{L}$-uniformized open sets form a basis of open sets for M.

Two families of local uniformizing systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$ defining an orbifold M are said to be *equivalent* if $\mathfrak{L}_1 \cup \mathfrak{L}_2$ satisfies the conditions above. Equivalent families define the same orbifold structure on M. Thus when talking about family of local uniformizing systems for an orbifold M, we will mean a maximal family.

A *smooth function* on M is given locally on $\mathfrak{L}$-uniformized sets V by Γ_U-invariant smooth functions on U. Similarly, a *smooth p-form* on M is given locally on V by Γ_U-invariant smooth forms on U.

Note that a smooth manifold is an example of an orbifold where every group Γ_U for $\{U, \Gamma_U, \phi\} \in \mathfrak{L}$ is the trivial group. If $\tilde{M}$ is a smooth manifold and Γ is a properly discontinuous group of automorphisms of $\tilde{M}$, then the quotient $\Gamma\backslash\tilde{M}$ has a canonical structure of an orbifold.

10.2. $\Gamma\backslash D$ as an orbifold

Since Γ acts properly discontinuously on D, $X = \Gamma\backslash D$ has a canonical structure of an orbifold. Let π denote the projection $D \to X$.

Given a connected open set $V \subset X$ with local uniformizing system L, we can and will identify L with a triple $\{U, \mathrm{Stab}_\Gamma(U), \pi|_U\}$ where U is a connected component of $\tilde{U} = \pi^{-1}(V)$. A smooth function on V is given by a $\mathrm{Stab}_\Gamma(U)$-invariant smooth function on U. In particular, for an open subset $O \subset X$, a smooth function on O is given by a Γ-invariant function on $\pi^{-1}(O)$.

Since D_0 is a spine, there exists a Γ-equivariant deformation retract $\tilde{r} : D \to D_0$. Then there is an induced deformation retract $r : X \to X_0 = \Gamma\backslash D_0$.

10.3. Locally constant orbifold (fiber) bundles over $\Gamma\backslash D$

Let E be a Γ-module with Γ-action given by $\rho : \Gamma \to \mathrm{GL}(E)$. Let $D \times_\Gamma E$ denote the quotient space $D \times E/\{(x, v) \sim (\gamma \cdot x, \rho(\gamma)v) \mid \gamma \in \Gamma\}$. For a Γ-invariant deformation retract $D_0 \subseteq D$, we can similarly define $D_0 \times_\Gamma E$. This yields the following maps.

$$
\begin{tikzcd}
& D_0 \times E \arrow[r] \arrow[dl, "\tilde{\pi}_0"'] \arrow[d] & D \times E \arrow[d] \arrow[dr, "\tilde{\pi}"] & \\
D_0 \times_\Gamma E \arrow[d] \arrow[rrr, bend left=60] & D_0 \arrow[r, bend right] \arrow[dl, "\pi_0"'] & D \arrow[l, bend right, "\tilde{r}"'] \arrow[dr, "\pi"] & D \times_\Gamma E \arrow[d] \\
X_0 = \Gamma\backslash D_0 \arrow[rrr, bend right] & & & \Gamma\backslash D = X \arrow[lll, bend right, "r"']
\end{tikzcd}
$$

Suppose that E is an N-dimensional vector space. If Γ is torsion-free, then X is a smooth manifold and $D \times_\Gamma E$ is a flat rank-N vector bundle over X. If Γ has torsion, then X is an orbifold and $D \times_\Gamma E$ is called a *flat orbifold bundle* over X.

Consider the presheaf $\mathcal{E}$ on X defined as follows. For every open set $U \subset X$,

$$\mathcal{E}(U) = \left\{ f : \pi^{-1}(U) \to E \;\middle|\; \begin{array}{c} f \text{ is locally constant and} \\ f(\gamma \cdot x) = \rho(\gamma) f(x)\ \forall \gamma \in \Gamma,\ x \in \pi^{-1}(U) \end{array} \right\}$$

with the obvious restriction maps. Let $\mathbb{E}$ denote the sheafification of $\mathcal{E}$. Similarly define the presheaf $\mathcal{E}_0$ on X_0 and let $\mathbb{E}_0$ denote the sheafification of $\mathcal{E}_0$. Note that if Γ is torsion-free, $\mathbb{E}$ is the sheaf associated to the local system X defined by (E, ρ). We will extend this terminology to X, respectively X_0, when Γ is not torsion-free and say that $\mathbb{E}$, respectively $\mathbb{E}_0$, is the sheaf associated to the local system on X, respectively X_0, defined by (E, ρ).

Proposition 10.2. *The sheaf $\mathbb{E}_0$ on X_0 is isomorphic to the push-forward $r_*\mathbb{E}$ of the sheaf $\mathbb{E}$ on X.*

Proof. Let $V_0 \subset X_0$ be a contractible open set. Let U_0 be a connected component of $\tilde{V}_0 = \pi_0^{-1}(V_0)$. Since Γ acts properly discontinuously on D_0, by shrinking V_0 (and hence shrinking U_0) if necessary, one can arrange that $\Gamma_{U_0} \equiv \{\gamma \in \Gamma \mid \gamma \cdot U_0 \cap U_0 \neq \emptyset\}$ is a finite group. A section $f \in \mathbb{E}_0(V_0)$ is a locally constant map $f : \tilde{V}_0 \to E$ such that $f(\gamma \cdot x) = \rho(\gamma) f(x)$ for all $\gamma \in \Gamma$. Since V_0 is a connected open set, it follows that $\pi_0(U_0) = V_0$ and $\Gamma \cdot U_0 = \pi_0^{-1}(V_0)$, and hence f is determined by its value v_{U_0} on U_0. Furthermore, by the Γ-equivariance of f, $v_{U_0} \in E^{\Gamma_{U_0}}$, the subspace of E fixed by Γ_{U_0}.

Recall that $r_*\mathbb{E}(V_0) = \mathbb{E}(r^{-1}(V_0))$. Since $\pi_0 \circ \tilde{r} = r \circ \pi$, there is a connected component U of $\pi^{-1}(r^{-1}(V_0))$ such that $U = \tilde{r}^{-1}(U_0)$. Since Γ acts properly discontinuously on D, $\Gamma_U \equiv \{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\}$ is finite. Thus a section $\psi \in r_*\mathbb{E}(V_0)$ is determined by its value u_U on U. Furthermore, by the Γ-equivariance of ψ, $u_U \in E^{\Gamma_U}$, the subspace fixed by Γ_U.

Thus to show that the sheaves are isomorphic, it suffices to show that for sufficiently small $V_0 \subset X_0$, the groups Γ_{U_0} and Γ_U defined above are equal. It is clear that $\Gamma_{U_0} \subseteq \Gamma_U$. To show the opposite inclusion, suppose $\gamma \in \Gamma_U$. Then by definition $\gamma \cdot U \cap U \neq \emptyset$. Let $x \in \gamma \cdot U \cap U$. In particular, $x \in U$, so $\tilde{r}(x) \in U_0$. Furthermore, since $x \in \gamma \cdot U$, $\gamma^{-1} \cdot x \in U$. Thus $\tilde{r}(\gamma^{-1} \cdot x) \in U$ and the Γ-equivariance of $\tilde{r}$ implies that $\tilde{r}(x) \in \gamma \cdot U_0$ and hence $\gamma \in \Gamma_{U_0}$. □

10.4. Cohomology of subspaces

We recall without proof two results of sheaf cohomology. A reference for this section is [10].

Theorem 10.3. *(See [10, Theorem 10.6].) Let $\mathcal{F}$ be a sheaf on a paracompact space X, and let $A \subset X$ be a closed subspace. Let $\mathcal{N}$ be the set of all open neighborhoods of A. Then*

$$\varinjlim_{N \in \mathcal{N}} H^*(N; \mathcal{F}|_N) \cong H^*(A; \mathcal{F}|A).$$

Theorem 10.4. *(See [10, Theorem 11.7].) Let X be a paracompact space. Let $f : X \to Y$ a closed map and $\mathcal{F}$ a sheaf on X. Suppose that $H^p(f^{-1}(y); \mathcal{F}|_{f^{-1}(y)}) = 0$ for $p > 0$ and all $y \in Y$. Then the natural map*

$$f^\dagger : H^*(Y; f_*\mathcal{F}) \to H^*(X; \mathcal{F})$$

is an isomorphism.

Let $r : X \to X_0$ denote the deformation retraction arising from the Γ-equivariant deformation retraction of the symmetric space D to the spine D_0. Let (E, ρ) be a Γ-module and let $\mathbb{E}$ and $\mathbb{E}_0$ denote the associated local systems on X and X_0 respectively. We believe the following results are known, but since we could not find a reference, we provide a proof.

Theorem 10.5. $H^*(X; \mathbb{E}) \cong H^*(X_0; \mathbb{E}_0)$.

Lemma 10.6. *For every $z \in D$,*

$$\operatorname{Stab}_\Gamma(z) = \operatorname{Stab}_\Gamma\big(\tilde{r}_t(z)\big) \quad \textit{for } t < 1 \textit{ and}$$
$$\operatorname{Stab}_\Gamma(z) \subseteq \operatorname{Stab}_\Gamma\big(\tilde{r}_1(z)\big).$$

Proof. Let γ be an element of $\operatorname{Stab}_\Gamma(z)$. Then $\gamma \cdot \tilde{r}_t(z) = \tilde{r}_t(\gamma \cdot z) = \tilde{r}_t(z)$. Hence $\gamma \in \operatorname{Stab}_\Gamma(\tilde{r}_t(z))$. Notice that for each $z \notin D_0$, $c(t) = \tilde{r}_t(z), 0 \leqslant t \leqslant 1$, is a reparameterization of a geodesic, and Γ acts by isometries. Thus every $\gamma \in \operatorname{Stab}_\Gamma(z)$ fixes the geodesic through $\tilde{r}_1(z)$ and z. In particular, $\operatorname{Stab}_\Gamma(z') = \operatorname{Stab}_\Gamma(z)$ whenever $z = \tilde{r}_t(z')$ for some $t < 1$. $\square$

Proof of Theorem 10.5. We will show that $H^*(N; \mathbb{E}|_N) \cong H^*(r^{-1}(y); \mathbb{E}|_{r^{-1}(y)})$ for all contractible neighborhoods N of $y \in X_0$. Let $U \subset D$ be a connected component of $\pi^{-1}(r^{-1}(y))$. Let $\tilde{y}$ denote the unique lift of y in U. Note that U is a finite union of geodesic rays emanating from $\tilde{y}$. By Proposition 10.6, the stabilizer groups $\operatorname{Stab}_\Gamma(\tilde{x})$ are isomorphic for $\tilde{x}$ in a connected component of $U \setminus \tilde{y}$. Let $\tilde{N} \subseteq U$ be a contractible open subset containing $\tilde{y}$ and let $N = \pi(\tilde{N})$. Then it is clear that $H^*(r^{-1}(y); \mathbb{E}|_{r^{-1}(y)}) \cong H^*(N, \mathbb{E}|_N)$. Apply Theorem 10.3 with $X = r^{-1}(y)$, $\mathcal{F} = \mathbb{E}|_{r^{-1}(y)}$, $A = \{y\}$, and $\mathcal{N}$ the family of all contractible open sets containing y. Then

$$H^*\big(r^{-1}(y); \mathbb{E}|_{r^{-1}(y)}\big) \cong \varinjlim_{N \in \mathcal{N}} H^*(N; \mathbb{E}|_N) \cong H^*(y; \mathbb{E}|_y).$$

In particular, $H^p(r^{-1}(y); \mathbb{E}|_{r^{-1}(y)}) = 0$ for $p > 0$.

By Theorem 10.4, this implies that $H^*(X; \mathbb{E}) \cong H^*(X_0; r_*\mathbb{E})$. By Proposition 10.2, $r_*\mathbb{E} \cong \mathbb{E}_0$ and the result follows. $\square$

10.5. Computing the cohomology from the cell structure

The cell structure allows us to compute the cohomology $H^*(\Gamma \backslash D_0; \mathbb{E})$ combinatorially once Γ-representatives of cells and the stabilizers of those cells have been computed. This result is known, but rephrased here in a way that is convenient for our computations. We first set up some notation.

For each p, fix a set R_p of representatives of Γ-conjugacy classes of p-cells of D_0. Let $[\tau]$ denote the representative of the class of τ. For each representative $[\tau]$, fix a distinguished maximal flag of cells

$$F_{[\tau]} = \tau_0 < \tau_1 < \cdots < \tau_p, \quad \text{where } \tau_i \text{ is an } i\text{-cell and } \tau_p = [\tau].$$

For each cell τ, let $\gamma_\tau \in \Gamma$ be such that $\gamma_\tau \cdot F_{[\tau]}$ terminates in τ. Note that γ_τ is well-defined up to $\operatorname{Stab}_\Gamma(\tau)$.

For each fixed p-cell, let S_σ denote the simplicial complex arising from the poset of cells in σ with the partial ordering derived from containment. In particular, the vertices of S_σ are the cells contained in σ and the k-simplices are the $(k+1)$-flags $\sigma_0 < \sigma_1 < \cdots < \sigma_k$. Define a map $n_\sigma : \{p\text{-simplices of } S_\sigma\} \to \{\pm 1\}$ by the equation $\partial S_\sigma = \sum_{F \in S_\sigma} n_\sigma(F) \partial F$. Multiply by -1 if necessary so that $n_\sigma(\gamma_\sigma F_{[\sigma]}) = (-1)^p$. Then for each σ, define a map

$$\operatorname{sgn}_\sigma(\tau) = n_\sigma(\gamma_\tau F_{[\tau]} < \sigma). \tag{23}$$

Theorem 10.7. *The cohomology $H^*(X_0;\mathbb{E})$ can be computed from the complex*

$$0 \to \bigoplus_{\sigma\in R_0} E^{\mathrm{Stab}_\Gamma(\sigma)} \to \bigoplus_{\sigma\in R_1} E^{\mathrm{Stab}_\Gamma(\sigma)} \to \cdots \to \bigoplus_{\sigma\in R} E^{\mathrm{Stab}_\Gamma(\sigma)} \to 0,$$

where the differential

$$d:\bigoplus_{\sigma\in R_{p-1}} E^{\mathrm{Stab}_\Gamma(\sigma)} \to \bigoplus_{\sigma\in R_p} E^{\mathrm{Stab}_\Gamma(\sigma)}$$

is given by

$$(dv)_\sigma = \sum_{\tau\, a\ (p-1)\text{-}cell\ \in\partial\sigma} \mathrm{sgn}_\sigma(\tau)\rho(\gamma_\tau)v_{[\tau]}.$$

Here γ_τ, sgn, *and $[\cdot]$ are defined above and the vector $v_{[\tau]}$ is the $[\tau]$-component of the vector $v\in\bigoplus_{\sigma\in R_{p-1}} E^{\mathrm{Stab}_\Gamma(\sigma)}$.*

11. Cohomology

Recall that ϵ, w, σ, τ, and ξ were explicitly defined elements of G given in Section 2.

Theorem 11.1. *Let E be a Γ-module with the action of Γ given by $\rho:\Gamma\to \mathrm{GL}(E)$. Then $H^*(\Gamma\backslash D;\mathbb{E})$ can be computed from the following cochain complex.*

$$0\to C^0\to C^1\to C^2\to C^3\to 0,$$

where

$$\begin{aligned}
C^0 &= E^{\rho(\tau\epsilon w)}\oplus E^{\langle\rho(\epsilon),\rho(w)\rangle}\oplus E^{\rho(\epsilon w)}\oplus E^{\langle\rho(\epsilon w),\rho(\xi^2)\rangle}\oplus E^{\langle\rho(\epsilon w),\rho(\sigma\epsilon^2)\rangle}\oplus E^{\rho(\xi)},\\
C^1 &= E\oplus E^{\rho(\sigma\epsilon w\sigma^{-1})}\oplus E^{\rho(\epsilon w)}\oplus E\oplus E^{\rho(\epsilon w)}\oplus E^{\rho(\epsilon w)}\oplus E^{\rho(\epsilon)}\oplus E^{\rho(\xi^2)}\oplus E,\\
C^2 &= E\oplus E\oplus E\oplus E^{\rho(\epsilon w)}\oplus E\oplus E\oplus E,\\
C^3 &= E\oplus E.
\end{aligned}$$

Then for $(\kappa_i)\in C^0$, $(\lambda_i)\in C^1$, and $(\mu_i)\in C^2$, the differentials are given by

$$d_0(\kappa)=\begin{pmatrix} -\kappa_1+\rho(\xi^2)\kappa_3\\ \rho(\xi)\kappa_3-\kappa_5\\ \kappa_3-\kappa_5\\ -\kappa_3+\rho(\xi)\kappa_3\\ \kappa_3-\kappa_4\\ -\kappa_2+\kappa_4\\ \kappa_1-\kappa_2\\ \kappa_4-\kappa_6\\ \kappa_3-\kappa_6 \end{pmatrix},$$

$$d_1(\lambda) = \begin{pmatrix} -\lambda_1 + \rho(\tau\epsilon w)\lambda_1 + \rho(\xi)\lambda_4 \\ \lambda_2 - \lambda_3 - \lambda_4 \\ \lambda_4 + \rho(\xi)\lambda_4 + \lambda_5 - \rho(\xi^2)\lambda_5 \\ -\rho(\sigma\epsilon^2)\lambda_2 + \lambda_3 - \lambda_5 + \rho(\tau\sigma w\tau\sigma^{-1})\lambda_5 - \lambda_6 + \rho(\epsilon)\lambda_6 \\ \lambda_1 - \rho(\xi^2)\lambda_5 - \lambda_6 + \lambda_7 \\ -\lambda_5 - \lambda_8 + \lambda_9 \\ \lambda_4 + \lambda_9 - \rho(\xi)\lambda_9 \end{pmatrix},$$

$$d_2(\mu) = \begin{pmatrix} A\mu_1 + B\mu_2 - \mu_3 - \rho(\tau\sigma\epsilon w^{-1})\mu_3 - \mu_4 + \mu_5 - \rho(\epsilon)\mu_5 \\ -\mu_3 - \mu_6 + \rho(\xi^2)\mu_6 + \mu_7 + \rho(\xi)\mu_7 \end{pmatrix}$$

where

$$A = \rho(I) + \rho(\tau\epsilon w) + \rho(\tau\epsilon w)^2 \quad \textit{and} \quad B = -\rho(I) + \rho\big(\sigma\epsilon w\sigma^{-1}\big) - \rho\big(\sigma\epsilon w\sigma^{-1}\epsilon w\big).$$

An application of Theorem 11.1 is the following.

Corollary 11.2. *The group* $\Gamma = \mathrm{SU}(2,1;\mathbb{Z}[i])$ *is generated by* $\{\epsilon, w, \tau, \sigma\}$.

Lemma 11.3. *Let* H *be a subgroup of a group* G*. If* $H \neq G$*, then there exists a representation* (E,ρ) *of* G *such that* $E^{\rho(H)} \neq E^{\rho(G)}$.

Proof. Consider the representation (E,ρ) of functions $\phi : G/H \to \mathbb{C}$ with the left regular action of G on ϕ. The characteristic function χ_{eH} of the identity coset eH is fixed by H, but not fixed by G for $G \neq H$. □

Proof of Corollary 11.2. Let $\rho : \Gamma \to \mathrm{GL}(E)$ be a representation of Γ. Then the cohomology $H^0(X;\mathbb{E})$ is equal to the global sections $\mathbb{E}(X) \cong E^{\Gamma}$. Since $H^0(X;\mathbb{E})$ is the kernel of d_0 given in Theorem 11.1, $H^0(X;\mathbb{E}) \cong E^{\langle\epsilon,w,\tau,\sigma\rangle}$. It is well known that $H^0(X;\mathbb{E}) \cong E^{\rho(\Gamma)}$. The result then follows from Lemma 11.3. □

The following corollary follows immediately from the form of d_2 given in Theorem 11.1.

Corollary 11.4. *Let* E *be a* Γ*-module with action given by* $\rho : \Gamma \to \mathrm{GL}(E)$*. Let* $\mathbb{E}$ *denote the associated sheaf. Then*

$$\operatorname{rank}\big(H^3(\Gamma\backslash D;\mathbb{E})\big) \leqslant \operatorname{rank}(E) - \operatorname{rank}\big(E^{\rho(\epsilon w)}\big).$$

Corollary 11.5. *Let* E *be a finite-dimensional complex representation of* $\mathrm{SU}(2,1)$*. Let* E_{ij} *denote the* $i\omega_1 + j\omega_2$ *weight space of* E*, where* ω_1 *and* ω_2 *are the fundamental (complex) weights of* $\mathfrak{sl}_3\mathbb{C}$*, the complexification of* $\mathfrak{su}(2,1)$*. Then*

$$\dim H^3(\Gamma\backslash D;\mathbb{E}) \leqslant \sum_{\substack{i>j \\ i\equiv j\,(4)}} \dim E_{ij} + \sum \dim\big(E_{ii} \cap \ker\big(I + \rho(w)\big)\big).$$

Table 4
Cohomology for $\mathrm{Sym}^n(V)$

n	1	2	3	4	5	6	7	8	9	10
h^0	0	0	0	0	0	0	0	0	0	0
h^1	0	0	0	0	0	0	0	0	0	0
h^2	1	0	0	1	3	1	2	2	5	1
h^3	0	0	0	1	0	0	0	1	0	0

n	11	12	13	14	15	16	17	18	19	20
h^0	0	0	0	0	0	0	0	0	0	0
h^1	0	1	0	0	3	4	2	5	8	11
h^2	2	3	7	4	5	4	9	5	7	5
h^3	0	1	0	0	0	1	0	0	0	1

Proof. The dimension of $H^3(\Gamma\backslash D;\mathbb{E}) = 2\dim(E) - \operatorname{rank}(d_2)$. From the form of d_2 given in Theorem 11.1, it follows that

$$\dim H^3(\Gamma\backslash D;\mathbb{E}) \leqslant \dim(E) - \operatorname{rank}(\Phi), \tag{24}$$

where $\Phi : E^{\rho(\epsilon w)} \oplus E \to E$ is the linear map given by $\Phi(v) = v_1 + v_2 - \rho(\epsilon)v_2$. Let E_+ denote the $(+1)$-eigenspace of $\rho(\epsilon)$ and E_- denote the (-1)-eigenspace of $\rho(w)$. Then (24) implies that

$$\dim H^3(\Gamma\backslash D;\mathbb{E}) \leqslant \dim(E_+ \cap E_-). \tag{25}$$

The result follows from translating the eigenvalue condition into a condition on weights of E. □

First consider the trivial representation $E = \mathbb{Z}$. Then $H^*(\Gamma\backslash D;\mathbb{E})$ is isomorphic to the singular cohomology of $\Gamma\backslash D$. Proposition 11.1, allows us to explicitly compute the cohomology.

Theorem 11.6. *Let $\mathbb{Z}$ denote the constant sheaf of integers on $\Gamma\backslash D$. Then*

$$H^k(\Gamma\backslash D;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, 2, \\ 0, & k = 1 \text{ or } k \geqslant 3. \end{cases}$$

Theorem 11.7. *Let $E = \mathbb{Z}[i]^3$ with the natural action of Γ and let $\mathbb{E}$ denote the associated sheaf on $\Gamma\backslash D$. Then*

$$H^k(\Gamma\backslash D;\mathbb{E}) = \begin{cases} 0, & k = 0, 1, \\ \mathbb{Z}^2, & k = 2, \\ \mathbb{Z}/2\mathbb{Z}, & k = 3. \end{cases}$$

Theorem 11.8. *The dimensions h^i of the cohomology groups for Γ with coefficients in* $\mathrm{Sym}^n(V)$, $0 \leqslant n \leqslant 20$, *where $V \equiv \mathbb{C}^3$ is the standard representation is as tabulated in Table* 4.

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