Hyperbolic Tessellations Associated to Bianchi Groups

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Abstract. Let $F/\mathbb{Q}$ be a number field. The space of positive definite binary Hermitian forms over F form an open cone in a real vector space. There is a natural decomposition of this cone into subcones. In the case of an imaginary quadratic field these subcones descend to hyperbolic space to give rise to tessellations of 3-dimensional hyperbolic space by ideal polytopes. We compute the structure of these polytopes for a range of imaginary quadratic fields.

1 Introduction

Let $F/\mathbb{Q}$ be a number field. The space of positive definite binary Hermitian forms over F form an open cone in a real vector space. There is a natural decomposition of this cone into polyhedral cones corresponding to the facets of the Voronoï polyhedron [1, 11, 13]. This has been computationally explored for real quadratic fields in [16, 12] and the cyclotomic field $\mathbb{Q}(\zeta_5)$ in [23].

For F an imaginary quadratic field, the polyhedral cones give rise to ideal polytopes in $\mathbb{H}_3$, 3-dimensional hyperbolic space. In work of Cremona and his students [6, 7, 5, 14, 22], analogous polytopes have already been computed for class number one imaginary quadratic fields as well as a few fields with class number two and three using different methods. The structure of the polytopes was used to compute Hecke operators on modular forms for the Bianchi groups over those fields. These polytopes were used by Goncharov [10] in his study of Euler complexes on modular curves. The data of the polytope and stabilizer could also be used to give explicit presentations of $\mathrm{GL}_2(\mathcal{O})$ using results of Macbeath and Weil [15, 21]. Swan [20] has computed presentations of these groups, though not with the polytopes constructed here, for imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ for

$$-d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19\}.$$

Such explicit presentations have been used to compute cohomology of Bianchi groups of small discriminant with non-trivial coefficients in work of Berkove, Sengun, and Finis-Grunewald-Tirao [2, 3, 9, 19].

We remark that there are other ways to obtain the fundamental polytope data. Riley [18] wrote the first computer implementation of Poincaré's Polyhedron Theorem, which works in the more general setting of geometrically finite Kleinian

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groups. He computed the fundamental polytopes for many Bianchi groups. From this data, he computed presentations for the Bianchi groups and calculated the rank of their abelianizations. Another method is to use reduction theory. An algorithm of Swan [20] has been very recently implemented by Rahm and Fuchs [17], who used it to compute the integral homology groups of all Bianchi groups which are over imaginary quadratic fields of class number less than three.

In this paper, we investigate the structure of these ideal polytopes for a large range of imaginary quadratic fields. Our approach and implementation works for general imaginary quadratic fields, but we restrict the range to ease the computation. We compute the ideal polytope classes for all imaginary quadratic fields of class number one and two, as well as some fields of higher class number with small discriminant. Specifically, we compute the ideal polytopes for the fields $\mathbb{Q}(\sqrt{d})$ for square-free d, where

$$-d \in \{1, \cdots, 100, 115, 123, 163, 187, 235, 267, 403, 427\}.$$

There is no theoretical obstruction to computing these tessellations for higher class number and higher discriminant.

The structure of the paper is as follows. We set the notation for the quadratic fields and Hermitian forms in Section 2. The implementation is described in Section 3. Finally, in Section 4, we summarize some of the data collected so far. Finally, we describe a general result of Macbeath on computing group presentations for groups of homeomorphisms, illustrating one possible use of this data. We use this technique to give an explicit presentation for $\mathrm{GL}_2(\mathbb{Q}(\sqrt{-14}))$ in Section 5.

2 Notation and Background

Let $F = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$ be an imaginary quadratic number field. We always take $d < 0$ to be a square-free integer. Let $\mathcal{O} \subset F$ denote the ring of integers in F. Then $\mathcal{O}$ has a $\mathbb{Z}$-basis consisting of 1 and ω, where

$$\omega = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \bmod 4, \\ \sqrt{d} & \text{if } d \equiv 2, 3 \bmod 4. \end{cases}$$

Let $\bar{\cdot}$ denote complex conjugation, the nontrivial Galois automorphism of F.

Definition 1. *A* binary Hermitian form over F *is a map* $\phi : F^2 \to \mathbb{Q}$ *of the form*

$$\phi(x, y) = ax\bar{x} + bx\bar{y} + \bar{b}\bar{x}y + cy\bar{y},$$

where $a, c \in \mathbb{Q}$ *and* $b \in F$ *such that* ϕ *is positive definite.*

By choosing a $\mathbb{Q}$-basis for F, ϕ can be viewed as a quadratic form over $\mathbb{Q}$. In particular, it follows that $\phi(\mathcal{O}^2)$ is discrete in $\mathbb{Q}$.

Definition 2. *The* minimum of ϕ *is*

$$m(\phi) = \inf_{v \in \mathcal{O}^2 \setminus \{0\}} \phi(v).$$

A vector $v \in \mathcal{O}^2$ *is* minimal vector *for* ϕ *if* $\phi(v) = m(\phi)$. *The set of minimal vectors for* ϕ *is denoted* $M(\phi)$.

Definition 3. *A Hermitian form over* F *is* perfect *if it is uniquely determined by* $M(\phi)$ *and* $m(\phi)$.

3 Implementation

3.1 Cone of Hermitian Forms and Hyperbolic Space

The space of positive definite binary Hermitian forms over F form an open cone in a real vector space. There is a natural decomposition of this cone into polyhedral cones corresponding to the facets of the Voronoï polyhedron Π [11, 13, 1]. The top-dimensional cones of this decomposition correspond to perfect forms and descend to ideal polytopes in $\mathbb{H}_3$, 3-dimensional hyperbolic space. Details are given below.

Let $\mathbf{G}$ be the restriction of scalars $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$. Then the group of rational points $\mathbf{G}(\mathbb{Q}) = \mathrm{GL}_2(F)$, and the group of real points is $G = \mathbf{G}(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{C})$. Let $\mathbb{H}_3$ be hyperbolic 3-space:

$$\mathbb{H}_3 = \{(z,t) \ : \ z \in \mathbb{C}, \quad t \in \mathbb{R}_{>0}\}.$$

Then G acts on $\mathbb{H}_3$ by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot (z,t) = (z^*, t^*), \quad \text{where}$$

$$z^* = \frac{(\alpha z + \beta)(\overline{\gamma z + \delta}) + (\alpha t)(\overline{\gamma t})}{|\gamma z + \delta|^2 + |\gamma|^2 t^2} \quad \text{and} \quad t^* = \frac{|\alpha\delta - \beta\gamma| t}{|\gamma z + \delta|^2 + |\gamma|^2 t^2}$$

Note that diagonal matrices act trivially on $\mathbb{H}_3$, and the stabilizer of the point $(i, 1)$ is $U(2)$. Thus one gets an identification between $\mathbb{H}_3$ and the coset space $\mathrm{GL}_2(\mathbb{C})/(U(2) \cdot \mathbb{R}_{>0})$.

A binary Hermitian form can be identified with the 4-dimensional real vector space V of Hermitian 2×2 matrices. The group $\mathrm{GL}_2(\mathbb{C})$ acts on this space via

$$g \cdot A = gAg^*$$

and preserves the open cone $C \subset V$ of positive definite Hermitian matrices, and the stabilizer of I is $U(2)$. Thus one has identification $C \simeq \mathrm{GL}_2(\mathbb{C})/U(2)$. Modding out by homotheties, one gets

$$C/\mathbb{R}_{>0} \simeq \mathbb{H}_3. \tag{1}$$

3.2 Voronoï Decomposition

There is a map q from $\mathcal{O}^2$ to the closure $\bar{C}$ of $C \subset V$ given by $q(v) = vv^*$. The Voronoï polyhedron Π is the unbounded polytope gotten by taking the convex hull of $\{q(v) : v \in \mathcal{O}^2 \setminus 0\}$. Taking cones over the facets of Π, one gets a decomposition of C into polyhedral cones known as the *Voronoï decomposition* of C. By (1), this decomposition descends to a tessellation of $\mathbb{H}_3$ by ideal polytopes. Note that the group $\Gamma = \mathbf{G}(\mathbb{Z}) = \mathrm{GL}_2(\mathcal{O})$ acts on C and preserves this decomposition.

3.3 Perfect Forms

A perfect form ϕ is uniquely determined by its minimum $m(\phi)$ and set of minimal vectors $M(\phi)$. By scaling, we can assume $m(\phi) = 1$. Since each minimal vector defines a linear equation in V, and V is 4-dimensional, generically 4 minimal vectors will uniquely determine ϕ. Note that this does not imply that $\#M(\phi) = 4$. Indeed in many examples, one has $M(\phi) > 4$.

There is a bijection between perfect forms over F and the facets of Π. Let P be a facet of Π with vertices $\{w_1, \ldots, w_k\}$. Then there is a unique form $\phi_P \in C$ such that $m(\phi_P) = 1$ and

$$\{q(v) : v \in M(\phi_P)\} = \{w_1, \ldots, w_k\}.$$

There is an algorithm [11] that uses this bijection to compute the $\mathrm{GL}_2(\mathcal{O})$-equivalency classes of perfect forms. The algorithm uses linear algebra and convex geometry, but requires an initial input of a perfect form. To this end, we describe the method that we used to compute an initial perfect form.

For each field $F = \mathbb{Q}(\sqrt{d})$, we need only to find a single perfect form to begin the algorithm. Thus we limit our search to a particular family of quadratic forms. Specifically, let $S_0 \subset C$ be the subset of quadratic forms ϕ such that

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq M(\phi).$$

For $\phi \in S_0$, the Hermitian matrix A_ϕ associated to ϕ must have the form

$$A_\phi = \begin{bmatrix} 1 & \beta \\ \bar{\beta} & 1 \end{bmatrix}, \quad \text{where } \beta \in F \text{ with } \mathrm{Re}(\beta) = -\frac{1}{2} \text{ and } |\beta| < 1.$$

If $\phi \in S_0$ and ϕ has an additional minimal vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathcal{O}^2$, then

$$\beta = -\frac{1}{2} + \left(\frac{1 - {a_1}^2 + {a_2}^2 d + a_1 b_1 - a_2 d b_2 - {b_1}^2 + {b_2}^2 d}{2\, d a_1 b_2 - 2\, d a_2 b_1} \right) \sqrt{d}, \tag{2}$$

where $a = a_1 + a_2\sqrt{d}$ and $b = b_1 + b_2\sqrt{d}$. Combined with (2), this implies

$$-\frac{\left(1 - {a_1}^2 + {a_2}^2 d + a_1 b_1 - a_2 d b_2 - {b_1}^2 + {b_2}^2 d\right)^2 d}{\left(2\, d a_1 b_2 - 2\, d a_2 b_1\right)^2} < \frac{3}{4}. \tag{3}$$

Reduction theory, specifically the existence of Siegel sets, ensures that the values $\mathrm{N}_{F/\mathbb{Q}}(a)$, $\mathrm{N}_{F/\mathbb{Q}}(b)$, and $\mathrm{N}_{F/\mathbb{Q}}(b-a)$ for a solution are bounded above by a constant depending upon d. Thus we implement a brute force search over $a, b \in \mathcal{O}$ beginning at 0 and moving out. When a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is found satisfying (3), we check that the corresponding form ϕ satisfies

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\} \subseteq M(\phi).$$

This corresponds to a ideal polytope whose vertices contain $\{\infty, 0, 1, \frac{a}{b}\}$.

Once the initial form is found, we implement the algorithm of [11] to find all the perfect forms over F up to the action of $\mathrm{GL}_2(\mathcal{O})$ (and the corresponding structure of the Voronoï polyhedron) in Magma [4]. This descends, via (1), to give a tessellation of $\mathbb{H}_3$ by ideal polytopes.

4 Polytope Data

In this section we collect the results of the computations of the $\mathrm{GL}_2(\mathcal{O})$-conjugacy classes of the ideal Voronoï polytopes.

4.1 Example: $d = -14$

Let $F = \mathbb{Q}(\sqrt{-14})$. Then F has class number four and ring of integers $\mathcal{O} = \mathbb{Z}[\omega]$, where $\omega = \sqrt{-14}$. There are 9 $\mathrm{GL}_2(\mathcal{O})$-classes of polytopes which are of 3 combinatorial types. There are 3 triangular prisms with cuspidal vertices

$$P_1 = \left\{\infty, 1, \frac{5+2\omega}{9}, \frac{2+\omega}{4}, \frac{4+2\omega}{9}, 0\right\}$$

$$P_2 = \left\{\frac{11+4\omega}{23}, 1, \frac{5+2\omega}{9}, \frac{4+2\omega}{9}, \frac{12+4\omega}{23}, 0\right\}, \quad \text{and}$$

$$P_3 = \left\{\frac{8+5\omega}{23}, \frac{2+\omega}{5}, \frac{1+\omega}{5}, \frac{2+\omega}{6}, \frac{3+2\omega}{10}, \frac{7+4\omega}{21}\right\},$$

and 5 tetrahedra with cuspidal vertices

$$T_1 = \left\{\frac{11+4\omega}{23}, \frac{2+\omega}{5}, \frac{4+2\omega}{9}, 0\right\},$$

$$T_2 = \left\{1, \frac{5+2\omega}{9}, \frac{3+\omega}{5}, \frac{12+4\omega}{23}\right\},$$

$$T_3 = \left\{\frac{11+4\omega}{23}, \frac{2+\omega}{5}, \frac{2+\omega}{6}, 0\right\},$$

$$T_4 = \left\{\frac{8+5\omega}{23}, \frac{2+\omega}{5}, \frac{4+2\omega}{9}, 0\right\}, \quad \text{and}$$

$$T_5 = \left\{\frac{4+\omega}{6}, 1, \frac{3+\omega}{5}, \frac{12+4\omega}{23}\right\},$$

and a square pyramid with cuspidal vertices

$$S = \left\{ \frac{8+5\omega}{23}, \frac{2+\omega}{5}, \frac{1+\omega}{5}, \frac{2+\omega}{6}, 0 \right\}.$$

Given the cuspidal vertices, one can easily compute the stabilizers of each polytope. The stabilizers are all cyclic in this case. For each stabilizer, we compute a generator. The results are given in Table 1.

Table 1. Stabilizer groups of Voronoï ideal polytopes for $\mathbb{Q}(\sqrt{-14})$

Polytope	Stabilizer	Generator
P_1	C_6	$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$
P_2	C_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
P_3	C_4	$\begin{bmatrix} \omega+1 & -\omega+6 \\ 2 & -\omega-1 \end{bmatrix}$
T_1	C_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
T_2	C_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
T_3	C_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
T_4	C_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
T_5	C_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
S	C_2	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4.2 Polytope Summary

We compute the Voronoï polytopes for all imaginary quadratic number fields $F = \mathbb{Q}(\sqrt{d})$ with class number one and two as well as higher class number for $d > -100$. Although there is no reason an arbitrary convex 3-dimensional polytope could not arise, in all of these cases only 8 combinatorial types show up. We give the names and F-vector ([#vertices, #edges, #faces]) for each in Table 2. We also note that the triangular dipyramid shows up in this range much less frequently than the other polytopes.

In Table 3, we give the number of $\mathrm{GL}_2(\mathcal{O})$-classes of each polytope type for F with class number one or two. In Table 4, we give the number of $\mathrm{GL}_2(\mathcal{O})$-classes of each polytope type for the remaining imaginary quadratic fields with $d > -100$.

Table 2. Combinatorial types of ideal polytopes that occur in this range

polytope	F-vector	picture
tetrahedron	$[4, 6, 4]$	
octahedron	$[6, 12, 8]$	
cuboctahedron	$[12, 24, 14]$	
triangular prism	$[6, 9, 5]$	
hexagonal cap	$[9, 15, 8]$	
square pyramid	$[5, 8, 5]$	
truncated tetrahedron	$[12, 18, 8]$	
triangular dipyramid	$[5, 9, 6]$	

5 Group Presentation

A general result of Macbeath [15] and analogous result of Weil [21] give a general method of computing group presentations for groups of homeomorphisms. For the convenience of the reader, we recall these results here and describe how the polytope data computed above can be used to compute explicit presentations of $\mathrm{GL}_2(\mathcal{O}_F)$.

Consider a connected space X acted upon by a group of homeomorphisms Γ. Let $U \subset X$ be an open set such that $\Gamma \cdot U = X$, and let $\Sigma \subset \Gamma$ denote the set

$$\Sigma = \{g \in \Gamma : g \cdot U \cap U \neq \emptyset\}.$$

Let $F(\Sigma)$ be the free group generated by Σ. For $g \in \Sigma$, let x_g denote the corresponding element of $F(\Sigma)$. Let $W \subset \Sigma \times \Sigma$ denote the set

$$W = \{(g, h) : U \cap g \cdot U \cap gh \cdot U \neq \emptyset\}.$$

Let $R \subset F(\Sigma)$ denote the subgroup generated by $x_g x_h x_{(gh)^{-1}}$ for $(g, h) \in W$. Suppose $\pi_0(X) = \pi_1(X) = \pi_0(U) = 1$. Then the subgroup R is a normal subgroup of $F(\Sigma)$ and $\Gamma \simeq F(\Sigma)/R$.

To apply this result to the polytope data computed above, choose $X = \mathbb{H}_3$. Fix representatives $P_1, \ldots, P_k$ of the $\mathrm{GL}_2(\mathcal{O})$ classes of polytopes such that $D = P_1 \cup \cdots \cup P_k$ is a connected set of polytopes meeting along facets. Let $U \subset \mathbb{H}_3$ be an open neighborhood of $D \cap \mathbb{H}_3$. We note that since the vertices D are at

Table 3. $GL_2(\mathcal{O})$-classes of Voronoï ideal polytopes for class number one and two

h_F	d								
1	−1	0	1	0	0	0	0	0	0
1	−2	0	0	1	0	0	0	0	0
1	−3	1	0	0	0	0	0	0	0
1	−7	0	0	0	1	0	0	0	0
1	−11	0	0	0	0	0	0	1	0
1	−19	0	0	1	1	0	0	0	0
1	−43	0	0	0	2	1	0	1	0
1	−67	0	1	0	2	1	2	1	0
1	−163	11	0	1	8	2	3	0	0
2	−5	0	0	0	2	0	0	0	0
2	−6	0	0	0	0	1	0	1	0
2	−10	0	1	0	1	0	2	0	0
2	−13	1	0	0	3	1	1	0	0
2	−15	1	1	0	0	0	0	0	0
2	−22	5	0	1	4	0	2	0	0
2	−35	3	4	0	1	0	2	0	0
2	−37	10	0	0	8	1	8	0	0
2	−51	1	0	1	2	1	0	1	0
2	−58	47	0	0	7	2	6	0	0
2	−91	5	1	0	5	0	3	0	0
2	−115	3	1	0	5	2	4	0	0
2	−123	1	1	1	6	3	3	1	0
2	−187	18	1	1	4	1	9	1	0
2	−235	13	1	0	12	4	11	0	0
2	−267	24	1	1	13	5	10	1	0
2	−403	66	1	0	16	2	20	0	2
2	−427	65	2	0	19	4	24	0	0

Table 4. $GL_2(\mathcal{O})$-classes of Voronoï ideal polytopes with $d > -100$

h_F	d								
3	−23	0	1	0	1	0	1	0	0
3	−31	0	0	0	3	0	1	0	0
3	−59	0	1	1	3	0	2	0	0
3	−83	6	0	0	2	2	1	1	0
4	−14	5	0	0	3	0	1	0	0
4	−17	5	0	0	2	1	3	1	0
4	−21	8	2	0	2	1	4	0	0
4	−30	6	0	0	6	4	4	0	0
4	−33	9	0	1	8	1	6	1	0
4	−34	20	0	0	3	1	6	1	0
4	−39	1	0	0	3	1	1	0	0
4	−46	32	1	0	5	0	9	0	0
4	−55	5	1	0	2	0	2	0	0
4	−57	33	1	0	10	3	14	2	0
4	−73	57	1	1	13	1	14	0	2
4	−78	69	1	0	11	4	18	0	0
4	−82	92	0	0	8	3	11	1	0
4	−85	56	0	0	17	0	28	0	0
4	−93	79	1	0	20	7	21	0	0
4	−97	95	0	1	19	3	19	0	0
5	−47	5	0	0	1	1	2	0	0
5	−79	9	0	0	5	0	4	0	0
6	−26	18	1	0	2	1	4	0	0
6	−29	15	0	0	6	0	6	0	0
6	−38	33	1	0	2	1	6	1	0
6	−53	45	0	0	7	2	13	0	0
6	−61	41	1	0	11	1	16	0	0
6	−87	6	0	0	6	2	3	0	0
7	−71	7	1	0	4	0	4	0	0
8	−41	31	0	1	9	0	8	0	0
8	−62	81	0	0	7	2	7	0	0
8	−65	69	2	0	9	0	19	0	0
8	−66	67	1	1	9	4	12	1	0
8	−69	51	2	0	15	2	21	0	0
8	−77	81	1	0	9	2	26	0	0
8	−94	125	1	0	10	2	17	0	0
8	−95	12	0	0	4	0	9	0	0
10	−74	105	1	0	9	1	12	0	0
10	−86	130	0	0	9	1	18	1	0
12	−89	136	0	0	14	1	21	1	0

infinity, the set U can be chosen so that if $g \in \Sigma$, then g takes an edge of D to another edge of D.

We remark that many redundant generators and relations are created when implementing this result, especially when the stabilizer groups of the polytopes are large. We can compensate for this using Magma's commands for simplifying finitely-presented groups. We illustrate the technique in the example below.

5.1 Example: $d = -14$

Theorem 1. *Let $F = \mathbb{Q}(\sqrt{-14})$ with ring of integers $\mathcal{O} = \mathbb{Z}[\omega]$, where $\omega = \sqrt{-14}$. Then the following is a presentation of $\mathrm{GL}_2(\mathcal{O})$:*

$$\mathrm{GL}_2(\mathcal{O}) = \langle g_1, \cdots, g_8 : R_1 = \cdots = R_{22} = 1 \rangle, \quad \textit{where}$$

$$R_1 = g_7^2, \quad R_2 = g_8^2, \quad R_3 = g_6^2, \quad R_4 = g_3^2,$$
$$R_5 = g_4^2, \quad R_6 = g_2^2, \quad R_7 = g_5^4, \quad R_8 = (g_2 g_1^{-1})^2,$$
$$R_9 = (g_4 g_1)^2, \quad R_{10} = g_5^{-1} g_1^{-3} g_5^{-1}, \quad R_{11} = (g_7 g_5^{-2})^2, \quad R_{12} = (g_8 g_5^{-2})^2,$$
$$R_{13} = (g_6 g_5^{-2})^2, \quad R_{14} = (g_4 g_5^{-2})^2, \quad R_{15} = (g_3 g_5^{-2})^2, \quad R_{16} = (g_6 g_1^{-1} g_5^{-1})^2,$$

$$R_{17} = (g_3 g_5^{-1} g_3 g_1 g_2)^2, \quad R_{18} = (g_3 g_7 g_1 g_8 g_1^{-1})^2, \quad R_{19} = g_4 g_5 g_4 g_1^{-1} g_5 g_1,$$

$$R_{20} = g_8 g_5^{-1} g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_3 g_7 g_1 g_8 g_3 g_5 g_7 g_5^{-1},$$
$$R_{21} = g_1 g_5 g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_1 g_5^{-1} g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7,$$
$$R_{22} = g_6 g_5 g_7 g_5^{-1} g_3 g_1^{-1} g_3 g_7 g_1 g_6 g_1^{-1} g_7 g_3 g_1 g_3 g_5 g_7 g_5.$$

Proof. We choose X, U, and D as described above. In fact, one can choose D to be the polytopes given in Section 4.1. Then $F(\Sigma)/R$ is defined by 235 generators and 3416 relations. We can simplify this presentation in Magma to get the presentation of $\mathrm{GL}_2(\mathbb{Z}[\sqrt{-14}])$ above, with

$$g_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$g_3 = \begin{bmatrix} \omega+3 & -\omega+1 \\ 6 & -\omega-3 \end{bmatrix}, \qquad g_4 = \begin{bmatrix} 4\omega & -2\omega+13 \\ 2\omega+13 & -4\omega \end{bmatrix},$$
$$g_5 = \begin{bmatrix} -2\omega-5 & 2\omega-3 \\ -10 & 2\omega+5 \end{bmatrix}, \qquad g_6 = \begin{bmatrix} -5\omega & 3\omega-15 \\ -3\omega-15 & 5\omega \end{bmatrix},$$
$$g_7 = \begin{bmatrix} \omega+9 & -2\omega-1 \\ -2\omega+10 & -\omega-9 \end{bmatrix}, \qquad g_8 = \begin{bmatrix} -2\omega-13 & 4\omega+4 \\ \omega-14 & 2\omega+13 \end{bmatrix}.$$

The presentation given in the theorem has torsion elements as generators. In particular, $\mathrm{GL}_2(\mathcal{O})$ is generated by elements of order 2, 4, and 6. Since any torsion-free quotient must map these generators to the identity, one immediately gets the following corollary.

Corollary 1. $\mathrm{GL}_2(\mathbb{Z}[\sqrt{14}])$ *has no torsion-free quotients.*

One finds similar results for $F = \mathbb{Q}(\sqrt{d})$ for $d = -1$ and $d = -3$ in [8].

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