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Binary Hermitian forms over a cyclotomic field

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ABSTRACT

Let ζ be a primitive fifth root of unity and let F be the cyclotomic field $F = \mathbb{Q}(\zeta)$. Let $\mathcal{O} \subset F$ be the ring of integers. We compute the Voronoï polyhedron of binary Hermitian forms over F and classify $\mathrm{GL}_2(\mathcal{O})$-conjugacy classes of perfect forms. The combinatorial data of this polyhedron can be used to compute the cohomology of the arithmetic group $\mathrm{GL}_2(\mathcal{O})$ and Hecke eigenforms.

1. Introduction

Let $F/\mathbb{Q}$ be a number field, and let $\mathcal{O}$ denote its ring of integers. There exists an algorithm to compute all of the $\mathrm{GL}_n(\mathcal{O})$-equivalence classes of perfect n-ary quadratic forms over F once an initial perfect form is found [14,16]. This is investigated in the totally real number field case in [13,17,19]. We remark that a different notion of perfection has been investigated in [3,6,20]. In this paper, we consider the cyclotomic field $\mathbb{Q}(\zeta_5)$.

Although Hermitian forms over number fields are of interest in their own right, our main motivation for this computation comes from an investigation of a Taniyama–Shimura type correspondence over F, relating Hecke eigenforms over F with integer eigenvalues and elliptic curves over F. Due to the work of Wiles et al. such a correspondence is known to exist for $F = \mathbb{Q}$ [9]. It is an open problem for $[F : \mathbb{Q}] > 1$. This has been investigated for F an imaginary quadratic field in [5,7,8,18,21,22] and for F a real quadratic field in [10,11]. In an ongoing project joint with P. Gunnells, F. Hajir, and D. Ramakrishnan, we are investigating the complex quartic field $F = \mathbb{Q}(\zeta_5)$. To this end, we first compute the Voronoï polyhedron associated to binary Hermitian forms over $\mathbb{Q}(\zeta_5)$. The Voronoï polyhedron provides a combinatorial structure in which to perform the Hecke eigenvalue computations.

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The results of the computations in this paper could also be used to compute the elliptic points of $\mathrm{GL}_2(\mathcal{O})$ using techniques of [23]. Since the forms and their minimal vectors are given explicitly here, one could also extract invariants such as an additive analogue of the Hermite constant for $\mathbb{Q}(\zeta_5)$ [3,20].

The paper is organized as follows. In Section 2, we set notation. In Sections 3–5 we recall the Voronoï polyhedron and its relation to Hermitian forms over $\mathbb{Q}(\zeta_5)$. The Voronoï polyhedron for $\mathbb{Q}(\zeta_5)$ is computed in Section 6.

2. Notation

First we set notation and recall and collect a few basic facts from algebraic number theory that will be used later.

2.1. Field

Let $\zeta = \zeta_5 = e^{2\pi i/5}$ be a primitive fifth root of unity, and let F be the cyclotomic field $F = \mathbb{Q}(\zeta)$. Let $\mathcal{O} \subset F$ denote the ring of integers, and let $\bar{\cdot}$ denote complex conjugation. Let $k \subset F$ denote the real subfield $k = \mathbb{Q}(\sqrt{5})$, and let $\mathcal{O}_k$ denote its ring of integers. Then $u_5 = (1+\sqrt{5})/2$ is a fundamental unit for k. Let $\mathcal{O}_k^+$ denote the totally positive elements of $\mathcal{O}_K$.

Let $\iota = (\iota_1, \iota_2)$ denote the (non-complex conjugate) embeddings

$$\iota : F \to \mathbb{C} \times \mathbb{C}$$

given by sending $\sqrt{5}$ to $(\sqrt{5}, -\sqrt{5})$, or equivalently given by sending ζ to (ζ, ζ^3). Denote the non-trivial embedding by $\cdot'$. Specifically, for $\alpha \in F$, let (α, α') denote $\iota(\alpha)$.

2.2. Binary Hermitian forms over F

Definition 2.1. A *binary Hermitian form over* F is a map $\phi : F^2 \to k$ of the form

$$\phi(x, y) = a x\bar{x} + b x\bar{y} + \bar{b}\bar{x} y + c y\bar{y},$$

where $a, c \in k$ and $b \in F$ such that ϕ and ϕ', viewed as forms on $F \otimes_{\mathbb{Q}} \mathbb{R}$ are positive definite.

Note that $\hat{\phi} = \phi + \phi'$ takes values in $\mathbb{Q}$. Indeed, $\hat{\phi}$ is precisely the composition $\mathrm{Tr}_{k/\mathbb{Q}} \circ \phi$, and by choosing a $\mathbb{Q}$-basis for F, $\hat{\phi}$ can be viewed as a quadratic form over $\mathbb{Q}$. In particular, it follows that $\hat{\phi}(\mathcal{O}^2)$ is discrete in $\mathbb{Q}$.

Using $\hat{\phi}$, we can define minimal vectors and perfection. Specifically, the *minimum of* ϕ is

$$m(\phi) = \inf_{v \in \mathcal{O}^2 \setminus \{0\}} \hat{\phi}(v).$$

A vector $v \in \mathcal{O}^2$ is *minimal vector* for ϕ if $\phi(v) = m(\phi)$. The set of minimal vectors for ϕ is denoted $M(\phi)$. A Hermitian form over F is *perfect* if it is uniquely determined by $M(\phi)$ and $m(\phi)$.

3. Self-adjoint homogeneous cone

3.1. Symmetric space

Let $\mathbf{G}$ be the $\mathbb{Q}$-group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$ and let $G = \mathbf{G}(\mathbb{R})$ the corresponding group of real points. Let $K \subset G$ be a maximal compact subgroup, and let A_G be the identity component of the maximal $\mathbb{Q}$-split torus in the center of G. Then the symmetric space associated to G is a 7-dimensional space $X = G/KA_G$.

3.2. Cone of Hermitian forms over $\mathbb{C}$

Every Hermitian form over $\mathbb{C}$ can be represented by a Hermitian matrix. Let C be the cone of positive definite binary complex Hermitian forms, viewed as a subset of V, the $\mathbb{R}$-vector space of 2×2 complex Hermitian matrices. Let $\cdot^*$ denote complex conjugate transpose. Then the usual action of $\mathrm{GL}_2(\mathbb{C})$ on C is given by

$$(g \cdot \phi)(v) = \phi(g^* v), \quad \text{where } g \in \mathrm{GL}_2(\mathbb{C}) \text{ and } \phi \in C. \tag{1}$$

3.3. Cone of Hermitian forms over F

Let $\mathbf{V}$ be the Weil restriction $\mathrm{Res}_{F/\mathbb{Q}} H_2$ of the variety H_2 of 2×2 complex Hermitian matrices defined over F. Then $\mathcal{V} := \mathbf{V}(\mathbb{R}) \simeq V \times V$ and

$$\mathbf{V}(\mathbb{Q}) \simeq H_2(F) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} : a, c \in k \text{ and } b \in F \right\}.$$

Let $\mathcal{C}$ be the cone $\mathcal{C} = C \times C \subset \mathcal{V}$. Since C is the space of positive-definite Hermitian forms on $\mathbb{C}^2$, we can use ι to view $\mathcal{C}$ as the space of forms on F^2. Specifically, for $\phi = (\phi_1, \phi_2) \in \mathcal{C}$ and $v \in F^2$, we define

$$\phi(v) = \phi_1\big(\iota_1(v)\big) + \phi_2\big(\iota_2(v)\big).$$

Definition 3.1. Let ϕ be a Hermitian form over F. A 2×2 matrix A with coefficients in F is *associated to* ϕ if $\phi = \iota(A)$.

Since the map ι is injective, this matrix is unique and will be denoted A_ϕ. For $v \in F^2$, then $\phi(v)$ is just the trace. Specifically, if $\phi \in \mathbf{V}(\mathbb{Q})$ with associated matrix A_ϕ, then

$$\phi(v) = \mathrm{Tr}_{k/\mathbb{Q}}(v^* A_\phi v).$$

3.4. $\mathcal{C}$ as a symmetric space

The embedding ι gives an isomorphism

$$G \to \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}). \tag{2}$$

Under this identification, $\iota(A_G) = \{(rI, rI) \mid r > 0\}$, where I is the 2×2 identity matrix.

Combining (1) and (2), we get an action of G on $\mathcal{C}$. Let ϕ_0 denote the binary Hermitian form represented by I. Then the stabilizer in Γ of $\iota(\phi_0)$ is a maximal compact subgroup K. The group A_G acts on $\mathcal{C}$ by positive real homotheties, and we have

$$X = \mathcal{C}/\mathbb{R}_{>0} \simeq X_{\mathbb{C}} \times X_{\mathbb{C}} \times \mathbb{R},$$

where $X_{\mathbb{C}}$ is the symmetric space for $\mathrm{SL}_2(\mathbb{C})$.

3.5. $\mathcal{C}$ as a self-adjoint homogeneous cone

Let $\bar{\mathcal{C}}$ denote the closure of $\mathcal{C}$ in $\mathcal{V}$. Each vector $w \in \mathbb{C}^2$ gives a rank 1 Hermitian form ww^* (here w is viewed as a column vector). Combined with ι, we get a map $q : F^2 \to \bar{\mathcal{C}}$ given by

$$q(v) = (vv^*, v'v'^*).$$

The cone $\mathcal{C}$ is a *self-adjoint homogeneous cone* [1]. In particular, $\mathcal{C}$ is endowed with a scalar product, and the interpretation of $\mathcal{C}$ as a space of forms over F is reflected in the scalar product. Specifically, suppose $\phi \in \mathbf{V}(\mathbb{Q})$ and $v \in \mathcal{O}^2$. Then $\langle \phi, q(v) \rangle = \phi(v)$.

4. Voronoï polyhedron

In this section we recall properties of the Voronoï polyhedron. This section follows [15], and more details can be found there.

4.1. Rational structure of $\mathcal{V}$

Fix $\Lambda \subset \mathcal{V}$ to be the lattice generated by $q(v)$ for $v \in \mathcal{O}^2$, and let $\Lambda' = \Lambda \setminus \{0\}$. We will refer to points of $\mathcal{C} \cap \mathbf{V}(\mathbb{Q})$ as *Hermitian forms over F*.

Let $R(v)$ be the ray $\mathbb{R}_{>0} \cdot q(v) \subset \bar{\mathcal{C}}$. The set of *rational boundary components* $\mathcal{C}_1$ of $\mathcal{C}$ is the set of rays of the form $R(v)$, $v \in F^2$ [1].

Definition 4.1. The *Voronoï polyhedron Π* is the closed convex hull in $\bar{\mathcal{C}}$ of the points $\mathcal{C}_1 \cap \Lambda'$.

4.2. Voronoï decomposition

By construction $\mathrm{GL}_2(\mathcal{O})$ acts on Π. By taking the cones on the faces of Π, one obtains a *Γ-admissible decomposition* of $\mathcal{C}$ for $\Gamma = \mathrm{GL}_2(\mathcal{O})$ [1]. Since the action of $\mathrm{GL}_2(\mathcal{O})$ commutes with the homotheties, this decomposition descends to a $\mathrm{GL}_2(\mathcal{O})$-equivariant tessellation of X by ideal 7-dimensional polytopes.

5. Primitivity and minimal vectors

There is another notion of minimal vectors that is explored in [1,14]. In this section, we show that our notion of minimality agrees with theirs in this case by first defining primitive points for the case of interest and examining both notions of minimal vectors. This should follow from general results of [16].

The statements in this section are valid only because the class number of k is 1. The results should extend more generally to a CM field F with real maximal subfield k of class number 1.

5.1. Primitive points

There are several notions of primitivity that we will need.

Definition 5.1. A vector $v = \left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] \in \mathcal{O}^2$ is *primitive* if the ideal $(\alpha, \beta) = \mathcal{O}$.

Proposition 5.2. *Let $u, v \in \mathcal{O}^2$ be primitive vectors. Then $q(u) = q(v)$ if and only if $u = \tau v$ for some torsion unit $\tau \in \mathcal{O}$.*

Proof. One direction is immediate. Specifically, if $\tau \in \mathcal{O}$ is a torsion unit, then $\tau\bar{\tau} = \tau'\bar{\tau}' = 1$, and thus $q(\tau v) = q(v)$ follows from the definition of Λ.

For the converse, if $v = \left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]$ then $u = \left[\begin{smallmatrix}\xi\alpha \\ \eta\beta\end{smallmatrix}\right]$ for some $\xi, \eta \in F$. Since $q(u) = q(v)$,

$$\xi\bar{\xi} = \eta\bar{\eta} = 1 \quad \text{and} \quad \xi\bar{\eta} = \bar{\xi}\eta = 1.$$

It follows that $u = \xi v$ for some $\xi \in F$ that satisfies $\xi\bar{\xi} = 1$. Write ξ as $\xi = \lambda/\mu$ for some $\lambda, \mu \in \mathcal{O}$ with $(\lambda, \mu) = \mathcal{O}$. Since $\xi\alpha \in \mathcal{O}$, it follows that $\mu \mid \alpha$. Similarly, since $\xi\beta \in \mathcal{O}$, it follows that $\mu \mid \beta$. Since v is primitive, μ is a unit, and hence $\xi \in \mathcal{O}$. Then ξ must be a torsion unit because $\xi\bar{\xi} = 1$. □

Definition 5.3. A form $\phi \in \Lambda \cap \bar{C}$ is *primitive form* if there exists a matrix $A = \left[\begin{smallmatrix} a & c \\ \bar{c} & b \end{smallmatrix}\right]$ with $\gcd(a, b) = 1$ and $\iota(A) = \phi$.

Note that if $\phi = \iota(A)$ is primitive, then $c \in \mathcal{O}$ and $a, b \in \mathcal{O}_k^+$.

Lemma 5.4. *Let $a, b \in \mathcal{O}_k^\times$ be totally positive with $\gcd(a, b) = 1$. Then $ab \in N_{F/k}(F)$ if and only if $a \in N_{F/k}(F)$ and $b \in N_{F/k}(F)$.*

Proof. If a and b are norms, then their product is clearly a norm. For the other direction, suppose $m = ab = N_{F/k}(\xi)$ for some $\xi \in F$. Since $\gcd(a, b) = 1$, it suffices to consider the case where m is square-free in k. Factor (ξ) as a product of prime ideals $(\xi) = \prod \pi$ in F. Since m is square-free, we must have that each π lies over 5 or a rational prime p such that $p \equiv 1 \bmod 5$. It follows that (ξ) has a factorization in k

$$(\xi) = \rho \wp_1 \wp_2 \cdots \wp_n,$$

where the $\wp_i$ are prime ideals in k and ρ is either trivial or generated by $\sqrt{5}$. Each of these factors has a generator which is a norm, and so it follows that (a) and (b) have generators which are norms. Specifically, $e_1 a$ and $e_2 b$ are norms for some choice of units $e_1, e_2 \in \mathcal{O}_k^*$. The fundamental unit u_5 of k is positive in one embedding into $\mathbb{R}$ and negative in the other. In particular since a, b, $e_1 a$, and $e_2 b$ are totally positive, e_1 and e_2 must be even powers of u_5. It follows that a and b are themselves norms. □

Proposition 5.5. *Let $\phi \in \Lambda \cap \bar{C}$ be a primitive rank 1 form. Then there exists a primitive vector $v \in \mathcal{O}^2$ such that $q(v) = \phi$.*

Proof. Since ϕ is primitive, there exists a matrix $A = \left[\begin{smallmatrix} a & c \\ \bar{c} & b \end{smallmatrix}\right]$ with $\gcd(a, b) = 1$ and $\iota(A) = \phi$. Since ϕ is rank 1, $c\bar{c} = N_{F/k}(c) = ab$. By Lemma 5.4, there exist $\alpha, \beta \in F$ with $N_{F/k}(\alpha) = a$ and $N_{F/k}(\beta) = b$. Since $\gcd(a, b) = 1$, it follows that $v = (\alpha, \beta)^t$ is a primitive vector with $q(v) = \phi$ as desired. □

Remark. Since Λ is a lattice, there is another notion of primitivity for $\phi \in \Lambda$. Specifically, ϕ is *primitive in Λ* if ϕ can be extended to a basis of the lattice. Note that Proposition 5.5 shows that primitive rank 1 forms correspond to primitive vectors in $\mathcal{O}^2$ and primitive vectors in $\mathcal{O}^2$ give rise to primitive rank 1 forms. However, while primitive rank 1 forms are primitive in Λ, there are many vectors primitive in Λ that are not primitive as forms.

Even when we restrict ourselves to considering vectors that are primitive in Λ and rank 1, if we relax the condition of $\gcd(a, b) = 1$, we can no longer guarantee that this vector comes from a primitive vector in $\mathcal{O}^2$. For example, $4 + u_5$ is totally positive, but there does not exist $\alpha \in F$ such that $\alpha\bar{\alpha} = 4 + u_5$. Thus the point $\iota(\left[\begin{smallmatrix} 4+u_5 & 0 \\ 0 & 0 \end{smallmatrix}\right])$ is not in $q(\mathcal{O}^2)$. However, we do have the following result, which basically says that although there are rank 1 rational forms which are not in the image of $\mathcal{O}^2$, they are contained in $\mathcal{O}_k \cdot q(\mathcal{O}^2)$.

Proposition 5.6. *Let $\phi \in \Lambda' \cap C_1$ be a rank 1 form. Then there exists $\alpha \in \mathcal{O}_k^+$ and a primitive vector $v \in \mathcal{O}^2$ such that $\alpha \cdot q(v) = \phi$.*

Proof. Write ϕ as $\phi = \iota(A)$, where $A = \left[\begin{smallmatrix} a & c \\ \bar{c} & b \end{smallmatrix}\right]$ for some $a, b \in \mathcal{O}_k^+$ and $c \in \mathcal{O}$. If ϕ is primitive, the result follows from Proposition 5.5 with $\alpha = 1$. If ϕ is not primitive, choose $\alpha \in \mathcal{O}_k^+$ such that $\alpha = \gcd(a, b)$. Since ϕ is rank 1, it follows that $c\bar{c} = ab$. In particular $\alpha^2 \mid c\bar{c}$. Since $\alpha \in \mathcal{O}_k$, we have that $\alpha \mid c$ and $\alpha \mid \bar{c}$. Thus we have $\phi = \iota(\alpha A_0)$, where $A_0 = \alpha^{-1} A$. Since A_0 corresponds to a primitive rank 1 form, Proposition 5.5 implies that there exists a primitive vector $v \in \mathcal{O}^2$ such that $q(v) = \iota(A_0)$. The result follows. □

Proposition 5.7. *Let $b \in \mathcal{O}_k^+$. Then there exists an $\alpha \in \mathcal{O} \setminus \{0\}$ and $t \in \mathcal{O}_k^+ \cup \{0\}$ such that*

$$b = \alpha\bar{\alpha} + t.$$

Proof. The square of the fundamental unit $\eta = u_5^2 \in \mathcal{O}_k$ is totally positive. Since $\eta = u_5\bar{u}_5$, multiplication by η acts on $\mathcal{O}_k^+$ and preserves $N_{F/k}(\mathcal{O})$. In particular it suffices to show the result for a fundamental domain for the action of η on $\mathcal{O}_k^+$. Once can take the positive cone spanned by 1 and η^2. It is clear that every point in the cone has the form $\eta^2 + t$ for some $t \in \mathcal{O}_k^+$ except for 1 and 2. The condition is trivially satisfied for 1 and 2. □

5.2. Minimal vectors

There is another notion of minimal vectors and perfect forms described in [14]. Specifically one can define

$$\hat{m}(\phi) = \inf_{\psi \in \Lambda' \cap \mathcal{C}_1} \langle \phi, \psi \rangle$$

and

$$\hat{M}(\phi) = \{\psi \in \Lambda' \cap \mathcal{C}_1 \mid \langle \phi, \psi \rangle = \hat{m}(\phi)\}.$$

Notice that if $\psi \in \hat{M}(\phi)$, then ψ is primitive in Λ. It is clear that $\hat{m}(\phi) \leqslant m(\phi)$. If $\hat{m}(\phi) = m(\phi)$, then

$$\{q(v) \mid v \in M(\phi)\} \subseteq \hat{M}(\phi).$$

Proposition 5.8. *Let $\phi \in \mathcal{C}$. Then*

$$\hat{m}(\phi) = \min_{b \in \mathcal{O}_k^+} m(b \cdot \phi).$$

Proof. For $\psi \in \Lambda' \cap \mathcal{C}_1$, there exists $b \in \mathcal{O}_k^+$ and $v \in \mathcal{O}^2$ such that $\psi = b \cdot q(v)$ by Proposition 5.6. It follows that

$$\langle \phi, \psi \rangle = \langle \phi, b \cdot q(v) \rangle = \mathrm{Tr}_{k/\mathbb{Q}}(A_\phi b v v^*) = \mathrm{Tr}_{k/\mathbb{Q}}(v^* b A_\phi v).$$

Then

$$\hat{m}(\phi) = \inf_{\psi \in \Lambda' \cap \mathcal{C}_1} \langle \phi, \psi \rangle = \inf_{\substack{v \in \mathcal{O}^2 \\ b \in \mathcal{O}_k^+}} \mathrm{Tr}_{k/\mathbb{Q}}(v^* b A_\phi v) = \inf_{b \in \mathcal{O}_k^+} m(b \cdot \phi). \qquad \square$$

Proposition 5.9. *Let $\phi \in \mathcal{C}$. Then $m(\phi) = \hat{m}(\phi)$.*

Proof. By Proposition 5.8,

$$\hat{m}(\phi) = \inf_{b \in \mathcal{O}_k^+} m(b \cdot \phi).$$

In particular $\hat{m}(\phi) \leqslant m(\phi)$, and to prove the result it suffices to show that

$$m(\phi) \leqslant m(b \cdot \phi) \quad \text{for every } b \in \mathcal{O}_k^+. \tag{3}$$

By Proposition 5.7, there exists an $\alpha \in \mathcal{O}$ and $t \in \mathcal{O}_k^+$ such that $b = \alpha\bar{\alpha} + t$. We compute

$$\begin{aligned} \mathrm{Tr}_{k/\mathbb{Q}}(v^* b A_\phi v) &= \mathrm{Tr}_{k/\mathbb{Q}}\big(v^*(\alpha\bar{\alpha} + t)A_\phi v\big) \\ &= \mathrm{Tr}_{k/\mathbb{Q}}\big((\alpha v)^* A_\phi(\alpha v)\big) + \mathrm{Tr}_{k/\mathbb{Q}}(t v^* A_\phi v). \end{aligned}$$

We have that $\mathrm{Tr}_{k/\mathbb{Q}}((\alpha v)^* A_\phi(\alpha v)) \geqslant m(\phi)$. Furthermore $\mathrm{Tr}_{k/\mathbb{Q}}(t v^* A_\phi v) > 0$ since t and $v^* A_\phi v$ are both totally positive. The result follows. $\square$

6. Voronoï cones for F

In this section, we describe the $\mathrm{GL}_2(\mathcal{O})$-conjugacy classes of Voronoï cones. We implement the method described in [14].

6.1. Perfect forms

We find one perfect form ϕ, with associated matrix

$$A_\phi = \frac{1}{5}\begin{bmatrix} \zeta^3 + \zeta^2 + 3 & \zeta^3 - \zeta^2 + \zeta - 1 \\ -2\zeta^3 - \zeta - 2 & \zeta^3 + \zeta^2 + 3 \end{bmatrix}.$$

This is done using `Magma` [4] as follows.

(i) Fix a list $L \subset \mathcal{O}^2$ of vectors large enough so that the conditions

$$\big\{\phi(v) = 1 \,\big|\, v \in L\big\}$$

has a unique solution.

(ii) Ensure that ϕ is positive definite.

(iii) Check that $L \subseteq M(\phi)$.

Step (i) is accomplished by including more vectors into the list L until the linear system has a unique solution. Steps (ii) and (iii) are accomplished by picking a $\mathbb{Z}$-basis for $\mathcal{O}^2$ and expressing ϕ as a quadratic form on $\mathbb{Z}^8$.

The perfect form ϕ has 240 minimal vectors. It is clear that if $v \in M(\phi)$ then $\tau v \in M(\phi)$ for any torsion unit $\tau \in \mathcal{O}$. There are 24 minimal vectors (modulo torsion units). Let ω denote the unit $\omega = \zeta + \zeta^2$. Then the form ϕ has (modulo torsion units) minimal vectors

$$\begin{gathered} \begin{bmatrix} -\zeta + 1 \\ \zeta^3 + 1 \end{bmatrix}, \begin{bmatrix} -\zeta^3 + 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\omega \end{bmatrix}, \begin{bmatrix} 1 \\ -\zeta^2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \zeta^3 \end{bmatrix}, \begin{bmatrix} 1 \\ -\zeta^2 + 1 \end{bmatrix}, \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \zeta^3 + 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \zeta + 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \zeta^3 + \zeta + 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\zeta^4 \end{bmatrix}, \begin{bmatrix} \omega^{-1} \\ \zeta^4 \end{bmatrix}, \begin{bmatrix} \omega^{-1} \\ \zeta^4 - 1 \end{bmatrix}, \\ \begin{bmatrix} \omega^{-1} \\ -1 \end{bmatrix}, \begin{bmatrix} \omega^{-1} \\ -\zeta^3 - 1 \end{bmatrix}, \begin{bmatrix} \omega^{-1} \\ -\zeta^3 - \zeta^2 - 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega + 1 \end{bmatrix}, \begin{bmatrix} \omega \\ -\zeta^3 \end{bmatrix}, \begin{bmatrix} \omega \\ 0 \end{bmatrix}, \\ \begin{bmatrix} \omega \\ \zeta^2 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \omega \end{bmatrix}. \end{gathered} \tag{4}$$

By Proposition 5.2, these give rise to 24 distinct points in $\bar{C}$. These 24 points are vertices of a top-dimensional Voronoï cone. We show in Section 6.2 that this is the only top-dimensional Voronoï cone modulo $\mathrm{GL}_2(\mathcal{O})$.

Proposition 6.1. *There is* 1 $\mathrm{GL}_2(\mathcal{O})$*-conjugacy class of* 8*-dimensional cones. The corresponding perfect form* ϕ *has* (*modulo torsion units*) 24 *minimal vectors.*

6.2. Top cones

The perfect forms correspond to 8-dimensional Voronoï cones. Specifically, each top cone corresponds to a facet F of the Voronoï polyhedron. There is a unique point $\phi_F \in \mathcal{C} \cap \mathbf{V}(\mathbb{Q})$ [14] such that

(i) $F = \{x \in \Pi \mid \langle x, \phi_F \rangle = 1\}$, and
(ii) for all $x \in \Pi \setminus F$, we have $\langle x, \phi_F \rangle > 1$.

Let Z_F be the *minimal vertices of* ϕ_F, the finite set of points $x \in \Lambda' \cap \mathcal{C}_1$ such that $\langle x, \phi_F \rangle = 1$. Then F is the convex hull of Z_F. The form ϕ_F is a perfect form, and

$$\{q(v):\ v \in M(\phi_F)\} = Z_F.$$

For $\gamma \in \mathrm{GL}_2(\mathcal{O})$, let $\Theta\gamma = (\gamma^*)^{-1}$. The action of $\mathrm{GL}_2(\mathcal{O})$ on perfect forms, minimal vectors, and minimal vertices are related by the following.

Proposition 6.2. *Let F be a top cone and let ϕ_F be the corresponding perfect form with minimal vectors $M(\phi_F)$ and minimal vertices Z_F. For $\gamma \in \mathrm{GL}_2(\mathcal{O})$,*

(i) $\Theta\gamma \cdot \phi = \phi_{\gamma \cdot F}$, *and*
(ii) $M(\Theta\gamma \cdot \phi) = \gamma \cdot M(\phi)$.

In particular, two perfect forms are $\mathrm{GL}_2(\mathcal{O})$-conjugate if and only if their minimal vectors or minimal vertices are $\mathrm{GL}_2(\mathcal{O})$-conjugate.

Thus to classify the perfect forms modulo $\mathrm{GL}_2(\mathcal{O})$, one can instead classify the top-dimensional Voronoï cones. The top-dimensional cone corresponding to ϕ, denoted $\mathcal{C}_\phi$ is has faces given by the convex hull of $\{q(v) \mid v \in M(\phi)\}$. The program `Polymake` [12] is used to compute the convex hull. There are 118 codimension 1 faces, corresponding to 118 neighboring top-dimensional Voronoï cones and 118 perfect forms. There are 14 faces with 12 vertices, 80 faces with 9 vertices, and 24 faces with 7 vertices. Using `Magma`, the stabilizer S_ϕ of ϕ is computed. The group S_ϕ has order 600 and `Magma` type $\langle 600, 54\rangle$.

In order to cut down the number of computations that need to be made, we first classify the faces of $\mathcal{C}_\phi$ modulo S_ϕ, the stabilizer of $M(\phi)$. Indeed, let ψ be a perfect form such that $\mathcal{C}_\psi$ meets $\mathcal{C}_\phi$ in a codimension 1 face F. If $\gamma \in S_\phi$, then $\gamma \cdot \psi$ is the perfect form corresponding to top cone $\gamma \cdot \mathcal{C}_\psi$, which meets $\mathcal{C}_\phi$ in a codimension 1 face $\gamma \cdot F$.

It is clear that S_ϕ conjugate faces must have the same number of vertices. One computes that there are 3 orbits of faces with 12 vertices. One orbit $S_\phi \cdot F_1$ consists of 12 faces, and the other 2 orbits consist of 1 element each, denoted F_2 and F_3. One shows that F_2 is $\mathrm{GL}_2(\mathcal{O})$-conjugate to F_3 by computing a matrix that sends F_2 to F_3. There are 4 orbits of faces with 9 vertices. The orbits $S_\phi \cdot F_4$, $S_\phi \cdot F_5$, $S_\phi \cdot F_6$, and $S_\phi \cdot F_7$ consist of 20 faces each. There are 2 orbits of faces with 7 vertices. The orbits $S_\phi \cdot F_8$ and $S_\phi \cdot F_9$ consist of 12 faces each.

To show that there is only 1 $\mathrm{GL}_2(\mathcal{O})$-class of perfect form, it suffices to show that the perfect forms ϕ_i associated to the top cones that neighbor $\mathcal{C}_\phi$ at face F_i are each $\mathrm{GL}_2(\mathcal{O})$-conjugate to ϕ. To this end, we use the following lemma.

Lemma 6.3. *Let F, F' be two faces of $\mathcal{C}_\phi$ with associated perfect forms ψ, ψ'. If $\gamma \in \mathrm{GL}_2(\mathcal{O}) \setminus S_\phi$ and $\gamma \cdot F = F'$, then both ψ and ψ' are $\mathrm{GL}_2(\mathcal{O})$-conjugate to ϕ.*

Proof. Since $\mathcal{C}_\phi$ is a top-dimensional cone, and F and F' are codimension 1 faces of $\mathcal{C}_\phi$, we must have that $\gamma \cdot \mathcal{C}_\phi = \mathcal{C}_\phi$ or $\gamma \cdot \mathcal{C}_\phi = \mathcal{C}_{\psi'}$. Since $\gamma \notin S_\phi$, it follows that $\gamma \cdot \mathcal{C}_\phi = \mathcal{C}_{\psi'}$. Thus $\Theta\gamma \cdot \phi = \psi'$. Repeating the argument using γ^{-1} shows that $\Theta\gamma^{-1} \cdot \phi = \psi$. □

Table 1
$GL_2(\mathcal{O})$-conjugacy classes of Voronoï cones.

Type	# of v	Cone rank	Stabilizer
A	24	8	$\langle 600, 54\rangle$
B_1	9	7	$\langle 30, 4\rangle$
B_2	9	7	$\langle 30, 4\rangle$
B_3	12	7	$\langle 100, 14\rangle$
B_4	12	7	$\langle 600, 54\rangle$
B_5	7	7	$\langle 50, 5\rangle$
C_1	6	6	$\langle 30, 4\rangle$
C_2	6	6	$\langle 10, 2\rangle$
C_3	7	6	$\langle 10, 2\rangle$
C_4	6	6	$\langle 60, 11\rangle$
C_5	6	6	$\langle 50, 5\rangle$
C_6	6	6	$\langle 20, 2\rangle$
C_7	7	6	$\langle 10, 2\rangle$
C_8	6	6	$\langle 20, 2\rangle$
C_9	6	6	$\langle 30, 4\rangle$
C_{10}	6	6	$\langle 50, 5\rangle$
D_1	5	5	$\langle 10, 2\rangle$
D_2	5	5	$\langle 10, 2\rangle$
D_3	5	5	$\langle 20, 5\rangle$
D_4	5	5	$\langle 10, 2\rangle$
D_5	5	5	$\langle 10, 2\rangle$
D_6	5	5	$\langle 10, 2\rangle$
D_7	5	5	$\langle 10, 2\rangle$
D_8	5	5	$\langle 10, 2\rangle$
D_9	5	5	$\langle 20, 5\rangle$
D_{10}	5	5	$\langle 100, 14\rangle$
D_{11}	6	5	$\langle 20, 5\rangle$
E_1	4	4	$\langle 20, 5\rangle$
E_2	4	4	$\langle 20, 2\rangle$
E_3	4	4	$\langle 20, 2\rangle$
E_4	4	4	$\langle 10, 2\rangle$
E_5	4	4	$\langle 10, 2\rangle$
E_6	4	4	$\langle 20, 5\rangle$
E_7	4	4	$\langle 20, 5\rangle$
E_8	4	4	$\langle 10, 2\rangle$
E_9	4	4	$\langle 200, 31\rangle$
F_1	3	3	$\langle 60, 11\rangle$
F_2	3	3	$\langle 20, 5\rangle$
F_3	3	3	$\langle 100, 16\rangle$
F_4	3	3	$\langle 20, 5\rangle$
G_1	2	2	$\langle 200, 31\rangle$
G_2	2	2	Γ_∞

6.3. *Lower cones*

In an analogous way, one classifies the lower-dimensional faces.

Theorem 6.4. *There is exactly* 1 $GL_2(\mathcal{O})$*-class of* 8*-cones,* 5 $GL_2(\mathcal{O})$*-classes of* 7*-cones,* 10 *classes of* 6*-cones,* 11 *classes of* 5*-cones,* 9 *classes of* 4*-cones,* 4 *classes of* 3*-cones, and* 2 *classes of* 2*-cones.*

Table 1 gives the $GL_2(\mathcal{O})$-classes of Voronoï cones along with the number of Π vertices, the rank of the cone, and the stabilizer in $GL_2(\mathcal{O})$ of the cone. The `Magma` small group type of each stabilizer is given for the finite groups. In particular, the first component is the order of the group. Let U^*

denote the subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathcal{O})$ such that the top left entry is a torsion unit.

6.4. Classification of forms

Note that Theorem 6.4 gives a classification of binary Hermitian forms over F based on the configuration of the minimal vectors. Indeed, by duality, the vertices of the cones that arise above correspond to $\mathrm{GL}_2(\mathcal{O})$-classes of configurations of minimal vectors for forms over F [2]. For example, there are 4 distinct classes of cones with 3 vertices. Hence there are 4 distinct $\mathrm{GL}_2(\mathcal{O})$-types of binary Hermitian forms over F with exactly 3 minimal vectors. One can distinguish the types by studying the configuration of minimal vectors.

Let $[\cdot]$ denote the subset of the minimal vectors of the perfect form in the order given in (4). For example, $[5,23]=\{e_1,e_2\}$. Since F has class number one, every primitive vector in $\mathcal{O}^2$ is $\mathrm{GL}_2(\mathcal{O})$-conjugate to e_1. Combined with Theorem 6.4, one computes the following.

Theorem 6.5. *Let ϕ be a binary Hermitian form over F. Then $M(\phi)$ is $\mathrm{GL}_2(\mathcal{O})$-conjugate to exactly one of the following*

$$[5],\quad [5,20],\quad [5,23],\quad [5,8,23],\quad [5,22,23],\quad [5,20,23],\quad [5,10,23],\quad [5,8,22,23],$$
$$[4,5,8,23],\quad [5,8,10,23],\quad [5,8,18,23],\quad [5,18,20,23],\quad [5,15,17,23],\quad [5,19,20,23],$$
$$[5,18,19,23],\quad [5,20,23,24],\quad [4,5,8,18,23],\quad [5,8,10,12,23],\quad [5,8,20,22,23],$$
$$[4,5,8,9,23],\quad [5,8,18,19,23],\quad [5,8,12,20,23],\quad [5,8,18,22,23],\quad [5,18,19,20,23],$$
$$[5,20,22\text{–}24],\quad [5,9,15,17,23],\quad [5,8,18,19,20,22],\quad [5,8,10,12,21,23],$$
$$[5,8,12,20,21,23],\quad [5,8,18,19,20,22,23],\quad [5,8,20,22\text{–}24],\quad [5,8,9,15,17,23],$$
$$[5,8,10,22\text{–}24],\quad [3\text{–}5,8,15,18,23],\quad [4,5,8,18,22,23],\quad [4,5,8,9,15,23],$$
$$[5,9,15\text{–}17,23],\quad [5,8,10,12,20\text{–}24],\quad [3,5,8,18\text{–}20,22\text{–}24],$$
$$[3\text{–}5,7,8,10,13,15,18,22\text{–}24],\quad [1,4,5,8,9,11,13\text{–}17,23],\quad [5,6,9,15\text{–}17,23],\quad [1\text{–}24]$$

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