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Elliptic points of the Picard modular group

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Abstract We explicitly compute the elliptic points and isotropy groups for the action of the Picard modular group over the Gaussian integers on 2-dimensional complex hyperbolic space.

Keywords Picard modular group · Fixed point · Elliptic element

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1 Introduction

Let $D = \mathbf{G}(\mathbb{R})/K$ be a symmetric space of non-compact type, where $\mathbf{G}$ is a semisimple algebraic group defined over $\mathbb{Q}$. An arithmetic group $\Gamma \subset \mathbf{G}(\mathbb{Z})$ acts on D by left translation, and one can study the elliptic points of this action, the points in the interior of D with non-trivial stabilizer.

One application of this computation is to the study of arithmetic quotients $\Gamma\backslash D$. The quotient is not smooth in general. It has orbifold singularities arising from the elliptic elements of Γ. An explicit knowledge of the fixed points in D with associated stabilizer groups in Γ allow one to study the types of singularities that occur.

When the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1, one can use a family of exhaustion functions to find elliptic points. In [3], we define one such family of exhaustion functions. These exhaustion functions come out of Saper's work on tilings in [2]. In fact, our exhaustion functions are nothing more than the composition of his *normalized parameters* (in the $\mathbb{Q}$-rank 1 case) with the rational root. In [4], we use the exhaustion functions to construct an

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explicit deformation retraction of D onto a spine D_0 in the case where $\mathbf{G} = \mathrm{SU}(2, 1; \mathbb{Z}[i])$ is the Picard modular group over the Gaussian integers.

In this paper, we use the spine from [4], or rather the Γ-invariant decomposition of D that it induces to study the elliptic elements of $\mathrm{SU}(2, 1; \mathbb{Z}[i])$. Section 2 recalls a general decomposition of D in the $\mathbb{Q}$-rank 1 case and outlines a procedure for using the decomposition to compute elliptic points. Section 3 specializes to the case where $\mathbf{G} = \mathrm{SU}(2, 1; \mathbb{Z}[i])$. Finally, we compute the stabilizer groups and elliptic points in Sect. 4.

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2 General $\mathbb{Q}$-rank 1

In this section we briefly describe a Γ-invariant decomposition of D into codimension 0 sets using exhaustion functions. This construction is described for the general $\mathbb{Q}$-rank 1 case in [3].

Let $G = \mathbf{G}(\mathbb{R})$ be the group of real points of a $\mathbb{Q}$-rank 1 semisimple algebraic group defined over $\mathbb{Q}$. Let $\mathcal{P}$ denote the set of proper rational parabolic subgroups of $\mathbf{G}$. To ease the notation, when there is no risk of confusion, we will use the same roman letter to denote an algebraic group and its group of real points.

2.1 The exhaustion functions

There exists an exhaustion function f_P for every rational parabolic subgroup $P \subseteq G$. Since the rational parabolic subgroups correspond to cusps, these functions can be thought of as height functions with respect to the various cusps.

The family of exhaustion functions defined above is Γ-invariant in the sense that

$$f_{\gamma P}(z) = f_P(\gamma^{-1} \cdot z) \quad \text{for } \gamma \in \Gamma. \tag{1}$$

2.2 Induced decomposition of D

These exhaustion functions are used to define a decomposition of D into sets $D(\mathcal{I})$ for $\mathcal{I} \subset \mathcal{P}$. For a parabolic subgroup P, define $D(P) \subset D$ to be the set of $z \in D$ such that $f_P(z) \geq f_Q(z)$ for every $Q \in \mathcal{P} \backslash \{P\}$. In other words, $D(P)$ consists of the points that are higher with respect to P than any other cusp. This gives a decomposition of the symmetric space parameterized by rational parabolic subgroups,

$$D = \bigcup_{P \in \mathcal{P}} D(P). \tag{2}$$

More generally, for a subset $\mathcal{I} \subseteq \mathcal{P}$,

$$D(\mathcal{I}) = \bigcap_{P \in \mathcal{I}} D(P) \tag{3}$$

$$D'(\mathcal{I}) = D(\mathcal{I}) \backslash \bigcup_{\tilde{\mathcal{I}} \supsetneq \mathcal{I}} D(\tilde{\mathcal{I}}). \tag{4}$$

It follows that $D'(\mathcal{I}) \subseteq D(\mathcal{I})$ and $D(\mathcal{I}) = \coprod_{\tilde{\mathcal{I}} \supseteq \mathcal{I}} D'(\tilde{\mathcal{I}})$.

Definition 2.1 A subset $\mathcal{I} \subset \mathcal{P}$ is called *admissible* if $D(\mathcal{I})$ is non-empty and *strongly admissible* if $D'(\mathcal{I})$ is non-empty.

Proposition 2.2 [3, Proposition 3.7] *Let $\mathcal{S}$ denote the collection of strongly admissible subsets of $\mathcal{P}$. Then the symmetric space has a Γ-invariant decomposition*

$$D = \coprod_{\mathcal{I} \in \mathcal{S}} D'(\mathcal{I}),$$

such that $\gamma \cdot D'(\mathcal{I}) = D'({}^{\gamma}\mathcal{I})$ for all $\gamma \in \Gamma$ and $\mathcal{I} \in \mathcal{S}$.

Definition 2.3 Given a family of Γ-invariant exhaustion functions, define a subset $D_0 \subset D$ by

$$D_0 = \coprod_{\substack{\mathcal{I} \in \mathcal{S} \\ |\mathcal{I}| > 1}} D'(\mathcal{I}).$$

2.3 Application to fixed points

One can use the decompositions above to study the fixed points of D. If $z \in D$ is a fixed point, then $z \in D'(\mathcal{I})$ for some strongly admissible set $\mathcal{I}$. The Γ-invariance of the decomposition implies that ${}^{\gamma}\mathcal{I} = \mathcal{I}$ for every $\gamma \in \Gamma$ that fixes z. In particular, if $z \in D \backslash D_0$, then $\mathcal{I} = \{P\}$, for some rational parabolic subgroup P. Parabolic subgroups are self-normalizing, which implies that $\mathrm{Stab}_{\Gamma}(z) \subset \Gamma \cap P$. On the other hand, if $z \in D_0$ and γ fixes z, then γ could permute the members of $\mathcal{I}$.

The retraction D_0 depends on the choice of exhaustion functions, and hence is not in general unique. In fact, there is a $(k-1)$-parameter family of different retractions, where k is the number of Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups. However, the Γ-invariance of the exhaustion functions ensures that this choice is immaterial for the purposes of computing elliptic points. In particular, suppose a point $z \in D$ has stabilizer $\Gamma_z \subset \Gamma$ that is not contained in $\Gamma \cap Q$ for any rational parabolic subgroup Q. Pick a family of exhaustion functions. Then $z \in D(P)$ for some rational parabolic subgroup P. Since Γ_z is not a subgroups of a rational parabolic subgroup, there exists a $\gamma \in \Gamma_z$ such that ${}^{\gamma}P \neq P$. Then (1) implies that $f_{{}^{\gamma}P}(z) = f_P(z)$. It follows that $z \in D_0$.

3 Background

3.1 The unitary group

Let G be the identity component of the real points of the algebraic group $\mathbf{G} = \mathrm{SU}(2, 1)$, realized explicitly as

$$G = \mathbf{G}(\mathbb{R}) = \mathrm{SU}(2, 1; \mathbb{C}) = \{g \in \mathrm{SL}(3, \mathbb{C}) \mid g^* C g = C\},$$

where $C = \begin{bmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{bmatrix}$. Let $\mathcal{O} = \mathbb{Z}[i]$ and let Γ be the arithmetic subgroup $\Gamma = \mathbf{G}(\mathbb{Z}) = G \cap \mathrm{SL}_3(\mathcal{O})$. Let θ denote the Cartan involution given by inverse conjugate transpose and let K be the fixed points under θ. Then K is the maximal compact subgroup $K = G \cap \mathrm{SU}(3)$.

Because these elements of Γ will be used frequently, set once and for all

$$w = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 1 & 1+i & i \\ 0 & 1 & 1+i \\ 0 & 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\epsilon = \begin{bmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{bmatrix}, \quad \text{and} \quad \xi = \begin{bmatrix} 1 & -1-i & i \\ 1-i & -1 & 0 \\ 1-i & -1-i & i \end{bmatrix}.$$

3.2 The symmetric space

Let $D = G/K$ be the associated Riemannian symmetric space of non-compact type. Then D is 2-dimensional complex hyperbolic space or the complex 2-ball with the Bergmann metric. We will put coordinates on D using Langlands decomposition.

Let $P_0 \subset G$ be the rational parabolic subgroup of upper triangular matrices,

$$P_0 = \left\{ \begin{bmatrix} y\zeta & \beta\zeta^{-2} & \zeta\left(r + i|\beta|^2/2\right)/y \\ 0 & \zeta^{-2} & i\overline{\beta}\zeta/y \\ 0 & 0 & \zeta/y \end{bmatrix} \;\middle|\; \begin{matrix} \zeta, \beta \in \mathbb{C}, \ |\zeta| = 1, \\ r \in \mathbb{R}, \ y \in \mathbb{R}_{>0} \end{matrix} \right\}. \tag{5}$$

Zink showed that Γ has class number 1 [5]. Thus $\Gamma \backslash \mathbf{G}(\mathbb{Q})/\mathbf{P_0}(\mathbb{Q})$ consists of a single point, and all the rational parabolic subgroups of G are Γ-conjugate to P_0.

P_0 acts transitively on D, and every point $z \in D$ can be written as $z = pK$ for some $p \in P_0$. When p is written as above, the point $z = pK$ is independent of ζ, and so we will denote such a point $z = (y, \beta, r)$. These are also known as *horospherical coordinates*.

A computation shows that every point $z \in D$ is conjugate under Γ_{P_0} to a point in S, where

$$S = \{(y, \beta, r) \in D | -1/2 < r \leq 1/2, \beta \in \diamond\}, \quad \text{where}$$

$\diamond$ is the square in the complex plane with vertices 0, $(1+i)/2$, i, and $(-1+i)/2$.

4 Implementation

Using Proposition 2.2, we divide D into a codimension 1 piece D_0 and a codimension 0 piece $D \backslash D_0$ and study the fixed points on each separately. We deal with $D \backslash D_0$ first.

4.1 Elliptic points in $D\backslash D_0$

Proposition 4.1 *Every non-trivial isotropy group of a point in* $D\backslash D_0$ *is* Γ*-conjugate to exactly one of*

$$\Gamma_1 = \langle \epsilon \rangle \cong \mathbb{Z}/4\mathbb{Z}, \quad \Gamma_2 = \langle \xi^2 \rangle \cong \mathbb{Z}/4\mathbb{Z}, \quad or \quad \Gamma_3 = \langle \sigma\epsilon^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof Notice that $D\backslash D_0 = \coprod_{P\in\mathcal{P}} D'(P)$. Since all of the rational parabolic subgroups are Γ-conjugate, Proposition 2.2 implies that every point in $D\backslash D_0$ is a Γ-translate of a point in $D'(P_0)$. Since parabolic subgroups are self-normalizing, the subgroup of Γ that stabilizes $D'(P_0)$ is exactly $\Gamma_{P_0} = \Gamma \cap P_0$.

Let $z \in D\backslash D_0$ be a fixed point. Then z is a Γ-translate of a point $z_0 = (y_0, \beta_0, r_0)$ in $D'(P_0)$. If p is an element of Γ that fixes z_0, then $p \in P_0$ and can be written in coordinates as in (5). Then

$$p \cdot z_0 = \Big(yy_0, y\zeta^3\beta_0 + \beta, y^2 r_0 + r - \mathrm{Im}(\beta\bar{\beta}_0\zeta^{-3}y)\Big).$$

Then since $y_0 > 0$ and $p \cdot z_0 = z_0$,

$$y = 1, \quad \beta = \beta_0(1-\zeta^3), \quad \text{and} \quad r = |\beta_0|^2\,\mathrm{Im}(\zeta^{-3}). \tag{6}$$

Since $p \in \Gamma$, we must have $r \in \mathbb{Z}$, $\zeta \in \mathcal{O}^*$, and $\beta \in \mathcal{O}$ such that $2 \mid |\beta|^2$. It follows that $\mathrm{Stab}_\Gamma(z_0)$ consists of the intersection $\{I, \gamma, \gamma^2, \gamma^3\} \cap \Gamma$, where

$$\gamma = \begin{bmatrix} i & -(1+i)\beta_0 & -(1-i)|\beta_0|^2 \\ 0 & -1 & -(1-i)\overline{\beta}_0 \\ 0 & 0 & i \end{bmatrix}. \tag{7}$$

The points of S for which the intersection is non-trivial have $\beta_0 \in \{0, (1+i)/2, i\}$. The intersection has order four for $\beta_0 \in \{0, i\}$ and order two for $\beta_0 = (1+i)/2$. Since every point of D is Γ-conjugate to a point in S, the result then follows. □

4.2 Isotropy groups of points in D_0

A fundamental domain for the action of Γ on D_0 is given in [4]. In particular, D_0 is given the structure of a cell complex such that the stabilizer of a cell fixes the cell pointwise. We mention that this cell structure is just a subdivision of the decomposition given in Definition 2.3.

The space D_0 is given the structure of a cell-complex such that the stabilizer of a cell fixes the cell pointwise. The cells of D_0 fall into 24 equivalence classes modulo Γ consisting of two 3-cells, seven 2-cells, nine 1-cells, and six 0-cells.

Since the stabilizer of a cell fixes it pointwise, the cells with non-trivial stabilizer form a set of Γ-representatives of the fixed points that we are looking for. The stabilizers are computed in [4, Table 3], and the only new ones up to Γ-conjugacy are precisely $\Gamma_4, \ldots, \Gamma_9$ in Table 1.

Table 1 Isotropy groups for points in D

	Stabilizer	Generators	Fixed points
Γ_1	$\mathbb{Z}/4\mathbb{Z}$	$\langle\epsilon\rangle$	$\beta = 0$
Γ_2	$\mathbb{Z}/4\mathbb{Z}$	$\langle\xi^2\rangle$	$\beta = i$
Γ_3	$\mathbb{Z}/2\mathbb{Z}$	$\langle\sigma\epsilon^2\rangle$	$\beta = (1+i)/2$
Γ_4	$\mathbb{Z}/2\mathbb{Z}$	$\langle\epsilon w\rangle$	$y^2 + \lvert\beta\rvert^2/2 = 1$
Γ_5	$\mathbb{Z}/12\mathbb{Z}$	$\langle\tau\epsilon w\rangle$	$(\sqrt[4]{3}/\sqrt{2}, 0, 1/2)$
Γ_6	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$\langle\epsilon w, \epsilon\rangle$	$(1, 0, 0)$
Γ_7	$\mathcal{G}_{31}^{a}$	$\langle\epsilon w, \xi^2\rangle$	$\left(1/\sqrt{2}, i, 0\right)$
Γ_8	$\mathfrak{S}_3$	$\langle\epsilon w, \sigma\epsilon^2\rangle$	$(\sqrt{3}/2, (1+i)/2, 0)$
Γ_9	$\mathbb{Z}/8\mathbb{Z}$	$\langle\xi\rangle$	$(1/\sqrt[4]{2}, i, 1/2)$

[a] This is the order 32 group with Hall-Senior number 31 [1] and Magma small group library number 11

Note that Γ_1, Γ_2, and Γ_3 occur as subgroups of the groups in Table 1 since $D^i = D^{\Gamma_i}$ contains points with y small and points with y large for $i = 1, 2, 3$. In particular, the surface D^i intersects the spine D_0 and so the isotropy group of some point in D_0 contains Γ_i.

Theorem 4.2 *The isotropy group of a point in D is Γ-conjugate to exactly one of the groups in Table 1. The fixed points for each group are also tabulated.*

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