EXPLICIT COMPUTATIONS OF HIGHER WEIGHT MODULAR FORMS

NATHAN FONTES
ADVISOR: DAN YASAKI

Abstract. Modular forms are holomorphic functions on the complex upper half plane with special conditions imposed upon them. We examine computational methods on these structures. This involves looking at modular symbols and Manin symbols, as well as the corresponding Hecke operators on them. I discuss computations on higher weight modular forms and highlight their differences from computing weight 2 modular forms.

1. Background

The goal of this paper is to give some insight into computing spaces of cusp forms as a whole, particularly leaning towards the differences between weight 2 cusp forms and those of weight higher than 2.

1.1. **Modular Forms.** Modular forms are holomorphic functions on the complex upper half plane with special conditions imposed on them. These conditions allow for interesting properties to occur, some of which will be studied here. A few basic definitions are necessary before beginning to study modular forms.

The complex upper half plane is defined as the part of the complex plane with positive imaginary part,

$$\mathcal{H} = \{z \in \mathbb{C} \mid z = a + bi, b > 0\}.$$

The special linear group of 2×2 integer matrices, denoted $\mathrm{SL}_2(\mathbb{Z})$, is defined as

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| ad - bc = 1, \text{ and } a, b, c, d \in \mathbb{Z} \right\}.$$

This group is generated by

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by fractional linear transformation. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$, we have

$$g(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \frac{az+b}{cz+d}$$

With this action, we can look at the definition of a modular form.

Definition 1.1. Let k be an integer. Then a *modular form of weight* k is a function $f : \mathcal{H} \to \mathbb{C}$ such that:

(i) f is holomorphic on $\mathcal{H}$.
(ii) $f(g(z)) = (cz+d)^k f(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathcal{H}$.
(iii) f is holomorphic at ∞.

The space of modular forms of weight k is written $\mathcal{M}_k$.

The second condition defines a symmetry on modular forms, where taking an $\mathrm{SL}_2(\mathbb{Z})$-translate of a point on the upper half plane pulls out a symmetry factor of $(cz+d)^k$. After finding a particular modular form, this can be confirmed by acting by the generators σ and τ of $\mathrm{SL}_2(\mathbb{Z})$.

In order to understand the meaning of f holomorphic at ∞, we will first consider the change of coordinates from the upper half plane to the punctured unit disk:

$$\begin{aligned} \mathcal{H} &\to D' = \{q \in \mathbb{C} \setminus \{0\} \mid |q| < 1\} \\ z &\mapsto e^{2\pi i z} = q. \end{aligned}$$

This map is well-defined because $\tau(z) = z+1$ and using the condition (ii), this gives $f(z) = f(z+1)$. Then f is holomorphic at ∞ if there exists a holomorphic function

$$\begin{aligned} g : D' &\to \mathbb{C} \\ q &\mapsto \sum_{n=0}^{\infty} a_n q^n. \end{aligned}$$

This means that if f is a modular form, then it has a Fourier expansion in q. We can write

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

We will look at two types of modular forms: Eisenstein series and cusp forms.

Definition 1.2. Let k be an even integer and $k \geq 4$. Then an *Eisenstein series of weight k* at $z \in \mathcal{H}$ is defined by

$$G_k(z) = \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(cz+d)^k}.$$

Definition 1.3. Let k be an integer. A *cusp form of weight k* is a modular form of weight k such that $f(\infty) = 0$.
The space of cusp forms of weight k is written $\mathcal{S}_k$.

The Fourier expansion of a cusp form has a zero constant term. If f is a cusp form, then its Fourier expansion can be written starting at $n = 1$:

$$f(z) = \sum_{n=1}^{\infty} a_n q^n.$$

Furthermore, we have that every modular form can be written as a sum of cusp forms and Eisenstein series [Ste07]. With this knowledge, if we can find the basis for $\mathcal{S}_k$ and G_k for a given weight k, then we know exactly the space $\mathcal{M}_k$ as well.

Definition 1.4. The *principal congruence subgroup of level* N, $\Gamma(N)$, is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod N \right\}.$$

Note that $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.

Any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma(N) \subset \Gamma$ is a congruence subgroup. We will commonly be using a specific congruence subgroup of level N, denoted $\Gamma_0(N)$, which is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod N \right\}.$$

We define a right action of a congruence subgroup Γ on a modular form f, called the weight k operator or slash operator, by

$$(f[g]_k)(z) = \det(g)^{k-1}(cz+d)^{-k} f(g(z)) \quad \text{for} \quad g \in \Gamma, z \in \mathcal{H}.$$

Since we will mostly be studying modular forms over congruence subgroups Γ of level N, we need to alter our definition of a modular form to take into account the new cusps that are introduced with the congruence subgroup.

Definition 1.5. Let k be an integer, and let Γ be a congruence subgroup of level N. Then a *modular form of weight k for* Γ is a function $f : \mathcal{H} \to \mathbb{C}$ such that:

(i) f is holomorphic on $\mathcal{H}$.
(ii) $f(g(z)) = (cz + d)^k f(z)$ for all $g \in \Gamma$, $z \in \mathcal{H}$.
(iii) f is holomorphic at its cusps.

The space of modular forms of weight k for Γ is written $\mathcal{M}_k(\Gamma)$.

The main difference between Definition 1.1 and Definition 1.5 is the inclusion of more cusps than just ∞. We determine that a function is holomorphic at those points in a similar manner to determining that a function is holomorphic at ∞. It's also possible to check that the product of two modular forms for Γ, one of weight k and one of weight l, is a modular form of weight $k + l$ for Γ.

Theorem 1.1. [Ste07] *Let $f_1 \in \mathcal{M}_k(\Gamma)$ and $f_2 \in \mathcal{M}_l(\Gamma)$. Then $f_1 f_2 \in \mathcal{M}_{k+l}(\Gamma)$.*

Proof. We need to check that the three properties of a modular form hold for the product $f_1 f_2$.

(i) The product of two holomorphic functions is holomorphic.
(ii) Let $g \in \Gamma$. Then we have $f_1(g(z)) = (cz + d)^k f_1(z)$ and $f_2(g(z)) = (cz + d)^l f_2(z)$, so

$$f_1(g(z))f_2(g(z)) = (cz + d)^k f_1(z)(cz + d)^l f_2(z) = (cz + d)^{k+l} f_1(z)f_2(z).$$

(iii) It is possible to show that $f_1 f_2$ is holomorphic at each of its cusps.

Therefore, $f_1 f_2$ is a modular form of weight $k + l$ for Γ. □

We can similarly define cusp forms of weight k for Γ.

Definition 1.6. Let k be an integer, and let Γ be a congruence subgroup of level N. A *cusp form of weight k for* Γ is a modular form of weight k and level N such that $f(a) = 0$ for any cusp a.
The space of cusp forms of weight k and level N is written $\mathcal{S}_k(\Gamma)$.

1.2. **Hecke Operators.** The next objects we need to define are Hecke Operators. Computing these operators allows us to construct bases for the spaces of modular forms, and so we must understand them before doing so. We will be referring to the congruence subgroup of level N, $\Gamma_0(N)$, for the following definitions.

Before we are able to talk about Hecke Operators, we must define another set of matrices.

$$X_p = \begin{cases} \left\{ \begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix} \middle| 0 \le j < p \right\} & \text{if} \quad p|N. \\ \left\{ \begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix} \middle| 0 \le j < p \right\} \cup \left\{ \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \right\} & \text{if} \quad p \nmid N. \end{cases}$$

Definition 1.7. Let p be prime. The p^{th} *Hecke operator of weight* k *for* $\Gamma_0(N)$ on $f \in \mathcal{M}_k(\Gamma_0(N))$, denoted T_p, is

$$T_p(f) = \sum_{g \in X_p} f[g]_k.$$

The following theorem by Hecke gives us several useful properties of Hecke operators, which causes the only necessary Hecke operator computations to be on the prime operators. I will refer to the proof of this theorem by Serre [Ser73].

Theorem 1.2. *For all* n, m *with* $n \neq m$*, the following properties about Hecke Operators hold:*

(i) $T_n T_m = T_m T_n$.
(ii) $T_{nm} = T_n T_m$ *if* $(n, m) = 1$.
(iii) $T_{p^n} = T_{p^{n-1}} T_p - p^{k-1} T_{p^{n-2}}$ *for* p *prime.*

We can construct all T_n for n composite using these properties if we have T_p for p prime. Each T_p operator is a linear transformation on the space of modular forms,

$$T_p : \mathcal{M}_k(\Gamma_0(N)) \to \mathcal{M}_k(\Gamma_0(N))$$

The transformation matrix $[T_p]$ has several nice properties. The set of $[T_p]$ operators are simultaneously diagonalizable, which allows us to look at specific modular forms which are determined by every Hecke operator simultaneously, called eigenforms. The eigenvalues of $[T_p]$ are exactly the p^{th} coefficients, a_p, in the q-expansions of these eigenforms.

Definition 1.8. An *eigenform of weight* k *for* $\Gamma_0(N)$ is a modular form which is a simultaneous eigenvector for every Hecke operator.

These weight k eigenforms for $\Gamma_0(N)$ generate the corresponding space of modular forms. Therefore, if we can compute these eigenforms, then we have our space of modular forms.

1.3. **Tensor Products.** We will be using tensor products in the next section, so we will define them here.

Definition 1.9. Let V and W be vector spaces over a field F. Then the *tensor product of V and W over F*, $V \otimes_F W$, is the set of formal linear combinations of elements in $V \times W$ such that the following relationships hold for all $v_1, v_2 \in V, w_1, w_2 \in W, c \in F$:

(i) $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$.
(ii) $(v, w_1 + w_2) = (v, w_1) + (v, w_2)$.
(iii) $(cv, w) = (v, cw)$.

The tensor product of two vector spaces over F yields another vector space over F, as a result of the relationships on tensor products. We can multiply by scalars as follows:

$$c \cdot (v, w) = (cv, w) = (v, cw).$$

We will sometimes denote elements of the tensor product $V \otimes_F W$ as $v \otimes_F w$. Furthermore, when the field F is understood, we may leave off the subscript on the tensor product.

2. Modular Symbols

Now that we understand the building blocks of modular forms, we can move on to computing them. This section examines modular symbols and how to compute with them through methods involving Hecke Operators on Manin symbols and Heilbronn matrices. Through the section, the example space of weight 4 modular symbols for $\Gamma_0(5)$ will be studied.

Our first goal is to define a space of modular symbols of weight k for $k \geq 4$ and even. In order to do that, we need to first understand the space of weight 2 modular symbols. We define modular symbols corresponding to spaces of weight 2 modular forms as follows:

Definition 2.1. For $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$, the *set of symbols S* is defined by

$$S = \{\{\alpha, \beta\} : \alpha, \beta \in \mathbb{P}^1(\mathbb{Q})\}.$$

For $g \in \mathrm{GL}_2(\mathbb{Q})^+$, g acts on S as follows:

$$g \cdot \{\alpha, \beta\} = \{g(\alpha), g(\beta)\}.$$

where $g(\gamma)$ is the action given by fractional linear transformation.

There are some occurrences that differ based on the congruence subgroup taken, which we will avoid in this paper by using Γ_0 in place of a general Γ going forward. Recall

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod N \right\}.$$

Definition 2.2. The *space of weight 2 modular symbols*, denoted $\mathbb{M}_2$, is the $\mathbb{Q}$-vector space generated by the set of symbols modulo the following relations:

(i) 2-term relation: $\{\alpha, \beta\} + \{\beta, \alpha\} = 0$ for all $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$.
(ii) 3-term relation: $\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$ for all $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$.

Furthermore, we can define the space of weight 2 modular symbols for $\Gamma_0(N)$ by modding out by an extra $\Gamma_0(N)$-action, where $\{\alpha, \beta\} - g\{\alpha, \beta\} = 0$ for $g \in \Gamma_0(N)$.

Definition 2.3. The *space of weight 2 modular symbols for* $\Gamma_0(N)$, denoted $\mathbb{M}_2(\Gamma_0(N)$, is the space $\mathbb{M}_2$ modulo the extra relation:

$\Gamma_0(N)$-action: $\{\alpha, \beta\} - g\{\alpha, \beta\} = 0$ for all $g \in \Gamma_0(N)$.

Next, for some integer $n > 0$, let $\mathbb{Q}[X, Y]_n$ be the $\mathbb{Q}$-vector space of homogeneous polynomials of degree n in two variables X and Y (e.g., for $n = 2$, this space is generated by the three degree 2 polynomials X^2, XY, and Y^2). Then $\mathbb{Q}[X, Y]_n$ is finitely generated with $n + 1$ basis elements. We define the left action of Γ_0 on $\mathbb{Q}[X, Y]_n$ by

$$(gP)(X, Y) = P(dX - bY, -cX + aY) \text{ for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0 \ , \ P(X, Y) \in \mathbb{Q}[X, Y]_n$$

In order to define higher weight modular symbols, we recall these previous matrices and define a new matrix in $\mathrm{SL}_2(\mathbb{Z})$ as follows:

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tau = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, J = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We now write the definition for a space of higher weight modular symbols:

Definition 2.4. For a fixed weight $k \geq 2$, the space of weight k modular symbols, denoted $\mathbb{M}_k$, is the tensor product $\mathbb{Q}[X, Y]_{k-2} \otimes_{\mathbb{Q}} \mathbb{M}_2$.

We use a shortened notation to represent elements in the space of modular symbols as follows:

$$[P, \{\alpha, \beta\}] \in \mathbb{Q}[X, Y]_{k-2} \otimes_{\mathbb{Q}} \mathbb{M}_2 \text{ for } P \in \mathbb{Q}[X, Y]_{k-2} \ , \ \{\alpha, \beta\} \in \mathbb{M}_2.$$

Note that this is consistent with our definition of the space of weight 2 modular symbols given above, as $\mathbb{Q}[X, Y]_{2-2}$ is the space generated by degree 0 polynomials, which has $\{1\}$ as its basis.

We define a right action on the space of weight k modular symbols by combining the actions on the two vector spaces in the tensor product as follows:
For $g \in \mathrm{SL}_2(\mathbb{Z})$ and $[P, \{\alpha, \beta\}] \in \mathbb{M}_k(\Gamma_0)$, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{M}_k$ by

$$[P, \{\alpha, \beta\}]g = [g^{-1}P, \{g(\alpha), g(\beta)\}].$$

We further define the space of weight k modular symbols for $\Gamma_0(N)$ in a similar way as for the weight 2 case, by modding out by a generalized $\Gamma_0(N)$-action, which is that $[P, \{\alpha, \beta\}] - [P, \{\alpha, \beta\}]g = 0$ for $g \in \Gamma_0(N)$.

Definition 2.5. The *space of weight k modular symbols for* $\Gamma_0(N)$, denoted $\mathbb{M}_k(\Gamma_0(N)$, is the space $\mathbb{M}_2$ modulo the extra relation:

$$\Gamma_0(N)\text{-action: } [P, \{\alpha, \beta\}] - [P, \{\alpha, \beta\}]g = 0 \text{ for all } g \in \Gamma_0(N).$$

2.1. **Manin Symbols.**

Definition 2.6. A unimodular symbol is a modular symbol $\{\alpha, \beta\} \in \mathbb{M}_2(\Gamma_0)$ which is a $\mathrm{SL}_2(\mathbb{Z})$ translate of the modular symbol $\{0, \infty\}$.

It is common to use Manin's Trick [Man72] at this point, which uses convergents of continued fractions to express any modular symbol as a finite sum of unimodular symbols.

Theorem 2.1 (Manin's Trick)**.** [Man72] *Let $\{0, \frac{a}{b}\} \in \mathbb{M}_2(\Gamma_0)$, and let the continued fraction expression of $\frac{a}{b}$ be given by $[[a_1, a_2, \cdots, a_r]]$, so the convergents are $\frac{p_1}{q_k} = [[a_1, a_2, \cdots, a_k]]$. Then we can write our modular symbol in terms of unimodular symbols as follows:*

$$\left\{0, \frac{a}{b}\right\} = \{0, \infty\} + \left\{\infty, \frac{p_1}{q_1}\right\} + \left\{\frac{p_1}{q_1}, \frac{p_2}{q_2}\right\} + \cdots + \left\{\frac{p_{r-1}}{q_{r-1}}, \frac{p_r}{q_r}\right\}$$

From the definition of a unimodular symbol, we can write any unimodular symbol $\left\{\frac{a}{b}, \frac{c}{d}\right\}$ as a $\mathrm{SL}_2(\mathbb{Z})$ translate of the symbol $\{0, \infty\}$ as follows:

$$\left\{\frac{a}{b}, \frac{c}{d}\right\} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \{0, \infty\}$$

Definition 2.7. For any prime ℓ, there is a bijection $\Gamma_0(\ell) \setminus \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{P}^1(\mathbb{F}_\ell)$ defined by $\Gamma_0(\ell) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (c : d)$. The symbols $[P, (c : d)]$ for $P \in \mathbb{Q}[X, Y]_{k-2}$ and for $(c : d) \in \mathbb{P}^1(\mathbb{Q})$ are called *Manin symbols.*

Theorem 2.2. [Mer94] *The Manin symbols generate $\mathbb{M}_k(\Gamma_0(N))$.*

I will refer to Manin's proof of the above theorem. Using that theorem, we can then compute with Manin symbols instead of unimodular symbols, as long as we redefine the relations in terms of Manin symbols. We can do this by noting the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \{0, \infty\} \mapsto (0 : 1) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then our relations are formed in the following theorem by Merel:

Theorem 2.3. [Mer94] *For $[P,(c:d)] \in \mathbb{M}_k(\Gamma_0(N))$, we have*

$$[P,(c:d)] + [P,(c:d)]\sigma = 0 \qquad \textit{(2-term relation)} \tag{1}$$

$$[P,(c:d)] + [P,(c:d)]\tau + [P,(c:d)]\tau^2 = 0 \qquad \textit{(3-term relation)} \tag{2}$$

$$[P,(c:d)] - [P,(c:d)]J = 0 \qquad \textit{(J-term relation)} \tag{3}$$

We now go through an explicit computation for $\mathbb{M}_4(\Gamma_0(5))$:

$$\mathbb{M}_4(\Gamma_0(5)) = \mathbb{Q}[X,Y]_2 \otimes_{\mathbb{Q}} \mathbb{M}_2(\Gamma_0(5)).$$

Both spaces in the tensor product are finitely generated. We write $\mathbb{Q}[X,Y]_2$ with the generating set $\{X^2, XY, Y^2\}$ and $\mathbb{M}_2(\Gamma_0(5))$ is isomorphic to the $\mathbb{Q}$-vector space generated by $\mathbb{P}^1(\mathbb{F}_5)$, so it has the generating set $\{(0:1),(1:0),(1:1),(1:2),(1:3),(1:4)\}$.
Then before modding out by the 2-term, 3-term, and J-term relations, we have that $\mathbb{M}_4(\Gamma_0(5))$ is at most generated by the 18 element set:

$$\begin{array}{lll}
\{[X^2,(0:1)], & [XY,(0:1)], & [Y^2,(0:1)], \\
[X^2,(1:0)], & [XY,(1:0)], & [Y^2,(1:0)], \\
[X^2,(1:1)], & [XY,(1:1)], & [Y^2,(1:1)], \\
[X^2,(1:2)], & [XY,(1:2)], & [Y^2,(1:2)], \\
[X^2,(1:3)], & [XY,(1:3)], & [Y^2,(1:3)], \\
[X^2,(1:4)], & [XY,(1:4)], & [Y^2,(1:4)]\}
\end{array}$$

Next we need to take into consideration the three relations on this set. Via the 2-term relation, we can narrow our set down to 10 elements as follows:

$$\begin{aligned}
[X^2,(0:1)] + [X^2,(0:1)]\sigma = 0 &\implies [X^2,(0:1)] = -[Y^2,(1:0)] \\
[X^2,(1:0)] + [X^2,(1:0)]\sigma = 0 &\implies [X^2,(1:0)] = -[Y^2,(0:1)] \\
[X^2,(1:1)] + [X^2,(1:1)]\sigma = 0 &\implies [X^2,(1:1)] = -[Y^2,(1:4)] \\
[X^2,(1:2)] + [X^2,(1:2)]\sigma = 0 &\implies [X^2,(1:2)] = -[Y^2,(1:2)] \\
[X^2,(1:3)] + [X^2,(1:3)]\sigma = 0 &\implies [X^2,(1:3)] = -[Y^2,(1:3)] \\
[X^2,(1:4)] + [X^2,(1:4)]\sigma = 0 &\implies [X^2,(1:4)] = -[Y^2,(1:1)] \\
[XY,(0:1)] + [XY,(0:1)]\sigma = 0 &\implies [XY,(0:1)] = [XY,(1:0)] \\
[XY,(1:1)] + [XY,(1:1)]\sigma = 0 &\implies [XY,(1:1)] = [XY,(1:4)]
\end{aligned}$$

Via the 3-term relation, we can narrow the basis set down to 4 elements with the following relations:

$$\begin{aligned}
[X^2,(1:0)] &= [X^2,(1:1)] + 2[XY,(0:1)] \\
[X^2,(1:2)] &= [X^2,(1:1)] + 4[XY,(0:1)] + 2[XY,(1:3)] \\
[X^2,(1:3)] &= [X^2,(1:1)] + 12[XY,(0:1)] + 2[XY,(1:3)] \\
[X^2,(1:4)] &= [X^2,(1:1)] + 4[XY,(0:1)] \\
[XY,(1:1)] &= 2[XY,(0:1)] \\
[XY,(1:2)] &= -6[XY,(0:1)] - 3[XY,(1:3)]
\end{aligned}$$

The J-term relations give us no extra restrictions, so $\mathbb{M}_4(\Gamma_0(5))$ is isomorphic to the $\mathbb{Q}$-vector space generated by

$$\{[X^2,(0:1)],[X^2,(1:1)],[XY,(0:1)],[XY,(1:3)]\}.$$

2.2. **Heilbronn Computations.** Computing the eigenvalues for Hecke operators gives us the Fourier coefficents for our eigenforms. When we looked at Hecke operators in section 1.2, they were only computable by taking a sum over all slash operators from a set of matrices that were dependent on the prime and the level. There is a simpler method of computing Hecke operators using Heilbronn matrices, developed by Merel and Mazur at a conference in Heilbronn, Germany [Mer91]. These Heilbronn matrices are defined as follows:

$$Y_p = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \det(g) = p, a > b \geq 0, d > c \geq 0, a, b, c, d \in \mathbb{Z} \right\}.$$

These Heilbronn matrices are dependent only on the prime p corresponding to the p^{th} Hecke operators. As such, they only need to be computed once for every prime p. Then for $p \nmid N$, Hecke operators are defined on weight 2 Manin symbols by

$$T_p(c:d) = \sum_{g \in Y_p} (c:d)g.$$

Similarly, Hecke operators for $p \nmid N$ are defined on weight $k \geq 2$ Manin symbols by

$$T_p[P,(c:d)] = \sum_{g \in Y_p} [P,(c:d)]g.$$

The alternative to using Heilbronn matrices is to use Manin's trick to convert Manin symbols into unimodular symbols, and compute Hecke operators on those. The more efficient method is to compute with Heilbronn matrices, so we will use them.

We will go through a computation for the Hecke operator T_2 on $\mathbb{M}_4(\Gamma_0(5))$. First, we can compute the set of Heilbronn matrices for $p = 2$ as follows:

$$Y_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

In order to compute the 2^{nd} Hecke operator on, for example, the Manin symbol $[X^2, (0:1)]$, we need to act on it by every matrix in Y_2. We see what happens when we act on $[X^2, (0,1)]$ by the second matrix in the above set:

$$\begin{aligned} [X^2, (0:1)] \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} &= \left[\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} X^2, (0:1) \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \right] \\ &= [X^2, (1:2)] \\ &= [X^2, (1:1)] + 4[XY, (0:1)] + 2[XY, (1:3)]. \end{aligned}$$

The last equality comes from the previous relations on $\mathbb{M}_4(\Gamma_0(5))$. We can similarly act on $[X^2, (0,1)]$ by the other three matrices in Y_2 to get the Hecke operator in terms of the four generators of our space. Putting them all together, we get the following Hecke operator for $p = 2$.

$$\begin{aligned} T_2([X^2, (0:1)]) = [X^2, (0:1)] &\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + [X^2, (0:1)] \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + \\ &[X^2, (0:1)] \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + [X^2, (0:1)] \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \\ = [X^2, (0:1)] + [X^2, (1:1)] &+ 4[XY, (0:1)] + 2[XY, (1:3)] + \\ 4[X^2, (0:1)] + 4[X^2, (0:1)] &+ 4[X^2, (0:1)] - [X^2, (1:1)] - \\ 2[XY, (0:1)] & \\ = 9[X^2, (0:1)] + 6[XY, (0:1)] &+ 2[XY, (1:3)]. \end{aligned}$$

We proceed to write T_2 as a linear transformation on the basis we found using Manin symbols:

$$[T_2] = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 6 & 56 & -4 & 0 \\ 2 & 10 & 0 & -4 \end{bmatrix}.$$

The first column of this transformation matrix comes from the above computation for $T_2([X^2, (0:1)])$, and the following three columns come from computing the 2^{nd} Hecke operator on the other three elements we computed previously.

We previously determined that the eigenvalues of the p^{th} Hecke operators correspond exactly to the coefficients of the q-expansions of eigenforms. The eigenvalues of this Hecke operator can be read off the diagonal, so they are an 9 (with multiplicity 2) and -4 (with multiplicity 2). Therefore, there we notice that there are at minimum two distinct eigenforms in this space.

We noted that all of the transformation matrices of prime Hecke operators not dividing the level can be simultaneously diagonalized. This tells us that the coefficients of each Fourier series are in the same order in each Hecke transformation matrix. In order to see if there are any differences between the first and second eigenvalues, or between the third and fourth eigenvalues, we compute several more Hecke operators.

Through a similar computation for the primes 3 and 7, we get

$$[T_3] = \begin{bmatrix} 28 & 0 & 0 & 0 \\ 0 & 28 & 0 & 0 \\ 8 & 112 & 2 & 0 \\ 4 & 20 & 0 & 2 \end{bmatrix}, \quad [T_7] = \begin{bmatrix} 344 & 0 & 0 & 0 \\ 0 & 344 & 0 & 0 \\ 156 & 1456 & 6 & 0 \\ 52 & 260 & 0 & 6 \end{bmatrix}.$$

The eigenvalues corresponding to $[T_3]$ are 28 and 2 and the eigenvalues corresponding to $[T_7]$ are 344 and 6, again read off the diagonal.

Notice here that the first two eigenvalues are the same and the third and fourth eigenvalues are also the same, similarly to what we saw in $[T_2]$. This leads us to believe that there are two eigenforms for $\mathcal{M}_4(\Gamma_0(5))$. There is a difference between the first two eigenvalues for Hecke operators on primes dividing the level, however, which can be seen in the transformation matrix for T_5, as follows:

$$[T_5] = \begin{bmatrix} 126 & 0 & 0 & 0 \\ 20 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}.$$

This means that we should have three different eigenforms for $\mathcal{M}_4(\Gamma_0(5))$. We can find the q-expansions more completely by using Theorem 1.2 and by computing more prime Hecke operators. After computing Hecke operators, we get the following three Fourier series for

eigenforms in $\mathcal{M}_4(\Gamma_0(5))$:

$$f_1 = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + 344q^7 + \dots$$
$$f_2 = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + q^5 + 252q^6 + 344q^7 + \dots$$
$$f_3 = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 + \dots$$

In this example, f_1 and f_2 correspond to the Eisenstein series G_4 and a different weight 4 Eisenstein series, and f_3 corresponds to a cusp form in $\mathcal{S}_k(\Gamma_0(5))$. Therefore, these three modular forms generate the entire space of weight 4, level 5 modular forms.

3. Examples

Now that we have studied the techniques for computing spaces of modular forms, this section gives results associated with a pair of examples. Specifically, we will be looking at spaces of level 19 modular forms, of weight 2 and of weight 4.

3.1. $\mathcal{M}_2(\Gamma_0(19))$**.** The first space of modular forms we are studying is $\mathcal{M}_2(\Gamma_0(19))$, or the space of weight 2 and level 19 modular forms. The first thing we need to do is compute the basis of its corresponding space of modular symbols, $\mathbb{M}_2(\Gamma_0(19))$. We note that since our congruence subgroup is Γ_0, this is equivalent to taking Manin symbols from the projective line on $\mathbb{F}_{19}$. Therefore, before modding out by any of the relations, we have possible basis elements of our $\mathbb{Q}$-vector space from the set

$$\mathbb{P}^1(\mathbb{F}_{19}) = \{(0:1), (1:0), (1:1), \dots, (1,18)\}.$$

We can then find the basis of this vector space by using the 2-term and 3-term relations on this set. Doing so produces the 3-dimensional basis of $\mathbb{M}_2(\Gamma_0(19))$,

$$\{(0:1), (1:4), (1:5)\}.$$

Now that we have a basis for $\mathbb{M}_2(\Gamma_0(19))$, the next thing to do is to compute Hecke operators on each basis element. This will give us a set of 3×3 transformation matrices which represent the set of Hecke operators. I provide the transformation matrices for the Hecke operators for primes 2, 3, and 5 here:

$$[T_2] = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad [T_3] = \begin{bmatrix} 4 & 0 & 0 \\ 2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}, \quad [T_5] = \begin{bmatrix} 6 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$$

We can find the eigenvectors for one of these matrices, and use it to simultaneously diagonalize all of them. The eigenvectors for which we can do this are found from $[T_3]$:

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then we can immediately write down the basis which simultaneously diagonalizes the Hecke operators as follows:

$$\{3(0:1) + (1:4) - (1:5), (1:4), (1:5)\}.$$

Since all of the transformation matrices for our Hecke operators are lower triangular, we can read off the eigenvalues from the diagonal. The eigenvalues of $[T_2]$ are 3 and 0, the eigenvalues of $[T_3]$ are 4 and -2, and the eigenvalues of $[T_5]$ are 6 and 3. From these and a few more Hecke operators, it is possible to calculate the first several coefficients for the Fourier expansion of the eigenforms of $\mathbb{M}_2(\Gamma_0(19))$. Since there are two eigenvalues for each Hecke operator, and the eigenvalue corresponding the Eisenstein space is not repeated, we expect to have two eigenforms, described as follows:

$$f_1 = \frac{3}{4} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + \dots$$
$$f_2 = q - 2q^3 - 2q^4 + 3q^5 - q^7 + \dots$$

In this case, f_1 corresponds to an Eisenstein series, and f_2 is a cusp form. An important result, the modularity theorem, allows us to further use this cusp form. This result was originally stated by Taniyama, Shimura, and Weil between 1956 and 1967[Wei67], proven for some specific cases by several mathematicians between 1995 and 2001, and finally proven for all cases in 2001 by Breuil et al. [BCDT01].

Theorem 3.1 (Modularity Theorem)**.** *Every rational elliptic curve corresponds to a modular form.*

This result particularly applies to weight 2 cusp forms. Since we have a weight 2 cusp form in f_2 for $\mathcal{M}_2(\Gamma_0(19))$, we can determine the elliptic curves which correspond to it. We can do so by using converting each Hecke eigenvalue $a_p(f)$ to its corresponding point count $a_p(E)$ on the elliptic curve, using the equation

$$a_p(f) = p + 1 - a_p(E) \quad \text{for } f \in \mathcal{M}_2(\Gamma_0(N)) \text{ and } E \text{ an elliptic curve.}$$

Let E describe the isogeny class of elliptic curves corresponding to f_2. Then we can get some of the point counts for E, utilizing the previous equation and the cusp form $f_2 \in$

$\mathcal{M}_2(\Gamma_0(19))$:

$$\begin{aligned} a_2(f_2) = 0 = 2 + 1 - a_2(E), &\quad \text{so } a_2(E) = 3 \\ a_3(f_2) = -2 = 3 + 1 - a_3(E), &\quad \text{so } a_3(E) = 6 \\ a_5(f_2) = 3 = 5 + 1 - a_5(E), &\quad \text{so } a_5(E) = 3 \\ a_7(f_2) = -1 = 7 + 1 - a_7(E), &\quad \text{so } a_7(E) = 9 \end{aligned}$$

There are three elliptic curves which arise from this particular cusp form, all within the same isogeny class. This implies that all of them have the same point counts as described above. They are given here:

$$\begin{aligned} &\text{Elliptic curve 19.a1:} \quad y^2 + y = x^3 + x^2 - 769x - 8470 \\ &\text{Elliptic curve 19.a2:} \quad y^2 + y = x^3 + x^2 - 9x - 15 \\ &\text{Elliptic curve 19.a3:} \quad y^2 + y = x^3 + x^2 + x \end{aligned}$$

3.2. $\mathcal{M}_4(\Gamma_0(19))$**.** We could use the methods we have devoloped to determine the eigenforms which generate the space of modular forms of weight 4 and level 19, or $\mathcal{M}_4(\Gamma_0(19))$. The computations are too difficult to do by hand, so they were done in magma and the results are given here. The space of modular symbols corresponding to $\mathcal{M}_4(\Gamma_0(19))$ can be represented as a tensor product:

$$\mathbb{M}_4(\Gamma_0(19)) = \mathbb{Q}[X, Y]_2 \otimes \mathbb{M}_2(\Gamma_0(19)).$$

We wrote in the section above that that Manin symbols corresponding to $\mathbb{M}_2(\Gamma_0(19))$ can be represented with the basis elements in $\mathbb{P}^1(\mathbb{F}_{19})$. Also, the space of homogeneous degree 2 rational polynomials in two variables, $\mathbb{Q}[X, Y]_2$, can be represented by the basis set $\{X^2, XY, Y^2\}$, as in the example in section 2. Therefore, there are 60 possible basis elements for $\mathbb{M}_4(\Gamma_0(19))$. After modding out by the necessary relations, this is reduced down to a 10-dimensional space. The basis of $\mathbb{M}_4(\Gamma_0(19))$ space follows:

$$\begin{aligned} &\{[X^2, (0:1)], [4X^2 + 4XY + Y^2, (1:2)], [64X^2 + 16XY + Y^2(1:8)], \\ &[9X^2 + 6XY + Y^2, (1:3)], [225X^2 + 30XY + Y^2, (1:15)], \\ &[196X^2 + 28XY + Y^2], (1:14)], [36X^2 + 12XY + Y^2, (1:6)], \\ &[81X^2 + 18XY + Y^2, (1:9)], [324X^2 + 36XY + Y^2, (1:18)], [Y^2, (1:0)]\} \end{aligned}$$

We can compute Hecke operators in a similar way to the example in section 2 as well, and use those eigenvalues to compute the eigenforms for the space $\mathcal{M}_4(\Gamma_0(19))$. After doing so, we find there is one Eisenstein series and 4 cusp forms. The Eisenstein series is the same G_4 as the one mentioned for $\mathcal{M}_4(\Gamma_0(5))$, and so will not be written again. The four cusp forms can be written as follows:

$$
\begin{aligned}
f_1 &= q - 3q^2 - 5q^3 + q^4 - 12q^5 + 15q^6 + 11q^7 + \dots \\
f_2 &= q + \frac{1}{324}(\alpha^4 - 12\alpha^3 - 51\alpha^2 + 684\alpha + 738)q^2 \\
&\quad + \frac{1}{972}(-\alpha^5 + 9\alpha^4 + 87\alpha^3 - 423\alpha^2 - 3114\alpha + 2322)q^3 \\
&\quad + \frac{1}{324}(\alpha^5 - 13\alpha^4 - 39\alpha^3 + 627\alpha^2 + 378\alpha - 1062)q^4 \\
&\quad + \frac{1}{972}(\alpha^5 - 21\alpha^4 + 57\alpha^3 + 1035\alpha^2 - 5094\alpha - 2430)q^5 \\
&\quad + \frac{1}{972}(-7\alpha^5 + 93\alpha^4 + 249\alpha^3 - 4491\alpha^2 - 1278\alpha + 5346)q^6 \\
&\quad + \frac{1}{243}(-\alpha^5 + 15\alpha^4 + 15\alpha^3 - 729\alpha^2 + 990\alpha + 1647)q^7 + \dots \\
f_3 &= q + \frac{1}{324}(\alpha^4 - 12\alpha^3 - 51\alpha^2 + 360\alpha + 1710)q^2 \\
&\quad + \frac{1}{972}(\alpha^5 - 21\alpha^4 + 57\alpha^3 + 927\alpha^2 - 2502\alpha - 8910)q^3 \\
&\quad + \frac{1}{324}(-\alpha^5 + 17\alpha^4 - 9\alpha^3 - 723\alpha^2 + 1062\alpha + 6282)q^4 \\
&\quad + \frac{1}{972}(-\alpha^5 + 9\alpha^4 + 87\alpha^3 - 315\alpha^2 - 1818\alpha - 2862)q^5 \\
&\quad + \frac{1}{972}(7\alpha^5 - 117\alpha^4 + 39\alpha^3 + 4959\alpha^2 - 6714\alpha - 44118)q^6 \\
&\quad + \frac{1}{243}(\alpha^5 - 15\alpha^4 - 15\alpha^3 + 621\alpha^2 - 342\alpha - 3753)q^7 + \dots
\end{aligned}
$$

$$\begin{aligned} f_4 = q &+ \frac{1}{162}(-\alpha^4 + 12\alpha^3 + 51\alpha^2 - 522\alpha - 738)q^2 \\ &+ \frac{1}{81}(\alpha^4 - 12\alpha^3 - 42\alpha^2 + 468\alpha + 630)q^3 \\ &+ \frac{1}{81}(-\alpha^4 + 12\alpha^3 + 24\alpha^2 - 360\alpha + 639)q^4 \\ &+ \frac{1}{81}(\alpha^4 - 12\alpha^3 - 60\alpha^2 + 576\alpha + 1575)q^5 \\ &+ \frac{1}{81}(2\alpha^4 - 24\alpha^3 - 39\alpha^2 + 666\alpha - 2034)q^6 \\ &+ \frac{1}{9}(4\alpha^2 - 24\alpha - 237)q^7 + \dots \end{aligned}$$

In f_2, f_3, and f_4, α represents a root of the sixth degree irreducible polynomial $x^6 - 18x^5 + 9x^4 + 972x^3 - 1620x^2 - 11664x - 2052$, which is from the splitting field generated by the third degree polynomial $x^3 - 3x^2 - 18x + 38$.

From Theorem 1.1, we should be able to see the square of the cusp form from $\mathcal{M}_2(\Gamma_0(19))$ as a linear combination of the cusp forms in $\mathcal{M}_4(\Gamma_0(19))$. We can do this in magma as well, and see that the square of the weight 2 cusp form, f^2, can be written as

$$\begin{aligned} f^2 = &\frac{1}{509328}(-33\alpha^5 + 490\alpha^4 + 555\alpha^3 - 23802\alpha^2 + 29250\alpha + 145836)f_2 \\ &+ \frac{1}{509328}(33\alpha^5 - 500\alpha^4 - 435\alpha^3 + 20748\alpha^2 - 13086\alpha - 37224)f_3 \\ &+ \frac{1}{254664}(5\alpha^4 - 60\alpha^3 + 1527\alpha^2 - 8082\alpha - 54306)f_4. \end{aligned}$$

Therefore, f^2 is a linear combination of cusp forms in $\mathcal{M}_4(\Gamma_0(19))$, as theory suggests.

References

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939. MR 1839918

[Man72] Ju. I. Manin, *Parabolic points and zeta functions of modular curves*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 19–66. MR 0314846

[Mer91] Loï c Merel, *Opérateurs de Hecke pour* $\Gamma_0(N)$ *et fractions continues*, Ann. Inst. Fourier (Grenoble) **41** (1991), no. 3, 519–537. MR 1136594

[Mer94] ———, *Universal Fourier expansions of modular forms*, On Artin's conjecture for odd 2-dimensional representations, Lecture Notes in Math., vol. 1585, Springer, Berlin, 1994, pp. 59–94. MR 1322319

[Ser73] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York-Heidelberg, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7. MR 0344216

[Ste07] William Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells. MR 2289048

[Wei67] André Weil, *über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168** (1967), 149–156. MR 0207658