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On the existence of spines for $\mathbb{Q}$-rank 1 groups

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Abstract. Let $X = \Gamma\backslash G/K$ be an arithmetic quotient of a symmetric space of non-compact type. In the case that G has $\mathbb{Q}$-rank 1, we construct Γ-equivariant deformation retractions of $D = G/K$ onto a set D_0. We prove that D_0 is a spine, having dimension equal to the virtual cohomological dimension of Γ. In fact, there is a $(k-1)$-parameter family of such deformation retractions, where k is the number of Γ-conjugacy classes of rational parabolic subgroups of G. The construction of the spine also gives a way to construct an exact fundamental domain for Γ.

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1. Introduction

Let $D = G/K$ be a symmetric space of non-compact type, where G is the group of real points of a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. Let Γ be an arithmetic subgroup of the rational points $\mathbf{G}(\mathbb{Q})$. Let (E, ρ) be a Γ-module over R. If Γ is torsion-free, the locally symmetric space $\Gamma\backslash D$ is a $K(\Gamma, 1)$ since D is contractible, and the group cohomology of Γ is isomorphic to the cohomology of the locally symmetric space, i.e. $H^*(\Gamma, E) \cong H^*(\Gamma\backslash D; \mathbb{E})$, where $\mathbb{E}$ denotes the local system defined by (E, ρ) on $\Gamma\backslash D$. When Γ has torsion, the correct treatment involves the language of orbifolds, but the isomorphism of cohomology is still valid by using a suitable sheaf $\mathbb{E}$ as long as the orders of the torsion elements of Γ are invertible in R.

The *virtual cohomological dimension* (vcd) of $\mathbf{G}$ is the smallest integer p such that the cohomology of $\Gamma\backslash D$ vanishes in degrees above p, where $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is any torsion-free arithmetic subgroup. Borel and Serre [11] show that the discrepancy between the dimension of D and vcd($\mathbf{G}$) is given by the $\mathbb{Q}$-*rank* of $\mathbf{G}$, the dimension of a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$. Thus one can hope to find a Γ-equivariant defor-

mation retract $D_0 \subset D$ of dimension equal to the virtual cohomological dimension of **G**. When such a subset exists, it is called a *spine*.

Spines have been constructed for many groups [1,8,14,18,20,21,25,27]. In [3], Ash describes the *well-rounded retraction*, a method for constructing a spine for all linear symmetric spaces. Ash and McConnell extend [3] to the Borel-Serre compactification in [7]. The well-rounded retraction works for algebraic groups **G** where the real points are isomorphic to a product of the following groups [15]:

  (i) $\mathrm{GL}_n(\mathbb{R})$.
  (ii) $\mathrm{GL}_n(\mathbb{C})$.
  (iii) $\mathrm{GL}_n(\mathbb{H})$.
  (iv) $\mathrm{O}(1, n-1) \times \mathbb{R}^\times$.
  (v) The non-compact Lie group with Lie algebra $\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$.

The retraction has been used in the computation of cohomology [2,4–6,18,21,23,25–27].

The well-rounded retraction proves the existence and gives a method of explicitly describing spines in linear symmetric spaces. However, for non-linear symmetric spaces, no general technique to construct spines is known. In fact, there were no examples until MacPherson and McConnell [20] constructed a spine in the Siegel upper half-space for the $\mathbb{Q}$-rank 2 group $\mathrm{Sp}_4(\mathbb{R})$.

In this paper, we deal with the case when **G** has $\mathbb{Q}$-rank 1, and use a family of exhaustion functions to construct a deformation retraction of D onto a spine. Each exhaustion function can be thought of as a measure of height with respect to a cusp or a rational parabolic subgroup of **G**.

The existence of such functions is not new. For example, for $\mathbf{G} = \mathrm{SL}_2$, the associated symmetric space is the Poincaré upper half-plane $\mathfrak{H}$ and the cusps can be identified with $\mathbb{Q} \cup \{\infty\}$. Let $f_\infty(z) = \mathrm{Im}(z)$ and let $f_{\gamma \cdot \infty}(z) = \mathrm{Im}(\gamma^{-1} \cdot z)$. For $\lambda = p/q$ in reduced form, the level set $f_\lambda(z) = c$ is a circle of radius $(2cq^2)^{-1}$ that is tangent to the real axis at λ. For $c = 1$, all of the level sets together form the Farey circle packing. For $c > 1$, the level sets are disjoint. This family of exhaustion functions yields the same deformation retraction as the well-rounded retraction, which is the familiar infinite trivalent tree in $\mathfrak{H}$ (Figure 1). The retraction can be visualized by shrinking c and allowing the level sets to flatten against each other.

Siegel defined a notion of distance from a cusp when **G** is the restriction of scalars, $\mathbf{G} = \mathrm{Res}_{k/\mathbb{Q}}(\mathrm{SL}_2)$, where k is a number field [24]. The cusps can be
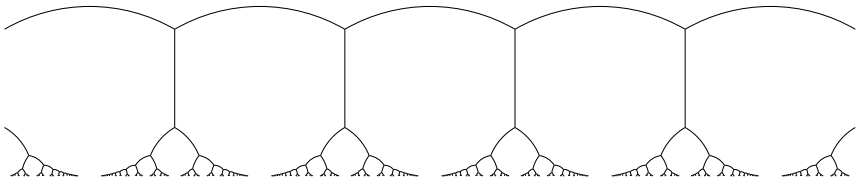

Fig. 1. Spine for $\mathrm{SL}_2(\mathbb{R})$

identified with $k \cup \{\infty\}$ and the distance can be expressed in terms of the norms of certain ideals on $\mathcal{O}_k$, the ring of integers in k. Siegel used these distance functions to construct a fundamental domain. One can show that his distance functions are a negative power of our exhaustion functions. For example, when k is an imaginary quadratic field, the associated symmetric space is hyperbolic 3-space

$$\mathbb{H}^3 = \{(z, r) \mid z \in \mathbb{C},\ r \in \mathbb{R} \text{ and } r > 0\}.$$

For a cusp $\lambda = \alpha/\beta$ in reduced form, the distance from (z, r) to λ is

$$d_\lambda(z, r) = \frac{|\beta z - \alpha|^2 + |\beta r|^2}{r N \langle \alpha, \beta \rangle},$$

where $N \langle \alpha, \beta \rangle$ is the norm of the ideal of $\mathcal{O}_k$ generated by α and β. The level set $d_\lambda(z, r) = 1/c$ is the sphere of radius $N \langle \alpha, \beta \rangle / (2c|\beta|^2)$ that is tangent to the plane $r = 0$ at the point $z = \alpha/\beta$. The level set $d_\infty(z, r) = 1/c$ is the plane $r = c$. When c is sufficiently large, these level sets are disjoint, and the retraction to the spine can again be visualized by shrinking c until the level sets flatten out against each other. The resulting spine is computed in [21, 27].

More directly, our exhaustion functions come out of Saper's work on tilings in [22]. In fact, our exhaustion functions are nothing more than the composition of his *normalized parameters* (in the $\mathbb{Q}$-rank 1 case) with the rational root. We use the exhaustion functions to construct a deformation retraction of D onto a spine. More precisely, we prove the following theorem:

Main Result. *Let $D = G/K$ be a symmetric space of non-compact type, where G is the group of real points of a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ with $\mathbb{Q}$-rank 1. Let Γ be an arithmetic subgroup of the rational points $\mathbf{G}(\mathbb{Q})$. There exists a Γ-equivariant retraction of the symmetric space onto a set $D_0 \subset D$ with the following properties:*

(i) *D_0 is a locally finite union of semi-analytic sets.*
(ii) *$\dim(\Gamma \backslash D_0) = \mathrm{vcd}(\Gamma)$.*
(iii) *D_0 has a decomposition $D_0 = \coprod D'(\mathcal{I})$, where $\mathcal{I}$ ranges over certain (at least 2-element) subsets of parabolic $\mathbb{Q}$-subgroups.*
(iv) *The decomposition satisfies $\gamma \cdot D'(\mathcal{I}) = D'(\gamma \mathcal{I})$ for every $\gamma \in \Gamma$.*
(v) *The quotient $\Gamma \backslash D_0$ is compact.*

In fact, there is a $(k-1)$-parameter family of different retractions, where k is the number of Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups.

Since each $D'(\mathcal{I})$ is a semi-analytic set, it follows from [19] that the decomposition of the spine $D_0 = \coprod D'(\mathcal{I})$ may be refined to a regular cell complex. Thus, the cohomology can be described by finite combinatorial data. In [29] we develop machinery for the computation of cohomology using D_0, and use it together with the results of this paper to investigate the cohomology of $\mathrm{SU}(2, 1)$ over the Gaussian integers.

Sections 2 and 3 set notation and define the main objects of this paper. Sections 4 and 5 give another interpretation of the exhaustion functions and prove

some of their properties. The main results are presented in Section 6. Section 7 introduces the notion of a strictly separated linear algebraic group, and show that for these groups, each $D'(\mathcal{I})$ is a smooth, contractible submanifold. Section 8 concludes by looking at a few examples of spines in low-dimensional cases and comparing to known results.

2. Notation and background

In order to set notation, we briefly recall without proof some standard results regarding algebraic groups over $\mathbb{Q}$, the geodesic action, and the Borel–Serre compactification [11]. We follow the exposition of [22]. Throughout this paper, G is the identity component of the real points of a semisimple $\mathbb{Q}$-rank 1 algebraic group defined over $\mathbb{Q}$. In order to lighten the notation and exposition, we will denote the algebraic group and its group of real points by the same Roman type ($G = G(\mathbb{R})$), and we may refer to the algebraic group when properly we should be referring to the group of real points. For example, we will speak of parabolic $\mathbb{Q}$-subgroups of G, when we properly should be referring to the group of real points of a parabolic $\mathbb{Q}$-subgroup of the algebraic group G.

2.1. Algebraic groups defined over $\mathbb{Q}$

For a reductive algebraic group H which is defined over $\mathbb{Q}$, let S_H denote the maximal $\mathbb{Q}$-split torus in the center of H, and set $A_H = S_H(\mathbb{R})^0$. Set

$$^0H = \bigcap_{\chi \in X(H)_{\mathbb{Q}}} \ker(\chi^2),$$

where $X(H)_{\mathbb{Q}}$ denotes the rational characters of H defined over $\mathbb{Q}$. Then H splits as a direct product

$$H = A_H \times {}^0H.$$

The group $^0H \subset H$ contains all compact and arithmetic subgroups of H.

For a parabolic subgroup $P \subset G$, let N_P denote its unipotent radical and let

$$\nu_P : P \to L_P = P/N_P$$

denote the projection to the Levi quotient. Let M_P denote the group ^0L_P. If $S_P \subset L_P$ denotes the maximal $\mathbb{Q}$-split torus in the center of L_P, then L_P splits as a commuting direct product $L_P = A_P M_P$, where $A_P = S_P(\mathbb{R})^0$. Note that L_P is the centralizer of A_P, and the connected component of the center of M_P is a $\mathbb{Q}$-anisotropic torus.

Let $_{\mathbb{Q}}\Delta_P = \{\alpha_P\}$ denote the simple root of the adjoint action of A_P on $\mathfrak{n}_P$, the Lie algebra of N_P. The roots will be viewed as characters of A_P and as elements of $\mathfrak{a}_P^*$. Since G has $\mathbb{Q}$-rank 1, α_P gives an isomorphism $A_P \to \mathbb{R}_{>0}$. Let

$$\psi_P : \mathbb{R}_{>0} \to A_P \tag{1}$$

be the isomorphism of groups given by $\psi_P(t) = \alpha_P^{-1}(t)$.

For two rational parabolic subgroups P and Q, there is a canonical isomorphism
$$A_P \xrightarrow{\sim} A_Q$$
induced by an element of $G(\mathbb{Q})$. In particular, the characters $\{\alpha_P\}_{P\in\mathcal{P}}$ can be identified and will be denoted α.

Any lift $i : L_P \to P$ determines a *Langlands decomposition*, which is a semi-direct product
$$P = N_P i(A_P M_P).$$

Let $K \subset G$ be a maximal compact subgroup and define $D = G/K$. There is a unique basepoint $x \in D$ such that $K = \mathrm{Stab}_G(x)$. This choice of K also determines the following data [17]:

(i) A maximal compact subgroup $K_P = K \cap P \subset G$ and a diffeomorphism $P/K_P \to D$.

(ii) A Cartan involution $\theta_x : G \to G$ such that K is the subgroup fixed by θ_x.

(iii) A unique θ_x-equivariant lifting $i_x : L_P \to P$. For a subset $T \subset L_P$, denote its lift by $T(x)$.

Definition 2.1 ([16]). Let $P \subset G$ be a rational parabolic subgroup and x a point in D. If the lift $L_P(x)$ is an algebraic subgroup of P, then x is a *rational basepoint for P*.

If the basepoint x can be chosen so that the associated maximal compact subgroup K is defined over $\mathbb{Q}$, then x is rational for all rational parabolic subgroups of G.

Definition 2.2. Let S be a maximal $\mathbb{Q}$-split torus of G. The *Weyl group over $\mathbb{Q}$* or *$\mathbb{Q}$-Weyl group* is the quotient of the normalizer $\mathcal{N}(S)$ of S by the centralizer $\mathcal{Z}(S)$ of S, and is denoted
$$_\mathbb{Q}W = \mathcal{N}(S)/\mathcal{Z}(S).$$

The following is the rational version of the standard *Bruhat decomposition* for Lie groups.

Theorem 2.3 ([9]). *Let $P \subset G$ be a minimal rational parabolic subgroup. Then $G(\mathbb{Q})$ is the disjoint union of the classes $P(\mathbb{Q})wP(\mathbb{Q})$ with $[w] \in {}_\mathbb{Q}W$. In particular, given $g \in G(\mathbb{Q})$, there exist $u_g \in N_P$, $[w] \in {}_\mathbb{Q}W$, and $p_g \in P$ such that $g = u_g w p_g$.*

2.2. Borel–Serre compactification [11]

Let x denote the point of D fixed by K. Then $P \in \mathcal{P}$ acts transitively on D, so every point $z \in D$ can be written as $z = p \cdot x$ for some $p \in P$. The geodesic action of A_P on D is given by
$$a \circ z = (p\tilde{a}) \cdot x,$$
where $\tilde{a}$ is the image of a in $A_P(x)$. The geodesic action commutes with the usual action of P on D and is independent of the choice of basepoint. Let $A_P \times {}^0P$ act

on D by $(a, p) \cdot z = p \cdot (a \circ z) = a \circ (p \cdot z)$. Then there is an analytic isomorphism of $A_P \times {}^0P$-homogeneous spaces

$$(a_{P,x}, q_P) : D \xrightarrow{\sim} A_P \times e(P), \tag{2}$$

where $e(P) = A_P \backslash D$ is the quotient of D by the geodesic action of A_P. Normalize $a_{P,x}$ so that $a_{P,x}(x) = e$. Via (2), D is a trivial principal A_P-bundle with *canonical cross-sections* given by the orbits of ${}^0P$.

   The Borel–Serre compactification is then constructed as follows: The simple root α_P induces an isomorphism $A_P \xrightarrow{\sim} \mathbb{R}_{>0}$ defined by $a \mapsto a^\alpha$. The *partial bordification* $D^c(P)$ associated to P is defined to be $\overline{A}_P \times_{A_P} D$. Equivalently, extend (2) to

$$(a_{P,x}, q_P) : D^c(P) \xrightarrow{\sim} \overline{A}_P \times e(P).$$

The Borel–Serre compactification $\overline{D} \equiv \bigcup_{P \in \mathcal{P}} D^c(P)$ is then given the unique structure of an analytic manifold with boundary so that each $D^c(P)$ is an open submanifold with boundary.

   The action of $G(\mathbb{Q})$ on D extends to an action on $\overline{D}$, and for an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, the quotient $\Gamma \backslash \overline{D}$ is compact.

2.3. Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups

Fix a proper parabolic $\mathbb{Q}$-subgroup P of G. The parabolic $\mathbb{Q}$-subgroups of G are all conjugate to P via elements of $G(\mathbb{Q})$. Thus, the set of $\mathbb{Q}$-parabolic subgroups $\mathcal{P}$ is in one-to-one correspondence with points of $G(\mathbb{Q})/P(\mathbb{Q})$, where $[g] \in G(\mathbb{Q})/P(\mathbb{Q})$ corresponds to the parabolic $\mathbb{Q}$-subgroup ${}^gP = P^{g^{-1}} \equiv gPg^{-1}$. For an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, the double coset space $\Gamma \backslash G(\mathbb{Q})/P(\mathbb{Q})$ is finite [10] and the number of elements in the space is known as the *class number*. In particular, there are only finitely many Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G. For $P \in \mathcal{P}$, let $\Gamma_P \equiv \Gamma \cap P \subset {}^0P$. Then there is a bijection between Γ/Γ_P and the parabolic subgroups that are Γ-conjugate to P via $\gamma\Gamma_P \mapsto {}^\gamma P$.

3. Main definitions

In this section, we define a family $\{f_P\}_{P \in \mathcal{P}}$ of *exhaustion functions* depending on a Γ-*invariant parameter*. We also define subsets of D associated to the family of exhaustion functions. In Section 6, these functions are used to give a Γ-equivariant retraction of D onto a codimension 1 set D_0.

Definition 3.1. A *parameter* is a family $\{O_P\}_{P \in \mathcal{P}}$ of closed submanifolds of D such that each O_P has the form

$$O_P = {}^0P \cdot x_P \quad \text{for some } x_P \in D.$$

A parameter is Γ-*invariant* if

$$\gamma \cdot O_P = O_{\gamma P} \quad \text{for all } \gamma \in \Gamma \text{ and } P \in \mathcal{P}.$$

Recall that by conjugation by $G(\mathbb{Q})$, one can canonically identify the A_P and corresponding simple roots α_P for different $P \in \mathcal{P}$. Denote the simple root by α and view it as a character on each A_P.

For $P \subset G$ a rational parabolic subgroup, let

$$a_P : D \times D \to A_P \tag{3}$$

be the map $a_P(z, x) = a_{P,x}(z)$. Thus $a_P(z, x)$ can be viewed as the amount of geodesic action required to push $^0P \cdot x$ to $^0P \cdot z$. The following is immediate from the definitions and the ^0P-invariance of $a_{P,x}$.

Proposition 3.2 ([11]). *Let $P \subset G$ be a rational parabolic subgroup. Then*

(i) $a_P(z, x) = a_P(x, z)^{-1}$ *for all $z, x \in D$.*
(ii) $a_{gP}(g \cdot z, g \cdot x) = a_P(z, x)$ *for all $z, x \in D$ and $g \in G(\mathbb{Q})$.*
(iii) $a_P(z, x) = a_P(p \cdot z, x) = a_P(z, p \cdot x)$ *for all $p \in {}^0P$.*
(iv) $a_P(z, s)a_P(s, x) = a_P(z, x)$ *for all z, x, and s in D.*

Given a point $x \in D$ and a rational parabolic subgroup $P \in \mathcal{P}$, consider the function $f : D \to \mathbb{R}_{>0}$ given by $f(z) = a_P(z, x)^\alpha$. Proposition 3.2 implies that f only depends on the orbit $^0P \cdot x$. This leads to the following definitions.

Definition 3.3. The *exhaustion functions associated to a parameter* $\{^0P \cdot x_P\}_{P \in \mathcal{P}}$ is the family $\{f_P\}_{P \in \mathcal{P}}$ of functions $f_P : D \to \mathbb{R}_{>0}$ given by

$$f_P(z) = a_P(z, x_P)^\alpha.$$

A family of exhaustion functions associated to a Γ-invariant parameter is called a family of Γ-*invariant exhaustion functions*.

Notice that Proposition 3.2 implies that

$$f_{gP}(g \cdot z) = f_P(z) f_{gP}(g \cdot x_P) \quad \text{for all } g \in G(\mathbb{Q}), \tag{4}$$

so that in particular, for exhaustion functions associated to a Γ-invariant parameter,

$$f_{\gamma P}(\gamma \cdot z) = f_P(z) \quad \text{for all } \gamma \in \Gamma. \tag{5}$$

For a parabolic P, define $D(P) \subset D$ to be the set of $z \in D$ such that $f_P(z) \geq f_Q(z)$ for every $Q \in \mathcal{P} \setminus \{P\}$. More generally, for a subset $\mathcal{I} \subseteq \mathcal{P}$,

$$E(\mathcal{I}) = \{z \in D \mid f_P(z) = f_Q(z) \text{ for every pair } P, Q \in \mathcal{I}\}, \tag{6}$$

$$D(\mathcal{I}) = \bigcap_{P \in \mathcal{I}} D(P), \tag{7}$$

$$D'(\mathcal{I}) = D(\mathcal{I}) \setminus \bigcup_{\mathcal{I}' \supsetneq \mathcal{I}} D(\mathcal{I}'). \tag{8}$$

It follows that $D'(\mathcal{I}) \subseteq D(\mathcal{I}) \subset E(\mathcal{I})$ and $D(\mathcal{I}) = \coprod_{\tilde{\mathcal{I}} \supseteq \mathcal{I}} D'(\tilde{\mathcal{I}})$.

Definition 3.4. The set $D'(\mathcal{I})$ will be called a *degenerate tile*. Let $f_\mathcal{I}$ denote the restriction to $E(\mathcal{I})$ of f_P for $P \in \mathcal{I}$.

Definition 3.5. Let $\mathcal{I} \subseteq \mathcal{P}$, $P \in \mathcal{I}$, and $z \in E(\mathcal{I})$. Then z is called a *first contact* for $\mathcal{I}$ if $f_{\mathcal{I}}(z)$ is a global maximum of $f_{\mathcal{I}}$ on $E(\mathcal{I})$.

Definition 3.6. A subset $\mathcal{I} \subset \mathcal{P}$ is called *admissible* if $D(\mathcal{I})$ is non-empty, and *strongly admissible* if $D'(\mathcal{I})$ is non-empty.

Proposition 3.7. *Let $\mathcal{S}$ denote the collection of strongly admissible subsets of $\mathcal{P}$. Then the symmetric space D has a Γ-invariant degenerate tiling*

$$D = \coprod_{\mathcal{I} \in \mathcal{S}} D'(\mathcal{I})$$

such that $\gamma \cdot D'(\mathcal{I}) = D'(^\gamma\mathcal{I})$ for all $\gamma \in \Gamma$ and $\mathcal{I} \in \mathcal{S}$.

Definition 3.8. Given a family of Γ-invariant exhaustion functions, define a subset $D_0 \subset D$ by

$$D_0 = \coprod_{\substack{\mathcal{I} \in \mathcal{S} \\ |\mathcal{I}| > 1}} D'(\mathcal{I}).$$

Let f_{D_0} denote the function on D_0 given by

$$f_{D_0}(z) = f_{\mathcal{I}}(z) \quad \text{for } z \in D(\mathcal{I}). \tag{9}$$

4. Exhaustion functions via representation theory

In this section we describe a systematic way to construct exhaustion functions.

Fix a parabolic $\mathbb{Q}$-subgroup $P \subset G$. Choose a rational basepoint $x^* \in D$ for P so that the θ_{x^*}-stable lift $A_P(x^*)$ of A_P to P is the connected component of the real points of a maximally $\mathbb{Q}$-split torus of G. Let $M_P(x^*)$ denote the θ_{x^*}-stable lift of $M_P = {}^0L_P$ to P, and let $\mathfrak{a}_P$ denote the Lie algebra of $A_P(x^*)$. The simple $\mathbb{Q}$-root α can be viewed as a linear functional on $\mathfrak{a}_P$. Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. Fix a Cartan subalgebra containing the complexification of $\mathfrak{a}_P$, and let $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\}$ be the simple roots. Order them so that $\tilde{\alpha}_j|_{\mathfrak{a}_P} = \alpha$ for $1 \leq j \leq l$ and $\tilde{\alpha}_j|_{\mathfrak{a}_P} = 0$ for $l + 1 \leq j \leq m$. Let $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_m\}$ be the associated fundamental weights. Consider the weight $\tilde{\omega} = s \sum_{j=1}^{l} \tilde{\omega}_j$. Then for s a sufficiently large positive integer, $\tilde{\omega}$ is the highest weight of a finite-dimensional (strongly rational) representation of G [12].

Let (V, π) denote this representation, and ω the restriction of $\tilde{\omega}$ to $\mathfrak{a}_P$. The restrictions of weights to $\mathfrak{a}_P$ are called *restricted $\mathbb{Q}$-weights*. Every restricted $\mathbb{Q}$-weight of V is of the form $\omega - j\alpha$, where j is a non-negative integer, and the lowest restricted $\mathbb{Q}$-weight is $-\omega$. Thus V has a decomposition as a direct sum of restricted $\mathbb{Q}$-weight spaces,

$$V = \bigoplus_\lambda V_\lambda = \bigoplus_{k=0}^{N} V_{\omega - k\alpha}, \quad \text{where } \omega - N\alpha = -\omega. \tag{10}$$

Fix an *admissible inner product* $\langle \cdot, \cdot \rangle$ on V [13], that is, one for which

$$\pi(g)^* = \pi(\theta_{x^*}g)^{-1} \quad \text{for all } g \in G,$$

so that in particular,

$$\pi(k)^* = \pi(k)^{-1} \quad \text{for all } k \in K.$$

Since the ℚ-Weyl group $_{\mathbb{Q}}W$ is finite, by averaging over $_{\mathbb{Q}}W$ one can arrange that the inner product is also $_{\mathbb{Q}}W$-invariant. The data $(V, \pi, \langle \cdot, \cdot \rangle, x^*)$ will be called a *P-adapted representation*.

Proposition 4.1. *Let $P \subset G$ be a parabolic ℚ-subgroup with rational basepoint x^*, and let $(V, \pi, \langle \cdot, \cdot \rangle, x^*)$ be a P-adapted representation. Then $M_P(x^*)$ preserves the restricted ℚ-weight spaces of V. The highest and lowest restricted weight spaces are one-dimensional, and $M_P(x^*)$ preserves length on both.*

Proof. Since $M_P(x^*)$ centralizes $A_P(x^*)$, it follows that $M_P(x^*)$ preserves the restricted weight space decomposition of V. Since $\tilde{\omega}$ is orthogonal to $\tilde{\alpha}_j$ for $l + 1 \le j \le m$, it follows that the highest weight space V_ω is 1-dimensional. Furthermore, this implies that the connected component of $M_P(x^*)$ acts trivially on V_ω. Now $M_P(x^*)$ has only finitely many connected components because $M_P \times A_P \cong L_P$, the real points of the Levi quotient, which has only finitely many connected components since it is Zariski connected [28]. It follows that $M_P(x^*)$ preserves length when restricted to V_ω. An analogous argument shows that $V_{-\omega}$ is 1-dimensional, and $M_P(x^*)$ preserves length on $V_{-\omega}$. $\qquad\square$

Write $z \in D$ as $z = p \cdot x$ with $p \in P$. Using Langlands decomposition, write p as $u\tilde{a}_P(z, x^*)m$, where $u \in N_P$, $\tilde{a}_P(z, x^*) \in A_P(x^*)$, and $m \in M_P(x^*)$. Fix $g \in G(\mathbb{Q})$. From (10) there exist unit vectors $v_k \in V_{\omega - k\alpha}$ and constants $c_k(z) \in \mathbb{C}$ such that

$$\pi(p^{-1}g)v = \pi(\tilde{a}_P(z, x^*))^{-1} \sum_k c_k(z)v_k. \tag{11}$$

Note that $c_k(z)$ is only defined up to multiplication by norm 1 scalars. Since $M_P(x^*)$ centralizes $A_P(x^*)$, the $c_k(z)$ only depend on m and u. Thus for each fixed g, the norm $|c_k|$ can be viewed as a function from D to $\mathbb{R}_{\ge 0}$ that is invariant under the geodesic action of A_P.

Proposition 4.2. *Fix $g \in G(\mathbb{Q})$ and two distinct rational parabolic subgroups P and $Q = {}^gP$. Fix a rational basepoint x^* for P and let $(V, \pi, \langle \cdot, \cdot \rangle, x^*)$ be a P-adapted representation. Then the functions $|c_k| : D \to \mathbb{R}$ (defined above) satisfy*

$$f_Q(z) = \frac{f_Q(g \cdot x_P)f_P(z)}{(\sum_{k=0}^{N} |c_k(z)|^2 f_P(x^*)^{-2k} f_P(z)^{2k})^{1/N}} \quad \text{for all } z \in D.$$

Proof. From the definition of the P-adapted representation, $\alpha = 2\omega/N$. Write $z \in D$ as $z = p \cdot x^*$, where $p \in P$. Let $v \in V$ denote a norm 1 highest weight vector. Since N_P fixes v, Proposition 4.1 implies

$$\|\pi(p^{-1})v\|^{-2/N} = \|\pi(m^{-1}\tilde{a}_P(z, x^*)^{-1}u^{-1})v\|^{-2/N} = a_P(z, x^*)^\alpha. \tag{12}$$

Then from the definition of f_P and (12),

$$f_P(z) = a_P(z, x^*)^\alpha f_P(x^*) = \|\pi(p^{-1})v\|^{-2/N} f_P(x^*). \tag{13}$$

Then (4), (13), and the orthogonality of the restricted weight spaces imply

$$f_Q(z) = f_P(g^{-1} \cdot z) f_Q(g \cdot x_P)$$

$$= f_Q(g \cdot x_P) f_P(x^*) \left(\left\| \pi(\tilde{a}_P(z, x^*))^{-1} \sum_{k=0}^{N} c_k(z) v_k \right\|^2 \right)^{-1/N}$$

$$= f_Q(g \cdot x_P) f_P(x^*) \left(\left\| \sum_{k=0}^{N} c_k(z) a_P(z, x^*)^{-\omega+k\alpha} v_k \right\|^2 \right)^{-1/N}$$

$$= f_Q(g \cdot x_P) f_P(x^*) \left(\sum_{k=0}^{N} |c_k(z)|^2 a_P(z, x^*)^{-2\omega+2k\alpha} \right)^{-1/N}$$

$$= f_Q(g \cdot x_P) f_P(x^*) a_P(z, x^*)^{2\omega/N} \left(\sum_{k=0}^{N} |c_k(z)|^2 a_P(z, x^*)^{2k\alpha} \right)^{-1/N}$$

$$= f_Q(g \cdot x_P) f_P(z) \left(\sum_{k=0}^{N} |c_k(z)|^2 f_P(x^*)^{-2k} f_P(z)^{2k} \right)^{-1/N}. \qquad \square$$

Proposition 4.3. *Let P and $Q = {}^g P$ be two distinct $\mathbb{Q}$-parabolic subgroups with $g \in G(\mathbb{Q})$. Let $g = u_g w p_g$ be the $\mathbb{Q}$-Bruhat decomposition of g as in Proposition 2.3. Use the Langlands decomposition to express $z \in D$ as $z = u\tilde{a}_P(z, x^*) m \cdot x^*$ for a rational basepoint x^* for P, $u \in N_P$, $m \in M_P(x^*)$, and $\tilde{a}_P(z, x^*) \in A_P(x^*)$. Then*

$$|c_N(z)| = a_P(p_g \cdot x^*, x^*)^\omega.$$

If $|c_k(z)| = 0$ for $k = 0, \dots, N-1$, then

(i) *$u = u_g$.*
(ii) *z is a first contact for $\{P, Q\}$.*
(iii) *$f_P(z) = f_Q(g \cdot x_P)^{1/2} a_P(p_g \cdot x^*, x^*)^{-\alpha/2} f_P(x^*)$.*

Proof. From the definition of $|c_k(z)|$,

$$|c_k(z)| = \|\mathrm{pr}_{V_{\omega-k\alpha}}(\pi(m^{-1}u^{-1}g)v)\| = \|\mathrm{pr}_{V_{\omega-k\alpha}}(\pi(m^{-1}u^{-1}u_g w p_g)v)\|$$

$$= a_P(p_g \cdot x^*, x^*)^\omega \|\mathrm{pr}_{V_{\omega-k\alpha}}(\pi(m^{-1}u^{-1}u_g w)v)\|. \tag{14}$$

Notice that $\pi(w)v \in V_{-\omega}$ and for $n \in N_P$,

$$\pi(nw)v = \pi(w)v + \text{higher weight vectors}.$$

Since $M_P(x^*)$ preserves weight spaces and preserves length on $V_{-\omega}$ by Proposition 4.1,

$$|c_N(z)| = a_P(p_g \cdot x^*, x^*)^\omega \|\mathrm{pr}_{V_{-\omega}}(\pi(m^{-1}u^{-1}u_g w)v)\| = a_P(p_g \cdot x^*, x^*)^\omega. \tag{15}$$

Now suppose $|c_k(z)| = 0$ for $k = 0, \ldots, N - 1$. Then from (14), $\pi(m^{-1}u^{-1}u_g w)v \in V_{-\omega}$. Since $\pi(m^{-1})$ preserves weight spaces by Proposition 4.1, $\pi(u^{-1}u_g)$ preserves $V_{-\omega}$. It follows that $u = u_g$.

Note that since $z \in E(\{P, Q\})$,

$$f_Q(g \cdot x_P)^N = \sum_{k=0}^{N} \frac{|c_k(z)|^2}{f_P(x^*)^{2k}} f_P(z)^{2k} \leq \frac{|c_N(z)|^2}{f_P(x^*)^{2N}} f_P(z)^{2N}.$$

Since $|c_N(z)| \neq 0$ and is independent of z from (15),

$$f_P(z) \leq \frac{f_Q(g \cdot x_P)^{1/2} f_P(x^*)}{|c_N(z)|^{1/N}} \quad \text{on } E(\{P, Q\}). \tag{16}$$

The bound is attained when $u(z) = u_g$ and

$$f_P(z) = \frac{f_Q(g \cdot x_P)^{1/2} f_P(x^*)}{|c_N(z)|^{1/N}} = \frac{f_Q(g \cdot x_P)^{1/2} f_P(x^*)}{a_P(p_g \cdot x^*, x^*)^{\alpha/2}}. \qquad \square$$

5. Properties of exhaustion functions

Notice that for a family of Γ-invariant exhaustion functions,

$$f_{\gamma P}(\gamma \cdot x_P) = 1 \quad \text{for every } \gamma \in \Gamma. \tag{17}$$

Proposition 3.2 implies the following.

Proposition 5.1. *Let $P \subset G$ be a rational parabolic subgroup and $\{f_P\}_{P \in \mathcal{P}}$ a family of exhaustion functions associated to a Γ-invariant parameter $\{^0P \cdot x_P\}_{P \in \mathcal{P}}$. Then*

 (i) $f_P(x_P) = 1$.
 (ii) $f_P(a \circ z) = a^\alpha f_P(z)$ *for all $z \in D$ and $a \in A_P$.*
 (iii) $f_P(z) = a_P(z, x)^\alpha f_P(x)$ *for all $z, x \in D$.*
 (iv) $f_P(p \cdot z) = f_P(z)$ *for all $z \in D$ and $p \in {}^0P$.*
 (v) $f_{gP}(z) = f_P(g^{-1}z) f_{gP}(g \cdot x_P)$ *for all $z \in D$ and $g \in G(\mathbb{Q})$.*

The following proposition follows from reduction theory [10].

Proposition 5.2. *Let $\{f_P\}_{P \in \mathcal{P}}$ be a Γ-invariant family of exhaustion functions. Then there exists a constant $C > 0$, depending on the Γ-invariant parameter, such that*

$$\sup_{P \in \mathcal{P}} f_P(z) \geq C \quad \text{for all } z \in D.$$

Lemma 5.3. *Let M be a Riemannian manifold and ψ an isometry of M. Let f and h be smooth functions on M such that $h(x) = f(\psi(x))$. Then*

$$\nabla h(x) = (d\psi^{-1})_{\psi(x)} \nabla f(\psi(x)) \quad \text{for all } x \in M.$$

Proposition 5.4. *Let $x, x' \in D$. If $f_P(x) = f_P(x')$ then $\|\nabla f_P(x)\| = \|\nabla f_P(x')\|$.*

Proof. If $f_P(x) = f_P(x')$, then there exists $p \in {}^0P$ such that $p \cdot x = x'$. Since f_P is ${}^0P$-invariant, $f_P(p \cdot z) = f_P(z)$ for all $z \in D$. The result then follows from Lemma 5.3 by noting that ${}^0P \subset G$ acts by isometries on D. $\qquad \square$

Lemma 5.5. *Let $P \subset G$ be a rational parabolic subgroup. The geodesic action of A_P induces an isometry on the 1-dimensional tangent space $T_z(A_P \circ z)$.*

Proof. Since the geodesic action of A_P is equal to the regular action of its θ_z-invariant lift on the orbit $A_P \circ z$, the geodesic action induces an isometry on the 1-dimensional tangent space $T_z(A_P \circ z)$. Furthermore, the geodesic action of A_P commutes with the regular action of P on D. The result follows. $\square$

Proposition 5.6. *Fix two distinct rational parabolic subgroups P and Q. Then*

$$f_P(z)\|\nabla f_Q(z)\| = f_Q(z)\|\nabla f_P(z)\| \quad \text{for all } z \in D.$$

In particular, if $z \in E(\{P,Q\})$, then $\|\nabla f_P(z)\| = \|\nabla f_Q(z)\|$.

Proof. There exists a $g \in G(\mathbb{Q})$ such that $Q = {}^g P$. By Proposition 5.1,

$$f_Q(z) = f_P(g^{-1} \cdot z) f_Q(g \cdot x_P). \tag{18}$$

Then Lemma 5.3 implies that

$$\nabla f_Q(z) = f_Q(g \cdot x_P)(dL_g)_{g^{-1} \cdot z} \nabla f_P(g^{-1} \cdot z), \tag{19}$$

where $L_g : D \to D$ is the isometry given by left translation by $g \in G$. Since f_P is ${}^0 P$-invariant,

$$\nabla f_p(g^{-1} \cdot z) = (dL_{p^{-1}})_{pg^{-1} \cdot z} \nabla f_P(pg^{-1} \cdot z) \quad \text{for any } p \in {}^0 P. \tag{20}$$

There exists a $p \in {}^0 P$ such that $pg^{-1}z = t \circ z$, where $t = \psi_P\left(\frac{f_P(g^{-1}z)}{f_P(z)}\right)$. Note that

$$f_P(t \circ z) = \left(\frac{f_P(g^{-1}z)}{f_P(z)}\right) f_P(z). \tag{21}$$

Then by Lemma 5.3 and (21),

$$\nabla f_P(pg^{-1} \cdot z) = \left(\frac{f_P(g^{-1}z)}{f_P(z)}\right) d(t \circ \cdot)_z \nabla f_P(z). \tag{22}$$

Combining (19), (20), and (22) yields

$$\nabla f_Q(z) = \frac{f_Q(z)}{f_P(z)}(dL_{gp^{-1}})_{pg^{-1} \cdot z} d(t \circ \cdot)_z \nabla f_P(z). \tag{23}$$

Thus by Lemma 5.5, $f_P(z)\|\nabla f_Q(z)\| = f_Q(z)\|\nabla f_P(z)\|$. $\square$

Proposition 5.7. *Let P and Q be distinct parabolic $\mathbb{Q}$-subgroups of G, and $z \in E(\{P,Q\})$. Then the following are equivalent:*

  (i) *z is a first contact for $\{P,Q\}$.*
  (ii) *z is a critical point for $f_P|_{E(\{P,Q\})}$.*
  (iii) *$\nabla f_P(z) = -\nabla f_Q(z)$.*

Proof. Let $E = E(\{P, Q\})$. By Proposition 5.6, $\|\nabla f_P(z)\| = \|\nabla f_Q(z)\|$ for $z \in E$. It follows that

$$\nabla f_P|_E(z) = \mathrm{pr}_{T_z E}(\nabla f_P(z)) = \frac{1}{2}(\nabla f_P(z) + \nabla f_Q(z)),$$

and hence (ii) is equivalent to (iii).

It is clear that if z is a first contact for $\{P, Q\}$, then $\nabla f_P(z)$ and $\nabla f_Q(z)$ point in opposite directions. Thus Proposition 5.6 implies that $\nabla f_P(z) = -\nabla f_Q(z)$ and hence (i) implies (ii).

Now suppose $\nabla f_P(z) = -\nabla f_Q(z)$ for some $z \in E(\{P, Q\})$. By Proposition 4.3, to show that z is a first contact for $\{P, Q\}$, it suffices to show that $|c_k| = 0$ for $k = 0, \ldots, N - 1$. Let $y = f_P(z)$ and fix $g \in G(\mathbb{Q})$ such that $Q = {}^g P$. Then by Proposition 4.2,

$$f_Q(z) = \frac{f_Q(g \cdot x_P) y}{(\sum_{k=0}^{N} |c_k(z)|^2 f_P(x^*)^{-2k} y^{2k})^{1/N}} \qquad \text{for all } z \in D.$$

Let $\frac{\partial}{\partial y}$ be the vector field on D which is tangent to the flow of the geodesic action of A_P such that $\frac{\partial}{\partial y}(f_P) = 1$. For $z \in E(\{P, Q\})$,

$$\sum_{k=0}^{N} |c_k(z)|^2 f_P(x^*)^{-2k} y^{2k} = f_Q(g \cdot x_P)^N, \tag{24}$$

and hence,

$$|c_N(z)|^2 f_P(x^*)^{-2N} y^{2N} = f_Q(g \cdot x_P)^N - \sum_{k=0}^{N-1} |c_k(z)|^2 f_P(x^*)^{-2k} y^{2k}. \tag{25}$$

If $\nabla f_P(z) = -\nabla f_Q(z)$, then in particular, $\frac{\partial f_P}{\partial y}(z) = -\frac{\partial f_Q}{\partial y}(z)$. This implies that

$$1 = -\left(1 - \frac{y}{N f_Q(g \cdot x_P)^N} \sum_{k=0}^{N} 2k |c_k(z)|^2 f_P(x^*)^{-2k} y^{2k-1}\right)$$

$$= -1 + \sum_{k=0}^{N} \frac{2k |c_k(z)|^2}{N f_Q(g \cdot x_P)^N f_P(x^*)^{2k}} y^{2k}. \tag{26}$$

Combining (25) and (26) gives

$$1 = -1 + 2\left(1 - \sum_{k=0}^{N-1} \frac{N-k}{N} \frac{|c_k(z)|^2}{f_Q(g \cdot x_P)^N f_P(x^*)^{2k}} y^{2k}\right). \tag{27}$$

Since $y > 0$, we have $|c_k(z)| = 0$ for $k = 0, \ldots, N - 1$. $\qquad \square$

Propositions 5.6 and 5.7 immediately yield the following.

Proposition 5.8. *Let P and Q be two distinct parabolics in $\mathcal{P}$. Then*

$$\langle \nabla f_P, \nabla f_P \rangle \geq |\langle \nabla f_P, \nabla f_Q \rangle|$$

on $E(\{P, Q\})$ with equality on the set of first contacts for $\{P, Q\}$.

6. The spine

For $z \in D$, denote by $M(z)$ the strongly admissible set $\mathcal{I}$ for which $z \in D'(\mathcal{I})$.

Proposition 6.1. *Let z be a point in D. Then the set $M(z) \subset \mathcal{P}$ is $\mathrm{Stab}_\Gamma(z)$-stable.*

Proof. Let $P \in M(z)$ and $\gamma^{-1} \in \mathrm{Stab}_\Gamma(z)$. Then $f_P(z) = f_P(\gamma^{-1} \cdot z) = f_{\gamma P}(z)$. It follows that $^\gamma P \in M(z)$. $\qquad \square$

Theorem 6.2. *D_0 is a Γ-invariant deformation retract of D.*

Proof. Since Γ acts on $\mathcal{P}$ by conjugation, it also acts on subsets $\mathcal{I}$ of $\mathcal{P}$ in the obvious way. Suppose $z \in D_0$. Then $z \in D'(\mathcal{I})$ for some $\mathcal{I}$. This means that

(i) $f_P(z) = f_Q(z)$ for every pair $P, Q \in \mathcal{I}$.
(ii) $f_P(z) > f_R(z)$ for every $P \in \mathcal{I}$ and $R \in \mathcal{P} \setminus \mathcal{I}$.

The Γ-invariance of $\{f_P\}$ implies that

(i) $f_{\gamma P}(\gamma \cdot z) = f_{\gamma Q}(\gamma \cdot z)$ for every pair $P, Q \in \mathcal{I}$.
(ii) $f_{\gamma P}(\gamma \cdot z) > f_{\gamma R}(\gamma \cdot z)$ for every $P \in \mathcal{I}$ and $R \in \mathcal{P} \setminus \mathcal{I}$.

Thus $\gamma \cdot z \in D'(^\gamma \mathcal{I})$ for every $\gamma \in \Gamma$, and so D_0 is Γ-invariant.

Proposition 3.7 and Definition 3.8 imply a decomposition of D as a disjoint union

$$D = D_0 \sqcup \coprod_{P \in \mathcal{P}} D'(P). \tag{28}$$

From Proposition 5.1,

$$f_P(a \circ z) = a^\alpha f_P(z) \quad \text{for all } z \in D \text{ and } a \in A_P.$$

Thus for $z \in D'(P)$, Proposition 5.2 implies that there exists an $a(z) \in A_P$ such that $a(z) \circ z \in D_0$. Thus we can define a function $\mu : D \to \mathbb{R}_{>0}$ by $\mu(z) = a(z)^\alpha$, where it is understood that if $z \in D_0$ then $\mu(z) = 1$.

Recall that there is a natural isomorphism $\psi_P : \mathbb{R}_{>0} \to A_P$ defined in (1). Define a family of maps $\tilde{r}_t : D \to D$ by

$$\tilde{r}_t(z) = \psi_P((1-t) + t\mu(z)) \circ z \quad \text{for } z \in D(P). \tag{29}$$

Note that this is well defined because $\mu \equiv 1$ on D_0. It is clear that $\tilde{r}_0 = \mathrm{id}_D$ and $\tilde{r}_1(D) \subset D_0$. The Γ-invariance of the exhaustion functions implies that

$$\gamma \cdot \tilde{r}_t(z) = \tilde{r}_t(\gamma \cdot z) \quad \text{for every } \gamma \in \Gamma \text{ and } z \in D.$$

It is clear that $\tilde{r}$ is a continuous function of t. To see that it is a continuous function of z, it suffices to note that μ is a continuous function. Thus $\tilde{r}$ gives a Γ-equivariant deformation retraction of D onto D_0. $\qquad \square$

Proposition 6.3. *For every $z \in D$,*

$$\mathrm{Stab}_\Gamma(z) = \mathrm{Stab}_\Gamma(\tilde{r}_t(z)) \quad \text{for } t < 1 \quad \text{and} \quad \mathrm{Stab}_\Gamma(z) \subseteq \mathrm{Stab}_\Gamma(\tilde{r}_1(z)).$$

Proof. Let γ be an element of $\text{Stab}_\Gamma(z)$. Then $\gamma \cdot \tilde{r}_t(z) = \tilde{r}_t(\gamma \cdot z) = \tilde{r}_t(z)$. Hence $\gamma \in \text{Stab}_\Gamma(\tilde{r}_t(z))$. Notice that for each $z \notin D_0$, $c(t) = \tilde{r}_t(z), 0 \le t \le 1$, is a reparameterization of a geodesic, and Γ acts by isometries. Thus every $\gamma \in \text{Stab}_\Gamma(z)$ fixes the geodesic through $\tilde{r}_1(z)$ and z. In particular, $\text{Stab}_\Gamma(z') = \text{Stab}_\Gamma(z)$ whenever $z = \tilde{r}_t(z')$ for some $t < 1$. $\square$

Theorem 6.4. *If $\mathcal{I} \subset \mathcal{P}$ is a 2-element subset then $E(\mathcal{I})$ is a contractible, smooth codimension 1 submanifold of D. If $\mathcal{I}$ has three elements, then $E(\mathcal{I})$ is a smooth codimension 2 submanifold of D.*

Proof. Let $\mathcal{I} = \{P, Q\}$ and $E = E(\mathcal{I})$. Then E is the zero set of $f_P - f_Q$. Thus to show smoothness, it suffices to show that $\nabla f_P(z) - \nabla f_Q(z) \neq 0$ for every $z \in E$. Suppose otherwise. Consider the geodesic $c(t)$ with $c(0) = z$ and $c'(0) = \nabla f_P(z)$. For t sufficiently large, $c(t)$ is contained in the degenerate tile $D'(P)$. Similarly, since $c'(0) = \nabla f_Q(z)$, for t sufficiently large, $c(t)$ is in $D'(Q)$. This contradicts the fact that $D'(P) \cap D'(Q) = \emptyset$.

To show contractibility, we will use the gradient flow of $f_P|_E$ to contract E to the set of first contacts for $f_P|_E$ and show that this set is contractible. Proposition 4.3 implies that on this set, $f_P(z)$ and $u(z)$ are constant, and $m(z)$ is free to range over $M_P(x)$. In particular, the set of critical points is diffeomorphic to the symmetric space D_{M_P} for M_P. Since D_{M_P} is contractible, this proves the result.

Now suppose $\mathcal{I} = \{P, Q, R\}$. The set $E(\mathcal{I})$ is defined by $h_1 = 0$, $h_2 = 0$, and $h_3 = 0$, where $h_1 = f_P - f_Q$, $h_2 = f_P - f_R$, and $h_3 = f_Q - f_R$. One must show that $\{\nabla h_1, \nabla h_2, \nabla h_3\}$ has constant rank on $E(\mathcal{I})$. Fix a point $z \in E(\mathcal{I})$. Note that $\nabla h_3 = \nabla h_2 - \nabla h_1$, so the rank is less than 3. All of ∇h_1, ∇h_2, and ∇h_3 are non-zero, so that the rank is either 1 or 2. We prove the rank is not 1 by contradiction. Suppose it is 1. Then for each $z \in E(\mathcal{I})$ there exist non-zero constants d_1 and d_2 such that

$$\nabla h_1(z) = d_1 \nabla h_3(z), \tag{30}$$

$$\nabla h_2(z) = d_2 \nabla h_3(z). \tag{31}$$

Using Proposition 5.8 and taking the inner product of (30) with $\nabla f_Q(z)$, it follows that $d_1 < 0$. Similarly, by taking the inner product of (31) with $\nabla f_R(z)$, it follows that $d_2 > 0$. Taking the inner product of (30) with $\nabla f_P(z)$ yields

$$\langle \nabla f_3(z), \nabla f_P(z) \rangle < 0.$$

Similarly, taking the inner product of (31) with $\nabla f_P(z)$ yields

$$\langle \nabla f_3(z), \nabla f_P(z) \rangle > 0,$$

which gives the desired contradiction. $\square$

Theorem 6.5. *D_0 is a spine for G. In particular,*
  (i) *$D_0 \subset D$ is a Γ-equivariant deformation retract.*
 (ii) *$\dim_{\mathbb{R}}(\Gamma \backslash D_0) = \text{vcd}(\Gamma)$.*
(iii) *$\Gamma \backslash D_0$ is compact.*

Proof. The first statement is Theorem 6.2. By [11], the virtual cohomological dimension of G is given by

$$\mathrm{vcd}(G) = \dim_{\mathbb{R}}(D) - \mathrm{rank}_{\mathbb{Q}}(G) = \dim_{\mathbb{R}}(D) - 1.$$

Theorem 6.4 implies that $\dim_{\mathbb{R}}(\Gamma \backslash D_0) = \mathrm{vcd}(G)$. To see that $\Gamma \backslash D_0$ is compact, notice that $D_0 \subset D \subset \overline{D}$ is closed and $\Gamma \backslash \overline{D}$ is compact by [11]. $\qquad\square$

7. Separated groups

One would like to know when the decomposition of D_0 given in Definition 3.8 is nice in some sense. To this end, Theorem 6.4 shows that the 2-element strongly admissible sets correspond to open subsets of contractible, smooth, codimension 1 submanifolds of D, and that the 3-element strongly admissible sets correspond to open subsets of smooth, codimension 2 submanifolds of D.

The example of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$, described in [14] and recalled in Section 8, shows that it is too much to hope that the strongly admissible sets always correspond to smooth contractible sets. Thus we try to give a criterion ensuring that this happens. First we prove some geometric lemmas, and then we apply these lemmas to the gradient vector fields for the exhaustion functions to define a *separation* condition that ensures that the strongly admissible sets correspond to smooth contractible sets. This condition is sufficient but not necessary. We hope to define a refinement of D_0 to give a more satisfactory solution in a future paper.

7.1. Geometric lemmas

For this section, let V be a finite-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$. Fix a basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of V such that $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$. Let $\hat{\Delta} = \{\beta_1, \ldots, \beta_n\} \subset V$ be defined by $\langle \beta_i, \alpha_j \rangle = \delta_{ij}$.

Definition 7.1. Call a vector $v \in V$ *dominant* (*with respect to* Δ) if $\langle v, \alpha_i \rangle \geq 0$ for all i. If these inequalities are strict for all i, then v is *strictly dominant*. Call a vector $v \in V$ *codominant* (*with respect to* Δ) if $\langle v, \beta_i \rangle \geq 0$ for all i. If these inequalities are strict for all i, then v is *strictly codominant*.

The dominant vectors in V form a convex cone generated by $\hat{\Delta}$, while the codominant vectors form the dual cone generated by Δ.

We recall without proof a geometric lemma due to Langlands [13].

Lemma 7.2. *The dominant cone is contained in the codominant cone. Equivalently, $\beta_i = \sum e_{ji}\alpha_j$ with $e_{ji} \geq 0$ for all $1 \leq i, j \leq n$. Consequently, $\langle \beta_i, \beta_j \rangle \geq 0$ for $1 \leq i, j \leq n$.*

Lemma 7.3. *Let $\{v_0, \ldots, v_k\} \subset V$ be a set of vectors such that $\langle v_i, v_j \rangle < 0$ for all $0 \leq i < j \leq k$. Then the span of $\{v_0, \ldots, v_k\} \subset V$ is either k- or $k+1$-dimensional. In particular, any k-element subset of $\{v_0, \ldots, v_k\}$ is linearly independent.*

Proof. Let W be the span of $S = \{v_0, \ldots, v_k\}$. Let $\Delta \subseteq \{v_0, \ldots, v_k\}$ be a linearly independent subset. Let C be the dominant cone in W with respect to Δ. Since $\langle v_i, v_j \rangle < 0$ for all $0 \le i \le j \le k$, every vector in $S \backslash \Delta$ must lie in the anti-dominant cone $-C$. By Lemma 7.2, the vectors in C and $-C$ have pairwise non-negative inner products. Hence $S \setminus \Delta$ has at most one element. $\qquad \square$

Lemma 7.4. *Let* $\{v_0, \ldots, v_k\} \subset V$ *be a set of vectors such that* $\langle v_i, v_j \rangle < 0$ *for all* $0 \le i < j \le k$. *Then the span of* $\{v_i - v_j\}_{0 \le i, j \le k}$ *has dimension* k.

Proof. It is clear that the span of $\{v_i - v_j\}$ is equal to the span of $\{\eta_1, \ldots, \eta_k\}$ where $\eta_i = v_0 - v_i$. Thus it suffices to show that the η_i are linearly independent. Assume the contrary. Then there exist $a_i \in \mathbb{R}$, not all zero, such that

$$\sum_{i=1}^{k} a_i \eta_i = \sum_{i=1}^{k} a_i (v_0 - v_i) = 0.$$

Then $(\sum_{i=1}^{k} a_i) v_0 = \sum_{i=1}^{k} a_i v_i$. By Lemma 7.3, $v_1, \ldots, v_k$ are linearly independent, so $\sum a_i \ne 0$. Thus

$$v_0 = \sum_{i=1}^{k} \left(\frac{a_i}{\sum a_j} \right) v_i.$$

By Lemma 7.2, $\frac{a_i}{\sum a_j} \le 0$ for $1 \le i \le k$, which contradicts $\sum_{i=1}^{k} \frac{a_i}{\sum a_j} = 1$. $\qquad \square$

Lemma 7.5. *Let* $\{v_0, \ldots, v_k\} \subset V$ *be a set of linearly independent vectors such that* $\langle v_i, v_j \rangle \le 0$ *for all* $0 \le i < j \le k$. *Let* $u \in V$ *be a vector such that* $\langle v_i, u \rangle \le 0$ *for all* $0 \le i \le k$. *Then the orthogonal projection* $\mathrm{pr}(v_0)$ *of* v_0 *to the orthogonal complement of* $\mathrm{span}\{v_i - v_j\}_{0 \le i, j \le k}$ *in* V *satisfies*

$$\langle \mathrm{pr}(v_0), v_i - u \rangle > 0 \quad \text{for all } 0 \le i \le k.$$

Proof. Let $\Delta = \{v_0, \ldots, v_k\}$ and $W \subset V$ the subspace spanned by Δ. Since the v_i are linearly independent, $\mathrm{pr}(v_0) \ne 0$. Furthermore, since $v_0 - v_i$ is perpendicular to $\mathrm{pr}(v_0)$,

$$\langle \mathrm{pr}(v_0), v_i \rangle = \langle \mathrm{pr}(v_0), v_i + (v_0 - v_i) \rangle = \langle \mathrm{pr}(v_0), v_0 \rangle$$
$$= \|\mathrm{pr}(v_0)\|^2 > 0 \quad \text{for all } i.$$

It follows that $\mathrm{pr}(v_0)$ lies in the dominant cone C in W with respect to Δ and has non-negative inner product with any other vector in C by Lemma 7.2. Since $\langle v_i, u \rangle \le 0$ for all $0 \le i \le k$, one can write u as $u = \epsilon - u_C$, where ϵ is perpendicular to W and u_C lies in C. Therefore

$$\langle \mathrm{pr}(v_0), v_i - u \rangle = \langle \mathrm{pr}(v_0), v_i - (\epsilon - u_C) \rangle$$
$$= \|\mathrm{pr}(v_0)\|^2 + \langle \mathrm{pr}(v_0), u_C \rangle > 0 \quad \text{for all } i. \qquad \square$$

7.2. Consequences

One can imagine the strongly admissible sets growing from their first contact as the parameter gets pushed lower. Thus a retraction to the set of first contacts should be possible by "going backward". More precisely, one would like to show that the negative gradient flow for f_P (for any $P \in \mathcal{I}$), restricted to $D'(\mathcal{I})$, defines a retraction onto the set of first contact points for $\mathcal{I}$. Thus we need the restriction of the gradient flow to push each boundary piece into $D'(\mathcal{I})$.

Lemma 7.6. *Let (M, g) be a Riemannian manifold and $f_0, \ldots, f_k$ be smooth functions defined on M. Then the set*

$$\{p \in M \mid f_0(p) = \cdots = f_k(p)\} \cap \{p \in M \mid \langle \nabla f_i, \nabla f_j \rangle_p < 0 \text{ for } 0 \le i < j \le k\}$$

is either empty or a smooth codimension k submanifold of M.

Proof. This is immediate from the implicit function theorem and Lemma 7.4. □

Definition 7.7. Given a strongly admissible set $\mathcal{I}$, let $\hat{\mathcal{I}}$ denote a minimal set of parabolics such that $D(\hat{\mathcal{I}}) = D(\mathcal{I})$. A strongly admissible set $\mathcal{I}$ is said to be *separated* if $\langle \nabla f_P, \nabla f_Q \rangle \le 0$ on $D(\mathcal{I})$ for every $P, Q \in \hat{\mathcal{I}}$, and *strictly separated* if the inequality is strict on $D(\mathcal{I})$. An algebraic group G is (*strictly*) *separated* if its strongly admissible sets are (strictly) separated.

Theorem 7.8. *Let G be a strictly separated group and $\mathcal{I}$ a strongly admissible set of order k. Then $D'(\mathcal{I})$ is a smooth codimension k submanifold of D. Furthermore, if $\overline{D'(\mathcal{I})}$ is compact and there is a unique critical point for $f_{\mathcal{I}}$, then $D'(\mathcal{I})$ is contractible.*

Proof. The first statement follows from the definition of a strictly separated group and Lemma 7.6.

   We claim that the gradient flow for $f_{\mathcal{I}}$ pushes the boundary of $D(\mathcal{I})$ into the interior. The boundary is a union of pieces of the form $D(\mathcal{I} \cup \{P\})$ for some rational parabolic subgroup P. This piece flows into the interior of $D(\mathcal{I})$ if

$$\langle \nabla f_{\mathcal{I}}, \nabla f_{\mathcal{I}} - \nabla f_P \rangle > 0. \tag{32}$$

Write $\mathcal{I} = \{Q_0, \ldots, Q_k\}$ and apply Lemma 7.5 with $v_i = \nabla f_{Q_i}$ for $i = 0, \ldots, k$ and $u = \nabla f_P$ to show that (32) is satisfied. Thus the gradient flow of $f_{\mathcal{I}}$ will define a deformation retraction of $D(\mathcal{I})$ onto the critical point. □

8. Examples

We apply the methods of Sections 3–5 to briefly describe the spines in some examples and compare our method with other known methods.

8.1. Bianchi groups

The arithmetic group $\mathrm{SL}_2(\mathcal{O}_k)$ for k an imaginary quadratic field, viewed as a subgroup of $\mathrm{SL}_2(\mathbb{C})$, acts on hyperbolic 3-space $\mathbb{H}^3$. $\mathbb{H}^3$ is a linear symmetric space, and hence the existence of a spine (the well-rounded retract) is guaranteed by [3]. Mendoza [21] and Vogtmann [27] use a notion of distance to a cusp due to Siegel [24] to explicitly compute a spine. Ash [1] shows that the spines they compute are the same as the well-rounded retract. The distance functions are a power of the exhaustion functions that we define in 3.3, and thus the spines also coincide with our spines.

8.2. Hilbert modular group

The arithmetic group $\mathrm{SL}_2(\mathcal{O}_k)$ for k a real quadratic field acts on the product of two upper half-planes $\mathfrak{H} \times \mathfrak{H}$. $\mathfrak{H} \times \mathfrak{H}$ is a linear symmetric space, and hence the existence of a spine (the well-rounded retract) is guaranteed by [3]. Brownstein again uses Siegel's notion of distance to a cusp to explicitly compute the spine in [14] for $k = \mathbb{Q}(\sqrt{5})$. He also conjectures a spine for $k = \mathbb{Q}(\sqrt{2})$. One can easily show that these spines are different from those obtained by Ash. On the other hand, the distance functions considered by Brownstein are powers of the exhaustion functions that we define in 3.3, and thus Brownstein's spines also coincide with ours.

We consider the example of $\mathbb{Q}(\sqrt{2})$ in some detail. Fix the following sets $\mathcal{L}_m^n$ of cusps in $k \cup \{\infty\} \simeq \mathcal{O}^* \backslash \mathcal{O}^2$:

$$\mathcal{L}_1^2 = \{\infty, 0\}, \quad \mathcal{L}_2^2 = \{\infty, 1/\sqrt{2}\},$$
$$\mathcal{L}_1^3 = \{\infty, 0, 1\}, \quad \mathcal{L}_2^3 = \{\infty, 0, 1/\sqrt{2}\},$$
$$\mathcal{L}_1^4 = \{\infty, 0, 1, 1/\sqrt{2}\}, \quad \mathcal{L}_2^4 = \{\infty, 0, \sqrt{2}, 1/\sqrt{2}\},$$
$$\mathcal{L}^5 = \{\infty, 0, 1, \sqrt{2}, 1/\sqrt{2}\}.$$

Let $\mathcal{K}_m^n$ denote the set of rational parabolics associated to $\mathcal{L}_m^n$. Then $D(\mathcal{K}_m^n) \subset \mathfrak{H} \times \mathfrak{H}$ is defined by the equality of n exhaustion functions and has codimension equal to $n - 1$.

These subsets of cusps correspond to geometrically different pieces of the spine. However, a given type may contain more than one Γ-conjugacy class of strongly admissible sets. $D(\mathcal{K}^5)$ is a vertex. $D(\mathcal{K}_1^4)$ is a line segment, and $D(\mathcal{K}_2^4)$ is the union of two transverse line segments. $D(\mathcal{K}_1^3)$ is a dodecagon, and $D(\mathcal{K}_2^3)$ is the union of two quadrilaterals joined at a vertex. Figure 2 shows that $D(\mathcal{K}_1^2)$ is an infinite tube with boundary faces of types $D(\mathcal{K}_1^3)$ and $D(\mathcal{K}_2^3)$, and $D(\mathcal{K}_2^2)$ is homeomorphic to an infinite string of 3-cells with boundary faces of type $D(\mathcal{K}_2^3)$. The strongly admissible sets of more than two elements have finite stabilizers. On the other hand, stabilizers of the 2-element strongly admissible sets contain a subgroup of finite index that is isomorphic to the group of units $\mathcal{O}_k^*$, acting by translation along the infinite direction. However, it is clear that the sets may be refined so that the spine has the structure of a regular cell complex, on which Γ acts with finite stabilizers.
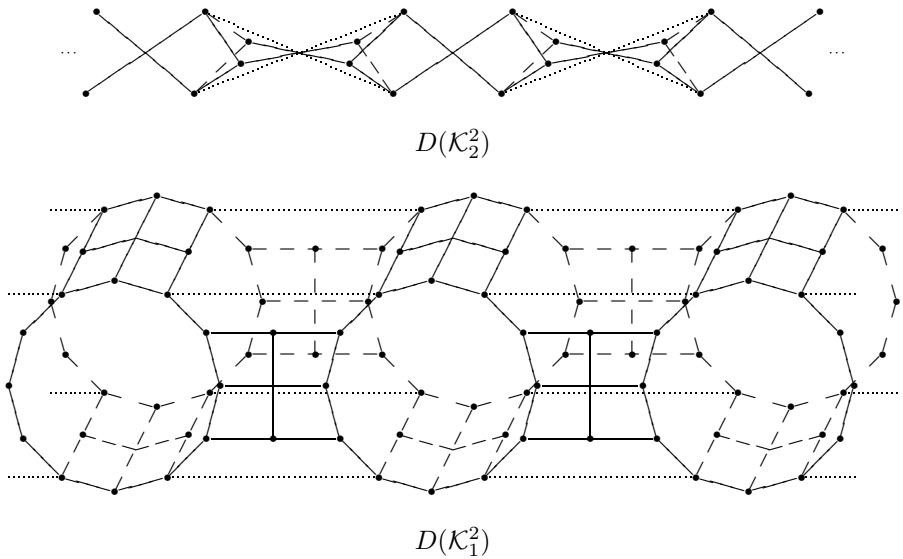
$$D(\mathcal{K}_2^2)$$

$$D(\mathcal{K}_1^2)$$

Fig. 2. The two $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$-conjugacy classes of 2-element strongly admissible sets are both infinite in length, and the boundary pieces can be identified with certain configurations of vectors in $\mathbb{Z}[\sqrt{2}]^2$. The boundary of $D(\mathcal{K}_2^2)$ consists of pieces which are the union of two quadrilaterals joined at a vertex. The boundary of $D(\mathcal{K}_1^2)$ consists of dodecagons and the union of quadrilaterals described above.

Table 1. Incidence types

	$\mathcal{K}_1^2$	$\mathcal{K}_2^2$	$\mathcal{K}_1^3$	$\mathcal{K}_2^3$	$\mathcal{K}_1^4$	$\mathcal{K}_2^4$	$\mathcal{K}^5$
$\mathcal{K}_1^2$	$*$	$*$	3	2	5	4	8
$\mathcal{K}_2^2$	$*$	$*$	0	1	1	2	2
$\mathcal{K}_1^3$	∞	0	$*$	$*$	2	0	4
$\mathcal{K}_2^3$	∞	∞	$*$	$*$	2	4	6
$\mathcal{K}_1^4$	∞	∞	12	4	$*$	$*$	4
$\mathcal{K}_2^4$	∞	∞	0	1	$*$	$*$	1
$\mathcal{K}^5$	∞	∞	12	6	2	4	$*$

The incidence table is given in Table 1, where the entry below the diagonal means that each column cell has that many row cells in its boundary, and the entry above the diagonal means the column cell appears in the boundary of this many row cells.

8.3. Picard modular group

The symmetric space of $\mathrm{SU}(2,1)$ is a complex 2-ball. It is not a linear symmetric space, and hence the well-rounded retraction does not apply for this group. Our method of exhaustion functions applied to this group exhibits the second known example of a spine in the non-linear case (after $\mathrm{Sp}_4(\mathbb{Z})$ by MacPherson and McConnell in [20]). Details about this spine, including stabilizer information and cohomology computations, can be found in [29].

　　　We can identify the cusps with isotropic vectors in $\mathbb{Z}[i]^3$ modulo scaling by $\mathbb{Z}[i]^*$. Fix the following sets $\mathcal{J}_m^n$ of cusps:

$$\mathcal{J}_1^2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{J}_2^2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 1+i \\ 1+i \end{pmatrix} \right\},$$

$$\mathcal{J}_1^3 = \mathcal{J}_1^2 \cup \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{J}_2^3 = \mathcal{J}_1^2 \cup \left\{ \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix} \right\}, \quad \mathcal{J}_3^3 = \mathcal{J}_1^2 \cup \left\{ \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{J}_1^4 = \mathcal{J}_1^3 \cup \left\{ \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix} \right\}, \quad \mathcal{J}_2^4 = \mathcal{J}_3^3 \cup \left\{ \begin{pmatrix} -1 \\ -1+i \\ 1+i \end{pmatrix} \right\},$$

$$\mathcal{J}^5 = \mathcal{J}_1^4 \cup \left\{ \begin{pmatrix} 1+i \\ 1+i \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{J}^8 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1+i \\ 1+i \end{pmatrix}, \begin{pmatrix} -1+i \\ 1+i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1-i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1+i \\ 1+i \end{pmatrix}, \begin{pmatrix} 2i \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 2 \\ 2 \end{pmatrix} \right\}.$$

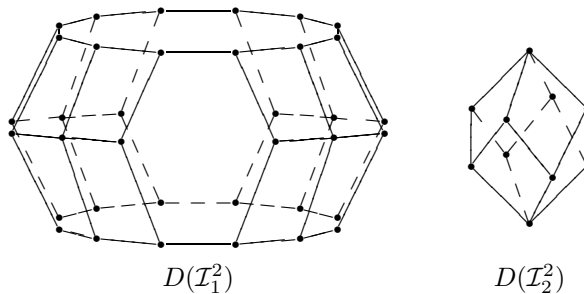Let $\mathcal{I}_m^n$ denote the set of rational parabolics associated to $\mathcal{J}_m^n$.

$$D(\mathcal{I}_1^2) \qquad\qquad\qquad D(\mathcal{I}_2^2)$$

FIG. 3. The two $\mathrm{SU}(2,1;\mathbb{Z}[i])$-conjugacy classes of 2-element strongly admissible sets. $D(\mathcal{I}_1^2)$ is homeomorphic to a polytope with dodecagon, hexagon and quadrilateral faces, while $D(\mathcal{I}_2^2)$ has only quadrilateral faces. The strongly admissible sets are parameterized by certain configurations of vectors in $\mathbb{Z}[i]^3$.

These subsets of cusps correspond to geometrically different pieces of the spine. Each class contains exactly one Γ-conjugacy class of strongly admissible sets. $D(\mathcal{I}^8)$ is a vertex, and vertices that occur where four quadrilaterals come together are all Γ-conjugate to $D(\mathcal{I}^8)$. The other vertices are conjugate to $D(\mathcal{I}^5)$. The Γ-translates of $D(\mathcal{I}_1^4)$ are the edges that do not terminate in a vertex of type $D(\mathcal{I}^8)$. The other edges have type $D(\mathcal{I}_2^4)$. $D(\mathcal{I}_1^3)$ is a dodecagon, and $D(\mathcal{I}_3^3)$ is a hexagon. Figure 3 shows $D(\mathcal{I}_1^2)$ and $D(\mathcal{I}_2^2)$. They are both homeomorphic to polytopes.

TABLE 2. Incidence types

	$\mathcal{I}_1^2$	$\mathcal{I}_2^2$	$\mathcal{I}_1^3$	$\mathcal{I}_2^3$	$\mathcal{I}_3^3$	$\mathcal{I}_1^4$	$\mathcal{I}_2^4$	$\mathcal{I}^5$	$\mathcal{I}^8$
$\mathcal{I}_1^2$	$*$	$*$	3	3	2	5	4	8	16
$\mathcal{I}_2^2$	$*$	$*$	0	0	1	1	2	2	8
$\mathcal{I}_1^3$	2	0	$*$	$*$	$*$	1	0	2	0
$\mathcal{I}_2^3$	4	0	$*$	$*$	$*$	1	0	2	0
$\mathcal{I}_3^3$	12	8	$*$	$*$	$*$	2	4	6	32
$\mathcal{I}_1^4$	40	8	12	6	2	$*$	$*$	4	0
$\mathcal{I}_2^4$	16	8	0	0	2	$*$	$*$	1	16
$\mathcal{I}^5$	32	8	12	6	3	2	1	$*$	$*$
$\mathcal{I}^8$	4	2	0	0	1	0	1	$*$	$*$

The incidence table is given in Table 2, where as in 8.2 the entry below the diagonal means that each column cell has that many row cells in its boundary, and the entry above the diagonal means the column cell appears in the boundary of this many row cells.

Acknowledgements

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