COMPUTING HECKE OPERATORS ON BIANCHI FORMS

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Abstract. Bianchi forms are generalizations of classical modular forms. The arithmetic significance of these forms is revealed through the system eigenvalues determined by a family of linear operators known as Hecke operators.

1. Introduction

The theme of modularity in number theory is a well-established one. In the 1950s and 60s, through the work of many mathematicians, there emerged a link between the existence of weight two cuspforms of level N with coefficients in the field of rational numbers, and the existence of elliptic curves of conductor N over the same field. This *Eichler-Shimura* correspondence was first discovered in the forms-to-curves direction. Some time later, Shimura and Taniyama asked what turned out to be a fundamental question: is there a reverse correspondence as well? This question received further treatment in the hands of Weil whose influential treatment and formulation in terms of L-series brought the problem to prominence as a celebrated conjecture. The development, spearheaded by Langlands, of a much more general theory in which Galois representations attached, on one hand to arithmetic-geometric objects and on the other to modular (or automorphic) ones, provided a general framework in which to expect and predict correspondences of this type.

1.1. **Automorphic forms.** Classical modular forms are holomorphic functions with certain growth conditions on the upper half-plane $\mathfrak{H} \subset \mathbb{C}$ that satisfy a specific transformation property with respect to the action of a congruence group $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. These forms are attached to $\mathbf{G} = \mathrm{SL}_2/\mathbb{Q}$. Automorphic forms are a further generalization attached to more general groups. The Langlands philosophy asserts that there should be a deep connection between automorphic forms and arithmetic, revealed through the action of Hecke operators on spaces of automorphic forms. There have been many great advances in this area, such as the work of Breuil, Conrad, Diamond, Taylor, and Wiles [4, 6, 12, 19, 21] proving the modularity of elliptic curves defined over $\mathbb{Q}$.

Specifically, as a consequence of their work, we now know that the L-function of an elliptic curve defined over $\mathbb{Q}$ equals the L-function of an appropriate modular form. A conjectural analogue of this theorem over number fields has been computationally explored in several examples [5, 7–11, 18, 20, 22]. We examine the case of Bianchi forms, which corresponds to looking at modular forms over imaginary quadratic fields.

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1.2. **Cohomology.** One way to study these spaces of automorphic forms is to relate them to cohomology. Certain automorphic forms are more intimately linked with arithmetic. For $\mathbf{G} = \mathrm{SL}_2$, these correspond to holomorphic modular forms. Rather than work directly with this space of automorphic forms, one can work with the cohomology of Γ with certain coefficient modules. In [13] Franke proves the Borel conjecture relating automorphic forms to the cohomology. For example, let $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of matrices that are upper triangular modulo N. Let V be the standard 2-dimensional representation of SL_2 and let $E_n = \mathrm{Sym}^n(V)$. The incarnation of the theorem in this case says

$$H^1(\Gamma_0(N); E_{k-2}) \simeq S_k(N) \oplus \overline{S_k(N)} \oplus \mathrm{Eis}_k(N),$$

where $S_k(N)$ is the space of weight k cusp forms on $\Gamma_0(N)$ and $\mathrm{Eis}_k(N)$ is the space of weight k Eisenstein series on $\Gamma_0(N)$.

For general $\mathbf{G}$ and Γ,

Theorem 1.1 (Franke)**.**

$$H^*(\Gamma; E) \simeq H^*(\mathfrak{g}, K; A(\Gamma, G) \otimes E),$$

where $A(\Gamma, G)$ *is space of automorphic forms.*

Thus we can think of the cohomology $H^*(\Gamma; E)$ as a concrete realization of certain automorphic forms.

Ash, Gunnells, and Lee-Szczarba [2, 15, 17] define a homology complex $S^*(\Gamma)$ known as the *sharbly complex*. A theorem of Borel-Serre [3] gives

$$H^{\nu-k}(\Gamma; \mathbb{C}) \simeq H_k(S_*(\Gamma)), \quad \text{where } \nu = \mathrm{vcd}(\Gamma).$$

There is a natural Hecke action on this complex which agrees with the Hecke action on the automorphic forms.

2. Imaginary quadratic field

Let $F = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$ be an imaginary quadratic number field. We always take $d < 0$ to be a square-free integer. Let $\mathcal{O} \subset F$ denote the ring of integers in F. Let $\mathbf{G}$ be the $\mathbb{Q}$-group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$ and let $G = \mathbf{G}(\mathbb{R})$ the corresponding group of real points. Let $K \subset G$ be a maximal compact subgroup, and let A_G be the identity component of the maximal $\mathbb{Q}$-split torus in the center of G. Then the symmetric space associated to G is $\mathbb{H}_3 = G/KA_G$. Let $\Gamma \subseteq \mathrm{GL}_2(\mathcal{O})$ be a finite index subgroup.

In this setting, the virtual cohomological dimension of Γ is 2. The Bianchi cusp forms of interest occur in $H^2(\Gamma; \mathbb{C})$, and so can be computed from $H_0(S^*(\Gamma))$. Thus the problem of computing Hecke operators on Bianchi cusp forms reduces to computing the action of Hecke operators on 0-sharblies.

2.1. **Voronoï polyhedron and the sharbly complex.** The space of positive definite binary Hermitian forms over F form an open cone in a real vector space. There is a natural decomposition of this cone into polyhedral cones corresponding to the facets of the Voronoï polyhedron [1,14,16]. These facets are in 1-1 correspondence with perfect forms over F. The polyhedral cones give rise to ideal polytopes in $\mathbb{H}_3$, 3-dimensional hyperbolic space.

The structure of the polytopes allows us to find a finite (modulo Γ) spanning set for the sharbly complex. These for the analogue of unimodular symbols in the classical case. The modular symbol algorithm, for describing the action of Hecke operators, can now be replaced by a 0-sharbly reduction algorithm. Unlike the usual modular symbol algorithm the number field F is no longer required to be Euclidean.

2.2. **Implementation in Magma.** Given an imaginary quadratic field F, we compute the structure of the Voronoï polyhedron by computing a complete set of $\mathrm{GL}_2(\mathcal{O})$-class representatives of perfect forms. Given a level $\mathfrak{n} \subseteq \mathcal{O}$ defining the level of the congruence subgroup, the polyhedron is used to compute $H^2(\Gamma_0(\mathfrak{n}))$. Given a principal prime ideal $\mathfrak{p} \subset \mathcal{O}$, the 0-sharbly reduction algorithm is used to compute the action of the Hecke operator $T_{\mathfrak{p}}$ on $H^2(\Gamma_0(\mathfrak{n}))$. By fixing $\mathfrak{n}$ and varying $\mathfrak{p}$, we can collect evidence for the conjectured link between cusp forms and elliptic curves.

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