ON THE GROWTH OF TORSION IN THE COHOMOLOGY OF ARITHMETIC GROUPS

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ABSTRACT. Let G be a semisimple Lie group with associated symmetric space D, and let $\Gamma \subset G$ be a cocompact arithmetic group. Let $\mathscr{L}$ be a lattice inside a $\mathbb{Z}\Gamma$-module arising from a rational finite-dimensional complex representation of G. Bergeron and Venkatesh recently gave a precise conjecture about the growth of the order of the torsion subgroup $H_i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}$ as Γ_k ranges over a tower of congruence subgroups of Γ. In particular they conjectured that the ratio $\log |H_i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}|/[\Gamma : \Gamma_k]$ should tend to a nonzero limit if and only if $i = (\dim(D) - 1)/2$ and G is a group of deficiency 1. Furthermore, they gave a precise expression for the limit. In this paper, we investigate computationally the cohomology of several (non-cocompact) arithmetic groups, including $\mathrm{GL}_n(\mathbb{Z})$ for $n = 3, 4, 5$ and $\mathrm{GL}_2(\mathscr{O})$ for various rings of integers, and observe its growth as a function of level. In all cases where our dataset is sufficiently large, we observe excellent agreement with the same limit as in the predictions of Bergeron–Venkatesh. Our data also prompts us to make two new conjectures on the growth of torsion not covered by the Bergeron–Venkatesh conjecture.

1. Introduction

1.1. Let $\mathbf{G}$ be a connected semisimple $\mathbb{Q}$-group with group of real points $G = \mathbf{G}(\mathbb{R})$. Suppose $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup and $\mathscr{M}$ is a $\mathbb{Z}\Gamma$-module arising from a rational finite-dimensional complex representation of G. The cohomology spaces $H^*(\Gamma; \mathscr{M})$ are important objects in number theory. By a theorem of Franke [19] they can be computed in terms of certain automorphic forms. Moreover the Langlands philosophy predicts connections between these automorphic forms and arithmetic geometry (counting points mod p of algebraic varieties, Galois representations, and so on).

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Now suppose that $\mathscr{M}$ has an integral structure, i.e., a lattice $\mathscr{L} \subset \mathscr{M}$ such that $\mathscr{M} = \mathscr{L} \otimes \mathbb{C}$ and $\Gamma\mathscr{L} \subset \mathscr{L}$. Then one can consider the groups $H^*(\Gamma; \mathscr{L})$; for each i the group $H^i(\Gamma; \mathscr{L})$ is a finitely generated abelian group, and thus has a torsion subgroup $H^i(\Gamma; \mathscr{L})_{\mathrm{tors}}$. In the past 30 years it has become understood that torsion classes, even when they do not arise as the reduction mod p of a characteristic 0 class, should also be connected to arithmetic. Indeed, already in the 1980s Elstrodt–Grünewald–Mennicke [17] observed relationships between Hecke eigenclasses in the torsion of abelianizations of congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])$ and the arithmetic of Galois extensions of $\mathbb{Q}(\sqrt{-1})$. Later one of us (AA) conjectured that any Hecke eigenclass $\xi \in H^*(\Gamma; \mathbb{F}_p)$, $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, should be attached to a Galois representation $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{F}_p)$, in the sense that for almost all primes l the characteristic polynomial of the Frobenius conjugacy class should equal a certain polynomial constructed from the l-Hecke eigenvalues of ξ [1]. This is now a theorem due to P. Scholze [28] (conditional on stabilization of the twisted trace formula). The torsion in the cohomology, even if it does not arise as the reduction of characteristic 0 classes and thus a priori has no connection with automorphic forms, does in fact play a significant role when one studies connections between cohomology of arithmetic groups and arithmetic.

1.2. In light of this, it is natural to ask what kind of torsion one expects in the cohomology of a given arithmetic group. For instance, one might ask for which $\mathbf{G}$ can one expect to find arithmetic subgroups Γ with $H^*(\Gamma; \mathscr{L})_{\mathrm{tors}}$ large? If one expects torsion in the cohomology, which degrees should be interesting, and how should the torsion grow as the index of Γ increases? In fact, it has been known for a long time that some arithmetic groups have little or no torsion in cohomology, whereas others have much. For example, consider a torsionfree congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Since Γ is torsionfree, $H^i(\Gamma; \mathbb{Z})$ vanishes unless $i = 0, 1$, and both cohomology groups are torsionfree: we have $H^*(\Gamma; \mathbb{Z}) \simeq H^*(\Gamma\backslash\mathbb{H}_2; \mathbb{Z})$, where $\mathbb{H}_2$ is the upper halfplane, and $\Gamma\backslash\mathbb{H}_2$ has a finite graph as a deformation retract. On the other hand, suppose Γ is a torsionfree congruence subgroup of $\mathrm{SL}_2(\mathscr{O})$, where $\mathscr{O}$ is the ring of integers in an imaginary quadratic field. Then $H^i(\Gamma; \mathbb{Z})$ vanishes unless $i = 0, 1, 2$, but now the torsion behavior is quite different. The group H^2, for instance, typically contains lots of torsion (cf. [27, 29]). In fact one observes what appears to be the initial stage of exponential growth in its torsion subgroup as the index of Γ goes to infinity. That there should be exponential growth in the torsion of the homology of such Γ was already observed in the 1980s in unpublished computations of F. Grünewald (of $H_1(\Gamma)$).

1.3. When should there be lots of torsion in the cohomology of an arithmetic group? One answer is suggested by Bergeron–Venkatesh in [8], who formulated a precise conjecture for the growth of the torsion in the *homology* of Γ when Γ is *cocompact*. To state it, we need more notation. Recall that $G = \mathbf{G}(\mathbb{R})$ is the group of real points of our algebraic group $\mathbf{G}$. Let $K \subset G$ be a maximal compact subgroup and let $D = G/K$ be the associated global symmetric space. Let δ be the *deficiency* of $\mathbf{G}$, defined by $\delta = \operatorname{rank} G - \operatorname{rank} K$, where rank denotes the absolute rank (i.e., the rank over $\mathbb{C}$). The deficiency is an important invariant in the representation theory of $\mathbf{G}$. For instance, G has discrete series representations if and only if $\delta = 0$. Then we have the following conjecture of Bergeron–Venkatesh:

Conjecture 1.1 ([8, Conjecture 1.3])**.** *Suppose* $\mathbf{G}$ *has* $\mathbb{Q}$*-rank* 0*. Let* $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ *be a decreasing family of (cocompact) congruence subgroups with* $\cap_k \Gamma_k = \{1\}$*. Then*

$$\lim_{k\to\infty} \frac{\log |H_i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]}$$

exists for each i *and is zero unless* $\delta = 1$ *and* $i = (d-1)/2$*, where* $d = \dim D$*. In that case, the limit is strictly positive and equals an explicit constant* $c_{G,\mathscr{L}}$ *times* $\mu(\Gamma)$*, the volume of* $\Gamma\backslash D$*.*

Thus the deficiency δ, which depends only on the group of real points of G and not on its $\mathbb{Q}$-structure, is the main quantity that conjecturally determines whether or not an arithmetic group Γ should be expected to have lots of torsion. Some heuristic motivation for this phenomenon, which ultimately is inspired by Bhargava's conjectures for the asymptotic counting of number fields [9], can be found in [8, §6.5].

In [8] the authors are able to obtain results in the direction of Conjecture 1.1 for certain G and certain modules $\mathscr{L}$. Let us say that $\mathscr{L}$ is *strongly acyclic* for the family $\{\Gamma_k\}$ if the spectra of the (differential form) Laplacian on $\mathscr{L} \otimes \mathbb{C}$-valued i forms on $\Gamma_k\backslash D$ are uniformly bounded away from 0 for all degrees i and all Γ_k. This implies in particular that the homology $H_i(\Gamma_k; \mathscr{L})$ is all torsion, i.e., the cohomology $H_i(\Gamma_k; \mathscr{L} \otimes \mathbb{Q})$ is trivial. Such modules can always be shown to exist for any Γ. Then if $\delta = 1$, under these assumptions Bergeron–Venkatesh prove [8, (1.4.2)] an "Euler characteristic" version of Conjecture 1.1:

$$\liminf_k \sum_i (-1)^{i+(d-1)/2} \frac{\log |H_i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]} = c_{G,\mathscr{L}} \operatorname{vol}(\Gamma\backslash D). \tag{1}$$

Further, they prove polynomial bounds on the torsion on H_0 and H_{d-1}, which allows them to isolate the contribution of the remaining homology groups in low-dimensional examples. In particular they show that Conjecture 1.1 is true for $G = \mathrm{SL}_2(\mathbb{C})$ (again with the assumptions of cocompact quotients and strongly acyclic coefficient modules).

1.4. In this paper, we investigate the growth of torsion in cohomology when the $\mathbb{Q}$-rank of $\mathbf{G}$ is nonzero and Γ is not cocompact. Moreover, we consider only the case of trivial coefficients. We look at these Γ and these coefficients because extensive computations become feasible for them. Based on our experimental evidence, in §7 we make two new conjectures, similar to Conjecture 1.1 but for all arithmetic groups, cocompact or not.

We treat a variety of examples with deficiencies $\delta = 1$ and 2:[1]

$\delta = 1$: We consider $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$ with $n = 3, 4$, and $\Gamma \subset \mathrm{GL}_2(\mathscr{O}_F)$, where F is the nonreal cubic field of discriminant -23. We also consider $\Gamma \subset \mathrm{GL}_2(\mathscr{O}_L)$ where L is imaginary quadratic, which complements work of Şengün [29] and Pfaff [26].

$\delta = 2$: We consider $\Gamma \subset \mathrm{GL}_5(\mathbb{Z})$ and $\Gamma \subset \mathrm{GL}_2(\mathscr{O}_E)$, where E is the field of fifth roots of unity.

We computed cohomology groups of congruence subgroups of these arithmetic groups, with several questions in mind:

- Do we see distinct qualitative behavior in the growth in the torsion in cohomology for $\delta = 1$ and $\delta = 2$?
- How fast does the torsion appear to grow as the level (or norm of the level) increases? For example, do we see exponential growth when $\delta = 1$? Does the analogue of Conjecture 1.1 for Γ non-cocompact seem to hold for cohomology? What about the analogue of Bergeron–Venkatesh's "Euler characteristic" theorem (1)?
- When the torsion does appear to grow exponentially, how does the prime factorization of the torsion order behave? For instance, do we see small prime divisors with large exponents, or large primes with small exponents?
- If the torsion does not appear to grow exponentially, what behavior do we see? For example, do the prime divisors of the torsion appear to get arbitrarily large?
- How does the torsion behave in different cohomological degrees? For instance, is one degree singled out, as in Conjecture 1.1, or do similar large amounts of torsion appear in other degrees? If a group has a cuspidal range, do we see different torsion behavior across the range? Does behavior outside the cuspidal range differ?
- The Eisenstein cohomology of an arithmetic group is, roughly speaking, the part of the complex cohomology that comes from lower rank groups via the boundary of a compactification of D [22]. Do we see Eisenstein cohomology phenomena for torsion classes?

In light of our experimental evidence, we make two new conjectures, along the lines of Conjecture 1.1 but more general. We remove the condition that the arithmetic groups in question be cocompact, and we allow them to increase in level without necessarily being arranged in a tower. One conjecture is for groups of prime level and the other groups of all levels. The exact conjectures may be found in §7.

[1]If a group is reductive, then we define its deficiency to be that of its derived subgroup.

We remark that in some cases, we have computed Hecke operators on some of these classes, in work reported on elsewhere, such as [4,6,7]. However, many of them are out of reach of our current programs. We can go much further with the ranks of cohomology groups than we can with Hecke operators because in the latter case we are forced to use dense matrix computations (to keep track of the bases) and the computations of each Hecke operator can take a very long time. We do plan to compute more Hecke operators on this data in future work.

1.5. We now give a guide to the paper. In §2 we describe the tools we use to compute cohomology. In §3 we explicitly compute the constant $c_{G,\mathscr{L}}$ appearing in Conjecture 1.1 for the G of deficiency 1 we consider. Section 4 gives an overview of the scope of our computations and gives the basic plots of our data. The next sections present analysis of our data: §5 discusses the variation of behavior of the torsion in various cohomological degrees for the same group, §6 discusses the "Euler characteristic" version of Conjecture 1.1, §7 discusses towers of congruence subgroups and makes two conjectures concerning the growth of their torsion, and §8 discusses torsion in Eisenstein cohomology. We present tentative conclusions in §9.

2. Background and computational tools

2.1. Let F be a number field of degree $r+2s$, where r is the number of real places and s is the number of complex places. Let $\mathscr{O} \subset F$ be the ring of integers of F. Let $\mathbf{G}$ be the algebraic group $\operatorname{Res}_{F/\mathbb{Q}} \operatorname{GL}_n$, and let $G = \mathbf{G}(\mathbb{R})$, the group of real points of $\mathbf{G}$. Then $G \simeq \prod_v \operatorname{GL}_n(F_v)$, where the product is over the real and complex embeddings v of F. Let $K \subset G$ be a maximal compact subgroup, and let A_G be the split component of $\mathbf{G}$, i.e., the identity component of the real points of the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{G}$. Let $D = G/A_G K$. Then D is a *type* $S-\mathbb{Q}$ *homogeneous space for* G, in the terminology of [11].

In particular, D is the Riemannian symmetric space for the Lie group G/A_G. Any congruence subgroup $\Gamma \subset \operatorname{GL}_n(\mathscr{O})$ acts on D by left multiplication. Note that, since $\mathbf{G}$ is reductive rather than semisimple, the space D is not necessarily a product of irreducible symmetric spaces of nonpositive curvature: in general, in addition to these there will be Euclidean factors that account for the nontrivial units in $\mathscr{O}$.

2.2. We now introduce a model for D that is more amenable to our computational techniques. Let $V_\mathbb{R}$ be the $n(n+1)/2$-dimensional $\mathbb{R}$-vector space of $n \times n$ symmetric matrices with real entries, and let $V_\mathbb{C}$ be the n^2-dimensional $\mathbb{R}$-vector space of $n \times n$ Hermitian matrices with complex entries. Let $C_\mathbb{R} \subset V_\mathbb{R}$, respectively $C_\mathbb{C} \subset V_\mathbb{C}$, denote the codimension 0 open cone of positive definite matrices. Let $V = \prod_v V_v$, where

$$V_v = \begin{cases} V_\mathbb{R} & \text{if } v \text{ is real, and} \\ V_\mathbb{C} & \text{if } v \text{ is complex.} \end{cases}$$

Define C_v analogously. There is a left action of the real group $\mathrm{GL}_n(F_v)$ on C_v defined by

$$g \cdot Q = gQg^*, \tag{2}$$

for each $g \in \mathrm{GL}_n(F_v)$ and $Q \in C_v$, where * is transpose if v is real and complex conjugate transpose if v is complex. Since $G \simeq \prod_v \mathrm{GL}_n(F_v)$, equation (2) gives an action of G on V. With this action, $A_G \simeq \mathbb{R}_{>0}$ acts by homotheties; that is, $h \in A_G$ acts on V by simultaneous scaling in each factor. This action preserves $C = \prod C_v$, and exhibits G as the full automorphism group of C. Thus we get an identification $D \simeq C/\mathbb{R}_{>0}$. A straightforward computation shows that D is a real manifold of dimension $d = ((r+2s)n^2 + rn - 2)/2$.

There is a map $q\colon \mathscr{O}^n \to V$ defined by

$$(q(x))_v = x_v x_v^*. \tag{3}$$

The rays defined by $q(x)$ for $x \in \mathscr{O}^n$ give us a notion of *cusps* for D.

Recall that a *polyhedral cone* in a real vector space V is a subset σ which has the form

$$\sigma = \left\{ \sum_{i=1}^{p} \lambda_i v_i \;\middle|\; \lambda_i \geq 0 \right\},$$

where $v_1, \ldots, v_p$ are vectors in V. In this case, we say that $v_1, \ldots, v_p$ span σ. We are most interested in polyhedral cones where each v_i has the form $q(w_i)$ for some $w_i \in \mathscr{O}^n$, and for the remainder of the paper, we will use the term polyhedral cone to include this additional condition.[2]

2.3. An explicit reduction theory due to Koecher [24], generalizing Voronoi's theory of perfect quadratic forms over $\mathbb{Q}$ [30], gives a tessellation of C by convex polyhedral cones, which we make precise below. Recall that a set Σ of polyhedral cones in a vector space forms a *fan* if (i) $\sigma \in \Sigma$ and $\tau \subset \sigma$ implies $\tau \in \Sigma$ and (ii) if $\sigma, \sigma' \in \Sigma$, then the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.[3]

Theorem 2.1 ([24]). *There is a fan $\tilde{\Sigma}$ in V with Γ-action such that the following hold:*

(i) *There are only finitely many Γ-orbits in $\tilde{\Sigma}$.*
(ii) *Every $y \in C$ is contained in the interior of a unique cone in $\tilde{\Sigma}$.*
(iii) *Given any cone $\sigma \in \tilde{\Sigma}$ with $\sigma \cap C \neq \emptyset$, the stabilizer $\mathrm{Stab}(\sigma) = \{\gamma \in \Gamma \mid \gamma \cdot \sigma = \sigma\}$ is finite.*

[2] We remark that at this point one would typically expect to see a rationality condition imposed on polyhedral cones. Our additional condition is an analogue of this, but it is not the same. Indeed, for general F the $\mathbb{Z}$-span of the points $\{q(w) \mid w \in \mathscr{O}^n\}$ will not be a lattice in V.

[3] A *face* of a rational polyhedral cone σ is the intersection of σ with a supporting hyperplane.

The structure of $\tilde{\Sigma}$ can be computed explicitly using a generalization of Voronoi's algorithm for enumeration of perfect quadratic forms. For details we refer to [21, §3] and the references given there; here we only sketch the situation.

Each point of C corresponds to a positive definite quadratic form on $\mathscr{O}^n$, given by summing the symmetric and Hermitian forms corresponding to the real and complex places of F. A form is called *perfect* if it can be reconstructed from the knowledge of its minimal nonzero value on $\mathscr{O}^n$ and of the vectors on which it attains this minimum. Then it is known that the perfect forms are in bijection with cones in $\tilde{\Sigma}$ of maximal dimension, which are therefore called *perfect pyramids*. Namely, a given perfect pyramid is generated as a convex cone by $q(w_1), \ldots, q(w_p)$, where $w_1, \ldots, w_p$ are the minimal vectors of the corresponding perfect form. Thus computing the structure of $\tilde{\Sigma}$ reduces to a classification problem in quadratic forms. Voronoi described an algorithm to solve this problem for $F = \mathbb{Q}$ that immediately generalizes to general F in this setting.

Modulo homotheties, the fan $\tilde{\Sigma}$ descends to a $\mathrm{GL}_n(\mathscr{O})$-stable tessellation of D by "ideal" polytopes. Specifically, let C^* be the union of the closures of the perfect pyramids in the cone $\overline{C}$ of positive semi-definite matrices (minus the origin). The $(k+1)$-dimensional cones in $\tilde{\Sigma}$ that intersect non-trivially with C descend to give k-dimensional cells in D. Let $\Sigma_k^*(\Gamma)$ denote a set of representatives, modulo the action of Γ, of these k-dimensional cells, and let $\Sigma^*(\Gamma) = \cup\, \Sigma_k^*(\Gamma)$.

For example, when $n = 2$ and $F = \mathbb{Q}$, the space D can be identified with the complex upper halfplane $\mathbb{H}_2$. Under this identification, the cusps are precisely $\mathbb{P}^1(\mathbb{Q})$ as expected. The Voronoi tessellation corresponds to the Farey tessellation [23] of $\mathbb{H}_2$ defined by $\mathrm{GL}_2(\mathbb{Z})$-translates of the ideal triangle with vertices $\{0, 1, \infty\}$. When $n = 2$ and $F = \mathbb{Q}(i)$, $i^2 = 1$, the space D can be identified with hyperbolic 3-space $\mathbb{H}_3$. The cusps can be identified with $\mathbb{P}^1(\mathbb{Q}(i))$. The cells in the tessellation are ideal octahedra [13,17,18] that are translates of the convex hull of $\{0, 1, i, i{+}1, (i{+}1)/2, \infty\}$.

2.4. We use the cells Σ^* to define a chain complex $\mathrm{Vor}(\Gamma) = (V_*, d_*)$, which we call the *Voronoi–Koecher complex*; we refer to [3, 14–16, 20, 21] for details and further examples. Over $\mathbb{C}$, the homology of the Voronoi–Koecher complex is isomorphic to $H^*(\Gamma; \tilde{\Omega}_{\mathbb{C}})$, where $\tilde{\Omega}_{\mathbb{C}}$ is the local coefficient system attached to $\Omega \otimes \mathbb{C}$, where Ω is the orientation module of Γ. Over $\mathbb{Z}$, the homology of the Voronoi–Koecher complex is not quite the integral cohomology of Γ (again with twisted coefficients corresponding to the orientation module), but does agree with the integral cohomology modulo certain small primes. More precisely, recall that a prime is a *torsion prime* of Γ if it divides the order of a torsion element of Γ. For any positive integer n let $\mathcal{S}_n$ be the Serre class of finite abelian groups with orders only divisible by primes less than or equal to n. Then we have the following theorem, which slightly generalizes results of [14, §3]. The proof given there holds in this more general setting.

Theorem 2.2. *Let b be an upper bound on the torsion primes for Γ and let d be the dimension of the symmetric space on which Γ acts. Then, modulo the Serre class $\mathcal{S}_b$, we have*

$$H^k(\Gamma; \tilde{\Omega}_{\mathbb{Z}}) \simeq H_{d-k}(\mathrm{Vor}(\Gamma); \mathbb{Z}) \tag{4}$$

for all k.

Hence, if one ignores small primes, and is willing to twist coefficients by the orientation module, the Voronoi–Koecher complex provides a tool to investigate torsion in the cohomology of Γ. Even for torsion primes, there is a Hecke action on the homology of $\mathrm{Vor}(\Gamma)$, as follows from the methods of [5] although such an action is not stated explicitly in that paper. The following theorem tabulates the torsion primes for the examples we consider. We omit the proof.

Theorem 2.3. *Let L be imaginary quadratic, let F be the complex cubic field of discriminant -23, and let E be the field of fifth roots of unity. The deficiency δ, dimension d of the symmetric space D, and the torsion primes for the arithmetic groups Γ we consider in this paper are as follows:*

Γ	δ	d	*torsion primes*
$\mathrm{GL}_2(\mathscr{O}_L)$	1	3	$2, 3$
$\mathrm{GL}_3(\mathbb{Z})$	1	5	$2, 3$
$\mathrm{GL}_2(\mathscr{O}_F)$	1	6	$2, 3$
$\mathrm{GL}_4(\mathbb{Z})$	1	9	$2, 3, 5$
$\mathrm{GL}_2(\mathscr{O}_E)$	2	7	$2, 3, 5$
$\mathrm{GL}_5(\mathbb{Z})$	2	14	$2, 3, 5$

2.5. There is one additional class of primes that should be mentioned here, since they are a potential source of torsion in the cohomology that is in some sense trivial: these torsion classes arise from the congruence covers of Γ. For more details, we refer to [12, §3.7]. For any arithmetic group Γ, let $\hat{\Gamma}$ be its congruence completion, that is the completion of Γ for the topology defined by congruence subgroups. The map $\Gamma \to \hat{\Gamma}$ induces a map in cohomology $H^*(\hat{\Gamma}) \to H^*(\Gamma)$, and the congruence cohomology is the image. From a practical point of view, when the group is $\Gamma_0(\mathfrak{n})$, the relevant primes are those dividing $\mathrm{Norm}(\mathfrak{p}) - 1$, as $\mathfrak{p}$ ranges over prime ideals dividing $\mathfrak{n}$. We call any such prime a *congruence prime*. Any prime that is not a torsion prime or a congruence prime will be called an *exotic prime*.

3. The conjectured limit in the case of deficiency 1

3.1. Let us call the constant $c_{G,\mathscr{L}}\mu(\Gamma)$ appearing in Conjecture 1.1 the *B-V limit*. In this section we compute the B-V limits for the groups of deficiency 1 that we consider.

Recall that Conjecture 1.1 states that if G is semisimple and $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ is a decreasing family of cocompact congruence subgroups with $\cap_k \Gamma_k = \{1\}$, then

$$\lim_{k\to\infty} \log |H_i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}|/[\Gamma : \Gamma_k]$$

should exist and vanish in all cases except $i = (d-1)/2$, in which case it should equal the constant $c_{G,\mathscr{L}}\mu(\Gamma)$, where $\mu(\Gamma) = \mathrm{vol}(\Gamma\backslash D)$. According to [8, §§3–5], the constant $c_{G,\mathscr{L}}$ is the L^2-torsion $t^{(2)}_D(\rho)$, where ρ is the rational representation giving rise to the local system $\mathscr{L}$. In our cases, we need to incorporate several features in our discussion.

First, the groups we work with are reductive rather than semisimple. This makes little conceptual difference, in that we simply have to be cognizant of how GL vs. SL affects the volume computation $\mu(\Gamma)$. For most cases this is not difficult, except when the unit group $\mathscr{O}^\times$ has nontrivial rank. In this case the symmetric space for GL has additional flat factors to accommodate the action of the units. These clearly have little arithmetic significance, but we must consider them when computing $\mu(\Gamma)$.

Second, since we work with GL instead of SL, we must incorporate the orientation module $\tilde{\Omega}_{\mathbb{Z}}$ into the discussion when we compare the homology of the Voronoi complex with the group homology (Theorem 2.2). This extra local system $\tilde{\Omega}_{\mathbb{Z}}$ has no arithmetic significance and is an artifact of our computational technique. In fact, if we worked instead with the subgroup $\Gamma' \subset \mathrm{GL}_n(\mathscr{O})$ of elements with totally positive determinant, then the orientation module restricts to the trivial representation. Since the index $[\mathrm{GL}_n(\mathscr{O}) : \Gamma']$ is a 2-power, only the 2-torsion in the homology is affected. Thus we will assume that the relevant representation ρ is the trivial representation when computing the analytic torsion.

3.2. We compute this value for the examples considered in this paper after some preliminary results.

Lemma 3.1. *Let $n \geq 1$ and let $\mathscr{O}$ be the ring of integers in a number field k. Then*

$$[\mathrm{PGL}_n(\mathscr{O}) : \mathrm{PSL}_n(\mathscr{O})] = [\mathscr{O}^* : (\mathscr{O}^*)^n].$$

Proof. We have the following commutative diagram, where η_n is the (possibly trivial) subgroup of all $a \in \mathscr{O}^*$ satisfying $a^n = 1$, and $(\mathscr{O}^*)^n$ is the subgroup of nth powers. In the diagram we realize η_n, $\mathscr{O}^*$, and $(\mathscr{O}^*)^n$ as subgroups of the matrix groups via the scalar matrices. One can check that the rows and columns are exact, where all of the maps are the obvious ones.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \eta_n & \longrightarrow & \mathrm{SL}_n(\mathscr{O}) & \longrightarrow & \mathrm{PSL}_n(\mathscr{O}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{O}^* & \longrightarrow & \mathrm{GL}_n(\mathscr{O}) & \longrightarrow & \mathrm{PGL}_n(\mathscr{O}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \\
1 & \longrightarrow & (\mathscr{O}^*)^n & \longrightarrow & \mathscr{O}^* & \longrightarrow & \mathscr{O}^*/(\mathscr{O}^*)^n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

It follows that

$$[\mathrm{PGL}_n(\mathscr{O}) : \mathrm{PSL}_n(\mathscr{O})] = |\mathscr{O}^*/(\mathscr{O}^*)^n| = [\mathscr{O}^* : (\mathscr{O}^*)^n].$$

□

In particular, if $\mathscr{O}$ is the ring of integers in an imaginary quadratic field, a short computation together with Lemma 3.1 implies

$$[\mathrm{PGL}_2(\mathscr{O}) : \mathrm{PSL}_2(\mathscr{O})] = 2. \tag{5}$$

3.3. Now we consider the volumes of our locally symmetric spaces. We have the following general result of Borel [10] that works for any number field k; in the special case of k imaginary quadratic, this result goes back to Humbert. Here

$$\zeta_k(s) = \sum_{I \subseteq \mathscr{O}} \mathrm{Norm}(I)^{-s}$$

denotes the Dedekind zeta function of k. Let $\mathbb{H}_2$ and $\mathbb{H}_3$ denote hyperbolic 2-space and 3-space, respectively.

Theorem 3.2 ([10, Theorem 7.3])**.** *Let k be a number field of degree $d = r + 2s$ and discriminant Δ with ring of integers $\mathscr{O}$. Let $\mathbb{H} = \mathbb{H}_2^r \times \mathbb{H}_3^s$. Then*

$$\mathrm{vol}(\mathrm{PSL}_2(\mathscr{O}) \backslash \mathbb{H}) = 2^{1-3s} \pi^{-d} |\Delta|^{3/2} \zeta_k(2).$$

We are now ready to compute the B-V limits attached to the torsion growth. We begin with imaginary quadratic fields.

Proposition 3.3. *Let L be an imaginary quadratic field of discriminant $\Delta < 0$ and $\zeta_L(s)$ its Dedekind zeta function. Then*

$$t_D^{(2)}(\mathrm{triv})\mu(\mathrm{GL}_2(\mathscr{O}_L)) = \frac{|\Delta|^{3/2}}{48\pi^3} \zeta_L(2).$$

Proof. Since L is imaginary quadratic, [8, §5.9.3, Example 1] shows $t^{(2)}_D(\mathrm{triv}) = \frac{1}{6\pi}$. It remains to compute $\mu(\mathrm{GL}_2(\mathscr{O}_L))$. In the notation of Theorem 3.2, we have $d = 2$, $s = 1$, and $\mathbb{H} = \mathbb{H}_3$. Since $\pm I$ acts trivially on $\mathbb{H}$, Theorem 3.2 implies

$$\mu(\mathrm{SL}_2(\mathscr{O}_L)) = \frac{|\Delta|^{3/2}}{4\pi^2}\zeta_L(2).$$

Note that the center of $\mathrm{GL}_2(\mathscr{O}_L)$ acts trivially on $\mathbb{H}$. It follows that to compute $\mu(\mathrm{GL}_2(\mathscr{O}_L))$ we need to divide not by the index $[\mathrm{GL}_2(\mathscr{O}_L) : \mathrm{SL}_2(\mathscr{O}_L)]$, but rather the index $[\mathrm{PGL}_2(\mathscr{O}_L) : \mathrm{PSL}_2(\mathscr{O}_L)]$. From (5), this index is independent of Δ and equals 2. Therefore

$$\mu(\mathrm{GL}_2(\mathscr{O}_L)) = \frac{1}{2}\mu(\mathrm{SL}_2(\mathscr{O}_L)) = \frac{|\Delta|^{3/2}}{8\pi^2}\zeta_L(2),$$

and multiplying by $t^{(2)}_D(\mathrm{triv})$ gives the desired result. □

3.4. Now we consider our cubic field of mixed signature:

Proposition 3.4. *Let F be a nonreal cubic field of discriminant $\Delta < 0$. Then*

$$t^{(2)}_D(\mathrm{triv})\mu(\mathrm{GL}_2(\mathscr{O}_F)) = \frac{|\Delta|^{3/2}\,\mathrm{reg}_F}{48\pi^5}\zeta_F(2), \tag{6}$$

where reg_F denotes the regulator of F.

Proof. The global symmetric space D for $\mathrm{GL}_2(\mathscr{O}_F)$ is $\mathbb{H}_2 \times \mathbb{H}_3 \times \mathbb{R}$, where the flat factor accounts for the nontrivial units in $\mathscr{O}_F$. We have $t^{(2)}_D(\mathrm{triv}) = \frac{1}{2\pi}\cdot\frac{1}{6\pi} = \frac{1}{12\pi^2}$. Since $\pm I$ acts trivially on $\mathbb{H}_2 \times \mathbb{H}_3$, Theorem 3.2 implies

$$\mathrm{vol}(\mathrm{SL}_2(\mathscr{O})\backslash(\mathbb{H}_2 \times \mathbb{H}_3)) = \frac{|\Delta|^{3/2}}{4\pi^3}\zeta_F(2). \tag{7}$$

In this case, the center does not act trivially on D, but instead acts on the flat factor. Thus we need to include the volume of the flat factor modulo the action of $\mathscr{O}^\times$. We normalize so that this volume is the regulator reg_F, and we multiply (7) by reg_F to pass to $\mu(\mathrm{GL}_2(\mathscr{O}))$. Finally, multiplying by $t^{(2)}_D(\mathrm{triv})$ gives the desired result. □

For our particular cubic field F of discriminant -23, we have

$$\zeta_F(2) = 1.110001006025\ldots,$$
$$\mathrm{reg}_F = 0.281199574322\ldots.$$

Plugging into (6), we find

$$t^{(2)}_D(\mathrm{triv})\mu(\mathrm{GL}_2(\mathscr{O}_F)) = 0.002343900569\ldots.$$

3.5. Next we consider $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{R})$.

Proposition 3.5 ([8]). *The L^2-analytic torsion $t_D^{(2)}(\mathrm{triv})$ for the trivial representation of $\mathrm{SL}_n(\mathbb{R})$ is*

$$t_D^{(2)}(\mathrm{triv}) = R_n \frac{\pi \operatorname{vol}(\mathrm{SO}(n))}{\operatorname{vol}(\mathrm{SU}(n))},$$

where $R_3 = \frac{1}{2}$, $R_4 = \frac{124}{45}$, and $R_n = 0$ otherwise.

Proof. From [8, Proposition 5.2], $R_n = 0$ for $n \neq 3, 4$. Bergeron and Venkatesh work out an explicit formula [8, §5.9.2] for the L^2-analytic torsion $t_D^{(2)}(\rho)$ for a representation ρ of $\mathrm{SL}_3(\mathbb{R})$ in terms of the highest weight $\lambda = p\epsilon_1 + q\epsilon_2 + r\epsilon_3$. Let

$$A_1 = \frac{1}{2}(p+1-q), \quad A_2 = \frac{1}{2}(p-r+2), \quad A_3 = \frac{1}{2}(q-r+1)$$
$$C_1 = \frac{1}{3}(p+q-2r+3), \quad C_2 = \frac{1}{3}(p+r-2q), \quad C_3 = \frac{1}{3}(2p-q-r+3).$$

Then

$$t_D^{(2)}(\rho) = \frac{\pi \operatorname{vol}(\mathrm{SO}(n))}{\operatorname{vol}(\mathrm{SU}(n))} \left(2A_1A_3C_1C_3 + 2A_2|C_2| \begin{cases} A_3C_3 & \text{if } C_2 \geq 0, \\ A_1C_1 & \text{if } C_2 \leq 0. \end{cases} \right) \tag{8}$$

For the trivial representation, we have $A_1 = A_3 = \frac{1}{2}$, $A_2 = C_1 = C_3 = 1$, and $C_2 = 0$. Then (8) gives

$$t_D^{(2)}(\mathrm{triv}) = \frac{\pi \operatorname{vol}(\mathrm{SO}(3))}{2 \operatorname{vol}(\mathrm{SU}(3))}$$

as desired.

3.6. Next we consider $n = 4$. From [8, Proposition 5.2, eq. (5.9.6)], we can express $t_D^{(2)}(\mathrm{triv})$ as

$$t_D^{(2)}(\mathrm{triv}) = c(D)R_4,$$

where R_4 is equal up to sign to a sum of integrals, and

$$c(D) = \frac{\pi \operatorname{vol}(\mathrm{SO}(4))}{\operatorname{vol}(\mathrm{SU}(4))}.$$

From [8, §5.8], the constant R_4 depends only on the inner form of the Lie algebra. Since $\mathrm{SL}_4(\mathbb{R})$ is isogenous to $\mathrm{SO}(3,3)$, we may use the computations in [8, §5.9.3 Example (1)] for $\mathrm{SO}(5,1)$ to get the constant. The computations there show that

$$R_4 = \frac{t_{\mathbb{H}^5}^{(2)}(\mathrm{triv})}{c(\mathbb{H}^5)} = \frac{\frac{31}{45\pi^2}}{\frac{2!}{8\pi^2}} = \frac{124}{45}.$$

∎

Proposition 3.6. *The conjectured limit for* $\mathrm{GL}_3(\mathbb{Z})$ *and* $\mathrm{GL}_4(\mathbb{Z})$ *is*

$$t_D^{(2)}(\mathrm{triv})\mu(\mathrm{GL}_3(\mathbb{Z})) = \frac{\sqrt{3}}{288\pi^2}\zeta(3)$$
$$t_D^{(2)}(\mathrm{triv})\mu(\mathrm{GL}_4(\mathbb{Z})) = \frac{31\sqrt{2}}{259200\pi^2}\zeta(3).$$

Proof. Note that the intersection of the center of $\mathrm{SL}_n(\mathbb{R})$ with $\mathrm{SO}(n,\mathbb{R})$ is trivial if n is odd and $\{\pm I\}$ if n is even. It follows that

$$\mu(\mathrm{SL}_n(\mathbb{Z})) = \begin{cases} \dfrac{\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R}))}{\mathrm{vol}(\mathrm{SO}(n,\mathbb{R}))} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{2\,\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R}))}{\mathrm{vol}(\mathrm{SO}(n,\mathbb{R}))} & \text{if } n \text{ is even.} \end{cases}$$

Next we pass to $\mathrm{GL}_n(\mathbb{Z})$. When n is odd, $-I \in \mathrm{GL}_n(\mathbb{Z})\setminus\mathrm{SL}_n(\mathbb{Z})$. It is in the center of $\mathrm{GL}_n(\mathbb{Z})$, so acts trivially on the symmetric space. Thus $\mu(\mathrm{GL}_n(\mathbb{Z})) = \mu(\mathrm{SL}_n(\mathbb{Z}))$ when n is odd. When n is even, $\mathrm{diag}(-1,1,1,...,1) \in \mathrm{GL}_n(\mathbb{Z})\setminus\mathrm{SL}_n(\mathbb{Z})$ is not central. It acts non-trivially on the symmetric space, thus dividing the volume of the quotient in half. From Lemma 3.1, we have the index $[\mathrm{PGL}_n(\mathbb{Z}) : \mathrm{PSL}_n(\mathbb{Z})]$ is equal to 1 if n is odd and equal to 2 if n is even so $\mu(\mathrm{GL}_n(\mathbb{Z})) = \frac{1}{2}\mu(\mathrm{SL}_n(\mathbb{Z}))$ when n is even. Therefore

$$\mu(\mathrm{GL}_n(\mathbb{Z})) = \frac{\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R}))}{\mathrm{vol}(\mathrm{SO}(n,\mathbb{R}))}$$

in either case.

Combining with Proposition 3.5, we have

$$t_D^{(2)}(\mathrm{triv})\mu(\mathrm{GL}_n(\mathbb{Z})) = \pi R_n \frac{\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R}))}{\mathrm{vol}(\mathrm{SU}(n))}$$

A classical computation (using the volume forms induced from the invariant forms $(X,Y) \mapsto \mathrm{Tr}(XY)$ on the Lie algebras) shows

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})) = 2^{(n-1)/2}\zeta(2)\zeta(3)\cdots\zeta(n), \tag{9}$$

$$\mathrm{vol}(\mathrm{SU}(n)) = \frac{\sqrt{n}(2\pi)^{(n^2+n-2)/2}}{1!2!\cdots(n-1)!}. \tag{10}$$

It follows that

$$t_D^{(2)}(\mathrm{triv})\mu(\mathrm{GL}_n(\mathbb{Z})) = \pi R_n \frac{\prod_{k=2}^{n}(k-1)!\zeta(k)}{2^{(n^2-1)/2}\pi^{(n^2+n-2)/2}\sqrt{n}}$$

Since $R_3 = \frac{1}{2}$, $R_4 = \frac{124}{45}$, $\zeta(2) = \frac{\pi^2}{6}$, and $\zeta(4) = \frac{\pi^4}{90}$, we have[4]

$$\begin{aligned} t_D^{(2)}(\mathrm{triv})\mu(\mathrm{GL}_3(\mathbb{Z})) &= \frac{\sqrt{3}}{288\pi^2}\zeta(3) \\ &= 0.00073247603662800481419124468 2033\ldots, \\ t_D^{(2)}(\mathrm{triv})\mu(\mathrm{GL}_4(\mathbb{Z})) &= \frac{31\sqrt{2}}{259200\pi^2}\zeta(3) \\ &= 0.000020599988405628878074264341 1677\ldots. \end{aligned}$$

□

3.7. We conclude by summarizing all the specific values for the the B-V limit computed in this section.

Theorem 3.7. *Let L be the imaginary quadratic field of discriminant Δ. Let F be the nonreal cubic field of discriminant -23. Then the quantities $c_{G,\mathrm{triv}}\mu(\Gamma)$ for the groups of deficiency 1 we consider are given in the following table:*

Γ	$\mathrm{GL}_2(\mathscr{O}_L)$	$\mathrm{GL}_2(\mathscr{O}_F)$	$\mathrm{GL}_3(\mathbb{Z})$	$\mathrm{GL}_4(\mathbb{Z})$
$c_{G,\mathrm{triv}}\mu(\Gamma)$	$\frac{\lvert\Delta\rvert^{3/2}}{48\pi^3}\zeta_L(2)$	$\frac{23^{3/2}\,\mathrm{reg}_F}{48\pi^5}\zeta_F(2)$	$\frac{\sqrt{3}}{288\pi^2}\zeta(3)$	$\frac{31\sqrt{2}}{259200\pi^2}\zeta(3)$

4. Overview of the data and basic plots

4.1. In this section we present plots of the torsion data we generated. We use the following conventions in our computations and our plots:

- The homology or cohomology group data we present is always indexed in terms of the Voronoi homology, that is, in terms of the homological labelling in Theorem 2.2. In particular, a label `Hk` on a graph indicates that this group is the kth Voronoi homology group, and therefore corresponds to cycles supported on the Voronoi cells in D of dimension k (i.e. the cells that are the images of the $(k+1)$-dimensional cones in C modulo homotheties).
- When different groups in different degrees appear in the same graph, the first group in the list corresponds, on the cohomology side of (4), to the cohomology group in the virtual cohomological dimension (vcd). Thus for example in Figure 1 (for subgroups of $\mathrm{GL}_3(\mathbb{Z})$) one sees groups labelled `H2`, `H3`, and `H4`. Since the dimension of $\mathrm{GL}_3(\mathbb{R})/(\mathbb{R}_{>0})\,\mathrm{O}(3)$ is 5, these Voronoi homology groups correspond to the cohomology groups H^3, H^2, and H^1, respectively.

[4]We remark that the value 0.000732... for $\mathrm{GL}_3(\mathbb{Z})$ here corrects the value in Example 2 of [8, §5.9.3].

- Data points have the same (shape|color)[5] based on where their group falls relative to the vcd. Thus for example the points labelled `H2` in Figure 1(a) for $\mathrm{GL}_3(\mathbb{Z})$ have the same (shape|color) as those in Figure 4(a) labelled `H1`. Both of these, under the isomorphism (4), correspond to the cohomology groups at the vcd.
- In our computations, we always worked with congruence subgroups of the form $\Gamma_0(\mathfrak{n})$ for some ideal $\mathfrak{n} \subset \mathscr{O}$, where $\mathscr{O}$ is the relevant ring of integers. By definition, $\Gamma_0(\mathfrak{n})$ is the subgroup of $\mathrm{GL}_n(\mathscr{O})$ with bottom row congruent to $(0, \ldots, 0, *)$ mod $\mathfrak{n}$.
- The plots in Figures 5 and 7 contain some vertical line segments of dots. These vertical lines are an artifact of the fact that a given composite number may multiple factorizations of a given composite number. In particular there can be a number of different ideals with the same norm.
- In all plots, the label `Index` refers to $[\mathrm{GL}_n(\mathscr{O}) : \Gamma_0(\mathfrak{n})]$, the index of the congruence group in the full group, and the label `Log torsion/Index` refers to the ratio $(\log |H_i(\Gamma)_{\mathrm{tors}}|)/[\Gamma : \Gamma_0(\mathfrak{n})]$.
- Each series of computations was run ordered by the norm of the level. Some very large levels were skipped because jobs crashed as overall memory usage became too large. It's not clear to us why this happened in certain cases. It could be that there were eventually giant numbers in the Smith normal form, and thus very large torsion in the cohomology, but it could also be the case that the matrices were simply too large for our resources. However, this happened infrequently, so our computations are nearly complete in the ranges of levels we considered.

We now make a remark about the cost of our computations. The largest computation, in terms of memory usage and time for a computation that finished, was to compute H_6 for GL_4 at level 66. The boundary matrix from 6 cells to 5 cells was 150491×256782, and the boundary matrix from 7 cells to 6 cells was 256782×216270. The computation took approximately 586.59 hours, with memory usage approximately 126 GB.

4.2. $\mathrm{GL}_n(\mathbb{Z})$**.** We were able to compute the Voronoi homology for the following groups/levels of congruence subgroups:

[5]Here one should read *shape* (respectively, *color*) if one is reading a black and white (resp., color) version of this paper.

Group	Deficiency δ	Level of congruence subgroup
$GL_3(\mathbb{Z})$	1	H_2: $N \leq 641$, H_3: $N \leq 641$, H_4: $N \leq 659$
$GL_4(\mathbb{Z})$	1	H_3: $N \leq 119$, H_4: $N \leq 99$, H_5: $N \leq 61$, H_6: $N \leq 74$, H_7: $N \leq 80$
$GL_5(\mathbb{Z})$	2	H_6: $N \leq 29$

As one can clearly see, the difficulty of the computation increases substantially as n increases. For GL_3 and GL_4, we computed a range of homology groups that included the cuspidal range (H_2—H_3 for GL_3, H_4—H_5 for GL_4). For $GL_5(\mathbb{Z})$, we only worked with H_6, which is the top of the cuspidal range. We highlight some of the results in this range. The torsion size is given in factored form with exotic torsion in **bold**.

4.2.1. $GL_3(\mathbb{Z})$.

- H_2**:** We computed H_2 for 639 levels less than or equal to 641. Of these, there is nontrivial exotic torsion at 457 levels. The largest torsion group occurs at level 570. The largest exotic torsion occurs at this same level. The torsion size is

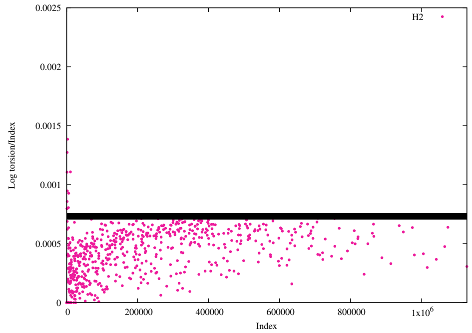
$$2^{154} \cdot 3^{27} \cdot \mathbf{5^2} \cdot \mathbf{7^6} \cdot \mathbf{11^6} \cdot \mathbf{17^{10}} \cdot \mathbf{37^6} \cdot \mathbf{47^2} \cdot \mathbf{131^2} \cdot \mathbf{619^6} \cdot \mathbf{3137^6} \cdot \mathbf{6113^6} \cdot \mathbf{2723737^2} \cdot \mathbf{242222857291^2} \cdot \mathbf{278917146364629278585122304155523929101710815974757^2}.$$

  All the exponents appearing in the exotic torsion are even because there is a non-degenerate alternating form on the mod p homology for an exotic prime p, as proved in [2].

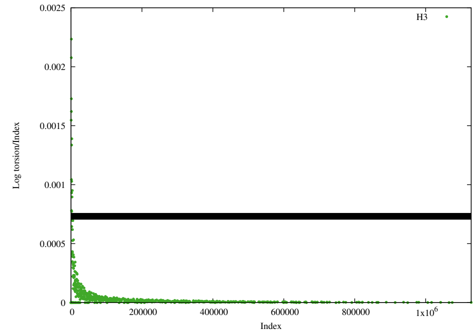
  The largest exotic prime occurs at level 638. The torsion size is

$$2^{106} \cdot 3^{22} \cdot 5^8 \cdot 7^2 \cdot \mathbf{11^2} \cdot \mathbf{31^6} \cdot \mathbf{8969^6} \cdot \mathbf{1537835313687313016296678426257182245274808716382194757792656443 61^2}.$$

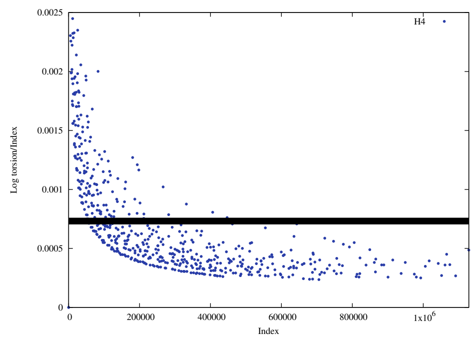
- H_3**:** We computed H_3 for 640 levels less than or equal to 641. Of these, there is nontrivial exotic torsion at 48 levels. The largest torsion group occurs at level 599. The torsion, none of which is exotic, has size $13 \cdot 23$. The largest exotic torsion occurs at level 625. The torsion, all of which is exotic, has size $\mathbf{5^3}$. The largest exotic prime is 23, which first occurs at level 529. The torsion size is $11 \cdot \mathbf{23}$.
- H_4**:** We computed H_4 for 658 levels less than or equal to 659. There were no levels with exotic torsion. The largest torsion group occurs at level 600. The torsion size is 2^{791}.



(a) H_2



(b) H_3



(c) H_4

FIGURE 1. All the Voronoi homology groups for subgroups of $\mathrm{GL}_3(\mathbb{Z})$, together with the predicted limiting constant (ordered by index of the congruence subgroup).

4.2.2. $\mathrm{GL}_4(\mathbb{Z})$.

- **H_3:** We computed H_3 for 118 levels less than or equal to 119. Of these, there is nontrivial exotic torsion at 3 levels. At level 114, we have the largest torsion group in this range. The torsion size is $2^{12} \cdot 3^7 \cdot \mathbf{11^4}$. The largest exotic torsion and largest exotic prime occurs at level 119. The full torsion is $2^4 \cdot 3^3 \cdot \mathbf{31^4}$.
- **H_4:** We computed H_4 for 98 levels less than 99. Of these, there is nontrivial exotic torsion at 2 levels. At level 49, the torsion is $3 \cdot \mathbf{7^2}$. At level 98, the torsion is all exotic and has size is $\mathbf{7^5}$.
- **H_5:** We computed H_5 for 55 levels less than 61. Of these, there is only nontrivial exotic torsion at level 49. At level 48, we have the largest torsion group in this range. The torsion size is $2^{418} \cdot 3$, all of which is nonexotic. The largest exotic torsion and largest exotic prime occur at level 49. The full torsion is $2^{126} \cdot 3^{10} \cdot \mathbf{7}$.

H_6**:** We computed H_6 for 71 levels less than 74. There were no levels with nontrivial exotic torsion. The largest torsion group occurs at level 50. The full torsion is 2^7, all of which is nonexotic.

H_7**:** We computed H_7 for 78 levels less than 80. There were no levels with nontrivial exotic torsion. The largest torsion group occurs at level 80. The full torsion is 2^{165} all of which is nonexotic.

4.2.3. $\mathrm{GL}_5(\mathbb{Z})$.

H_6**:** We computed H_6 for 27 levels less than or equal to 29. There were no levels with nontrivial exotic torsion. The largest torsion occurs at level 24. The torsion size is 2^{212}.

4.3. **Cubic of discriminant −23.** For the cubic field F of discriminant -23, we were able to compute homology groups up to the following level norms:

Group	Deficiency δ	Norm of level of congruence subgroup
$\mathrm{GL}_2(\mathscr{O}_F)$	1	H_1: $\mathrm{Norm}(\mathfrak{n}) \leq 5483$, H_2: $\mathrm{Norm}(\mathfrak{n}) \leq 11575$, H_3: $\mathrm{Norm}(\mathfrak{n}) \leq 5480$, H_4: $\mathrm{Norm}(\mathfrak{n}) \leq 5480$, H_5: $\mathrm{Norm}(\mathfrak{n}) \leq 5480$

We pushed the computation further for H_2 since that is the top of the cuspidal range from this group. We highlight some of the results of these computations. As before, the torsion size is given in factored form with exotic torsion in **bold**.

H_1**:** We computed H_1 for 2012 levels of norm less than or equal to 5483. There were no levels with exotic torsion in this range. The largest torsion group occurs at a level of norm 4481. The torsion size is $2^3 \cdot 7$.

H_2**:** We computed H_2 for 4240 levels of norm less than or equal to 11575. Of these, there is nontrivial exotic torsion at 3374 levels. The largest torsion group occurs at a level of norm 10600. The torsion size is

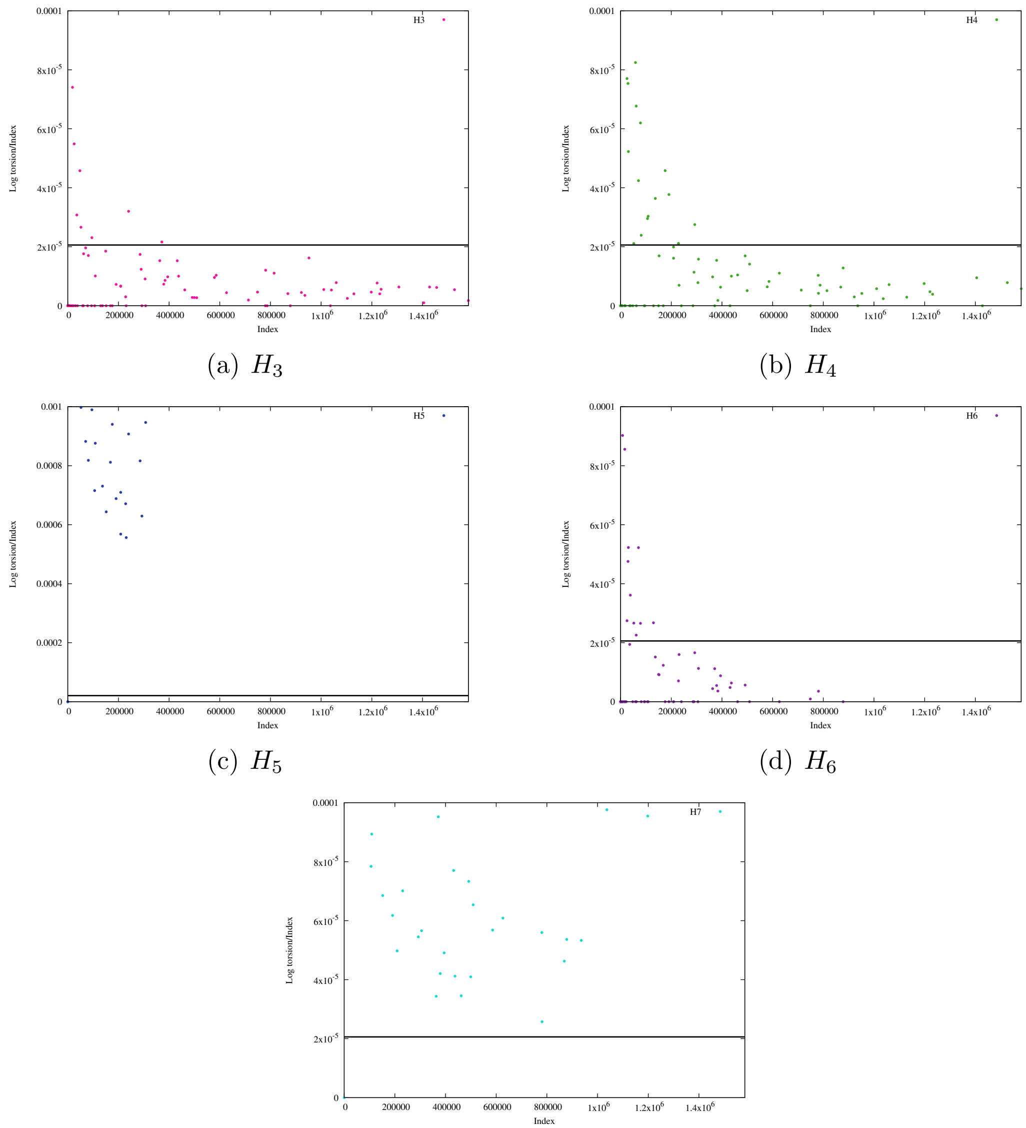
$$2^3 \cdot 3^{15} \cdot \mathbf{5^9} \cdot 7^2 \cdot \mathbf{11} \cdot \mathbf{103}.$$

The largest exotic torsion occurs at norm level 8575, where the full torsion has size

$$2^4 \cdot \mathbf{5^3} \cdot \mathbf{7^5} \cdot \mathbf{13^6} \cdot \mathbf{139}.$$

The largest exotic prime occurs at norm level 11443, where the size of the torsion is exactly the exotic prime contribution **7870506841**.

H_3**:** We computed H_3 for 2011 levels of norm less than or equal to 5480. There were 1234 levels with exotic torsion in this range. The largest torsion group occurs at a level of norm 4928. The torsion size is $2^{44} \cdot 3$, none of which is exotic. The largest exotic torsion occurs at norm level 4375. The full torsion has size $2^6 \cdot \mathbf{5^6} \cdot \mathbf{19} \cdot \mathbf{31}$. The largest exotic prime occurs at norm level 4597, where the torsion is $2 \cdot \mathbf{261529}$.



(a) H_3 (b) H_4

(c) H_5 (d) H_6

(e) H_7

FIGURE 2. All the Voronoi homology groups for subgroups of $\mathrm{GL}_4(\mathbb{Z})$, together with the predicted limiting constant (ordered by index of the congruence subgroup).

H_4: We computed H_4 for 2010 levels of norm less than or equal to 5480. There were no levels with exotic torsion in this range. The largest torsion group occurs at a level of norm 2560. The torsion size is 2^{16}.

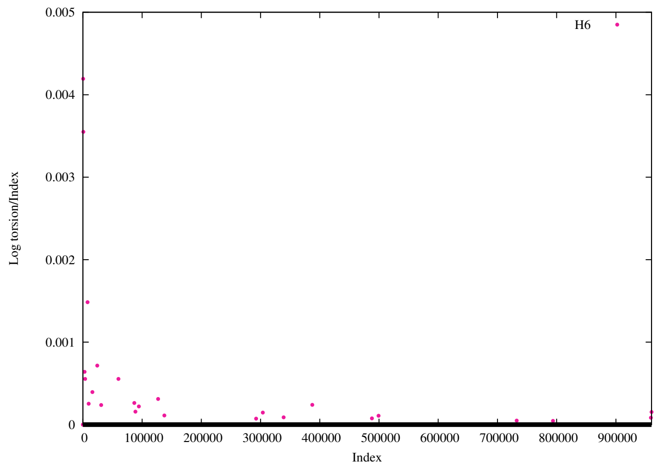


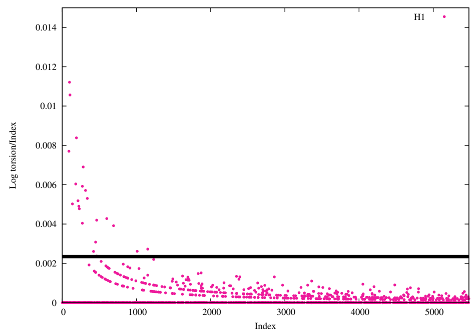
FIGURE 3. The Voronoi homology group H_6 for subgroups of $\mathrm{GL}_5(\mathbb{Z})$ (ordered by index of the congruence subgroup)

- H_5**:** We computed H_5 for 2010 levels of norm less than or equal to 5480. There were no levels with exotic torsion in this range. The largest torsion group occurs at a level of norm 3325. The torsion size is 3^4.
- H_6**:** We computed H_6 for 2010 levels of norm less than or equal to 5480. There were no levels with nontrivial torsion in this range.

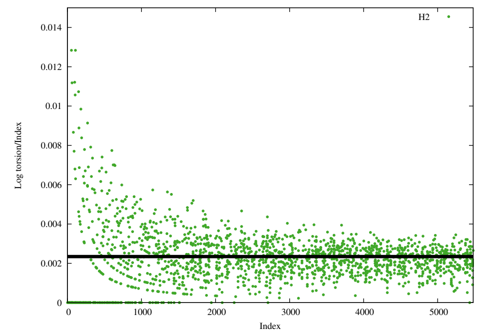
4.4. **Field of fifth roots of unity.** For this field, we only concentrated on H_2, which is the top of the cuspidal range. The deficiency of this group is $\delta = 2$. We computed H_2 for levels of norm up to 31805. This includes 2741 levels. Of these, there is nontrivial exotic torsion at 239 levels. We highlight some of the results of these computations. As before, the torsion size is given in factored form with exotic torsion in **bold**.

The largest torsion group occurs at a level of norm 15625 and is nonexotic. The order is 5^{15}. The largest exotic torsion occurs at norm level 24025, where the full torsion is $5^3 \cdot \mathbf{7} \cdot \mathbf{11} \cdot \mathbf{31^2} \cdot \mathbf{61}$. The largest exotic prime occurs at norm level 17161, where the size of the torsion is exactly the exotic prime contribution **2081**.

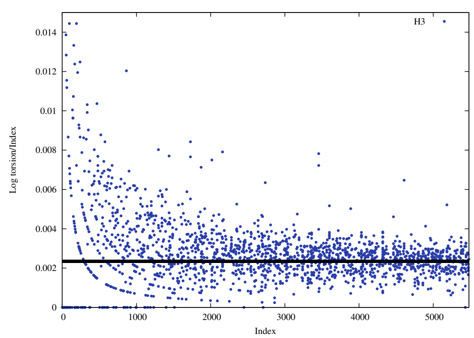
4.5. **Imaginary quadratic fields.** In a recent work, Şengün [29] computed H^2 of congruence subgroups of $\mathrm{PSL}_2(\mathscr{O}_L)$ for the euclidean imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11$. These groups all have deficiency $\delta = 1$. He computed cohomology for prime ideals of norm ≤ 5000, and also computed the cohomology for a variety of nontrivial coefficient systems. Our results when spot checked agree completely with his, although we have not systematically compared data. We give plots in Figure 7 for the other fields we considered, namely $\mathbb{Q}(\sqrt{-d})$, $d = 5, 6, 10, 13, 14, 15$. We remark that these fields have class number $2, 2, 2, 2, 4, 2$. As the plots show, the



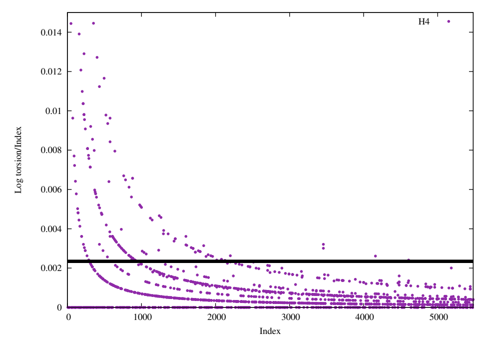
(a) H_1



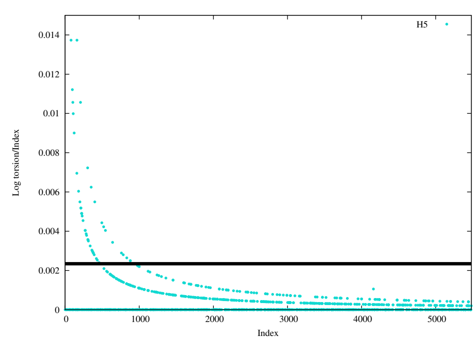
(b) H_2



(c) H_3



(d) H_4



(e) H_5

FIGURE 4. All the Voronoi homology groups for subgroups of $\mathrm{GL}_2(\mathscr{O}_F)$ for the cubic field of discriminant -23, together with the predicted limiting constant (ordered by index of the congruence subgroup).

torsion is enormous in H_1. For example, 8303 levels of norm less than or equal to 10103 were considered for $\mathbb{Q}(\sqrt{-15})$. In this range, the largest torsion group is on the order of 10^{1126}. The largest exotic contribution is on the order of 10^{924}. Because

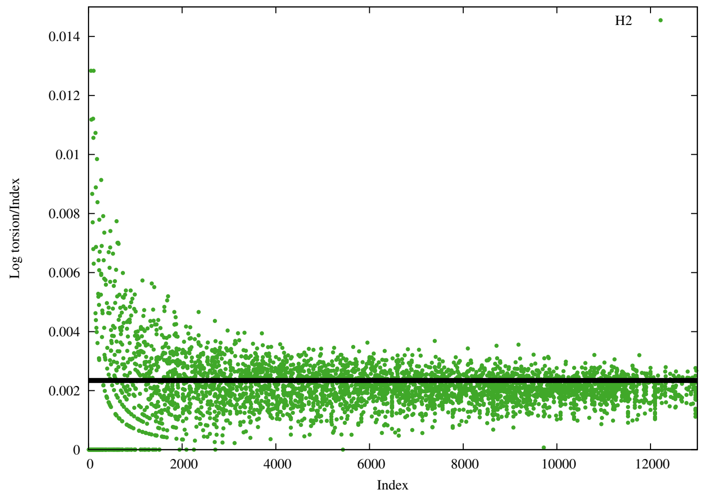


FIGURE 5. H_2 of subgroups of $\mathrm{GL}_2(\mathscr{O}_F)$ for the cubic field of discriminant -23, together with the predicted limiting constant (ordered by index of the congruence subgroup). This includes many more levels than Figure 4(b).

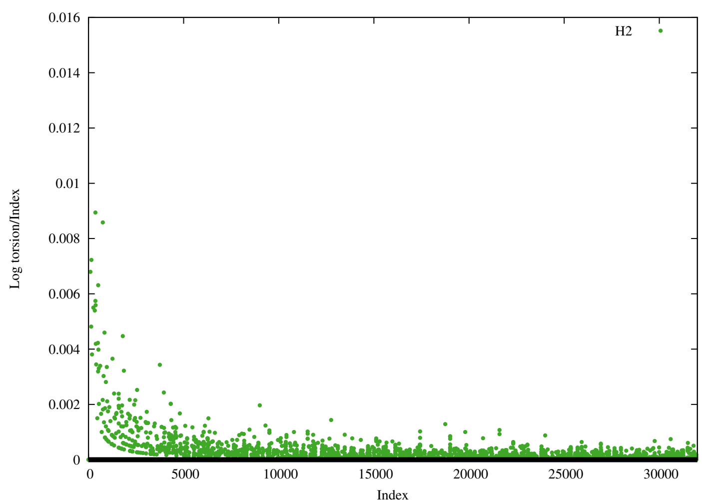
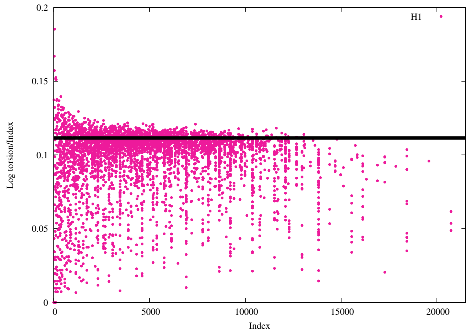
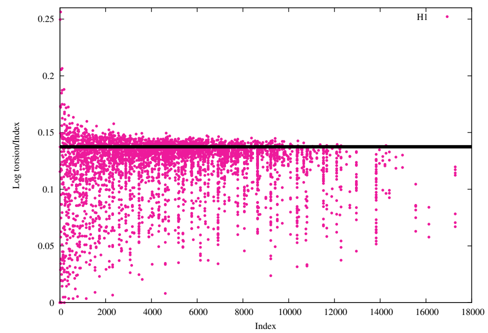


FIGURE 6. The Voronoi homology group H_2 for subgroups of $\mathrm{GL}_2(\mathscr{O}_E)$, E the field $\mathbb{Q}(\zeta_5)$ (ordered by index of the congruence subgroup).

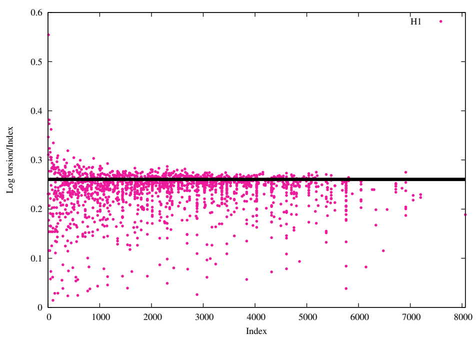
the torsion is so large, we are unable to factor the sizes to report on the largest exotic primes in the range of our computation.



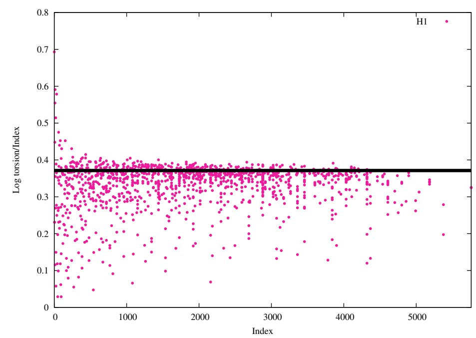
(a) $\mathbb{Q}(\sqrt{-5})$



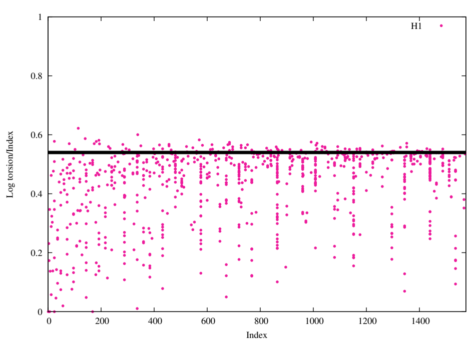
(b) $\mathbb{Q}(\sqrt{-6})$



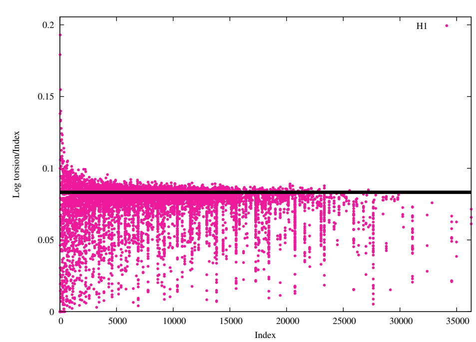
(c) $\mathbb{Q}(\sqrt{-10})$



(d) $\mathbb{Q}(\sqrt{-13})$



(e) $\mathbb{Q}(\sqrt{-14})$



(f) $\mathbb{Q}(\sqrt{-15})$

FIGURE 7. The Voronoi homology group H_1 for subgroups of $\mathrm{GL}_2(\mathscr{O}_L)$ for various imaginary quadratic fields L, together with the predicted limiting constant (ordered by index of the congruence subgroup). These fields have class number $2, 2, 2, 2, 4, 2$ respectively.

5. Cuspidal range

5.1. Recall that the *cuspidal range* for an arithmetic group is the set of cohomological degrees for which cuspidal automorphic forms can contribute to the corresponding cohomology. For a discussion, see [25, §2]. It is known that if a cuspidal automorphic form contributes to one degree in the cuspidal range, then it does to all, and the relevant Hecke eigensystems are determined by the contribution to the top (cf. [25, Theorem 2.6]). Thus it is of interest to see if similar phenomena occur for torsion in cohomology: if the deficiency δ is 1 and we see exotic torsion in the cohomology, do we see it across the cuspidal range, or is there an apparent difference among degrees in the range? What if the deficiency is greater than 1?

5.2. We observed in our computations that there is a distinction between the torsion phenomena across the cohomological degrees. In particular, we always observed an abundance of exotic torsion *in the top degree only.* This is consistent with earlier work on imaginary quadratic fields [27, 29]: the cuspidal range in that case is H^1—H^2, and only H^2 sees the big torsion. For an example from our data, consider the plots for $\mathrm{GL}_3/\mathbb{Q}$ (Figures 1(a)—1(c)). The cuspidal range corresponds to H_2 (Figure 1(a)) and H_3 (Figure 1(b)). These figures clearly show lots of torsion in H_2, little in H_3, and the top of the cuspidal range corresponds to H_2.

Thus our data suggests that, when Conjecture 1.1 is generalized to an arithmetic group Γ in a *semisimple* $\mathbb{Q}$-group $\mathbf{G}$ with nontrivial $\mathbb{Q}$-rank, the limit of the ratios

$$\frac{\log |H^i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]}$$

should vanish unless $\delta = 1$ and i is the *top degree of the cuspidal range*, and the nonzero limit should be the quantity $c_{G,\mathscr{L}}$. In the case of a *reductive* group $\mathbf{G}$ with positive-dimensional split component A_G, one should see more cohomology groups with a nonzero limit; see the discussion below in §6 for an explanation.

5.3. Given that the abundance of exotic torsion occurs in the top degree of the cuspidal range, what can one say about other degrees? In lower degrees, we observed that there can still be exotic torsion, as defined here, but we did not see the exponential growth or the appearance of gigantic torsion primes.

Our data behave differently for different groups. For example, our data shows that for $\mathrm{GL}_3/\mathbb{Q}$, *all the exotic primes in H_3 actually divide the level.* We don't have an explanation for this. On the other hand, for GL_2/F where F is the -23 cubic, we do see exotic primes in H_3 as well as H_2, but they coincide. More precisely, the torsion subgroups of H_3 and H_2 are in general different, but the portion of them corresponding to exotic primes has the same order in all cases. This should be another manifestation of the effect of the flat factor on the cohomology, as described in §6.4. In addition we note that H_4 for the -23 cubic, which is the bottom of the cuspidal range, has no exotic torsion at all.

5.4. Finally, we consider the case of deficiency $\delta > 1$. For the group $\mathrm{GL}_2(\mathscr{O}_E)$, $E = \mathbb{Q}(\zeta_5)$, we did observe exotic torsion in H_2, which is the top of the cuspidal range. These primes start out small, and don't appear to grow as fast as in the deficiency 1 case. However they do grow (although the data, which is omitted here, is somewhat equivocal). It remains an interesting problem to formulate precisely how rapidly the exotic torsion grows for $\delta > 1$.

6. Alternating sum ("Euler characteristic") plots

6.1. Recall that Bergeron–Venkatesh were able to prove an "Euler characteristic" version of their conjecture (1). In particular, although they were unable to show that their Conjecture 1.1 held, they could show that the alternating sum of the ratios $\log |H^i_{\mathrm{tors}}(\Gamma_k)|/[\Gamma : \Gamma_k]$ tended to the predicted constants. Thus we applied the alternating sum to our data to see if this provided better apparent matching with the predicted limiting constant. All groups we checked have deficiency $\delta = 1$.

We remark that these computations involve torsion classes of small orders, and in particular involve classes that potentially could come from Voronoi homology and not from group cohomology. However, as mentioned after Theorem 2.2, since we expect Galois representations to be attached to eigenclasses in the Voronoi homology, these Euler characteristic computations have arithmetic meaning.

6.2. For $\mathrm{GL}_3/\mathbb{Q}$, one notices that the ratios in Figure 1(a) for H_2 appear to creep up to the limiting constant, whereas those in Figure 1(c) for H_4 simultaneously fall down to the constant. The group H_4 consists of large amounts of 2- and 3-torsion, and is outside the cuspidal range, whereas the former is replete with exotic torsion and is at the top of the cuspidal range. The group H_3, on the other hand, quickly contributes nothing to the computation (Figure 1(b)). Thus together H_2 and H_4 combine to give a large contribution to the predicted limit, as seen in Figure 8. It is possible that for this group the limit of the Euler characteristic might be larger than the the B-V limit.

6.3. For $\mathrm{GL}_4/\mathbb{Q}$ we haven't enough data to show a clear picture of the trending of the Euler characteristic. We include the plot in Figure 9 for completeness.

6.4. Finally, we consider the group GL_2/F, F the -23 cubic. In this case we have an additional flat factor in the symmetric space, and the Euler characteristic computation causes an amusing cancellation: the exotic torsion that should properly be at the top of the cuspidal range in H_2 also appears in H_3, the group one degree below. As remarked at the end of §5, the order of the exotic torsion summand is the same for H_2 and H_3. This is caused by the Künneth theorem and the flat factor in the global symmetric space $D \simeq \mathbb{H}_2 \times \mathbb{H}_3 \times \mathbb{R}$.[6]

[6] We are being a bit sloppy here, since of course the quotient of D by Γ need not be the product of a quotient of the SL_2-symmetric space $\mathbb{H}_2 \times \mathbb{H}_3$ with a quotient of $\mathbb{R}$ by a central subgroup.

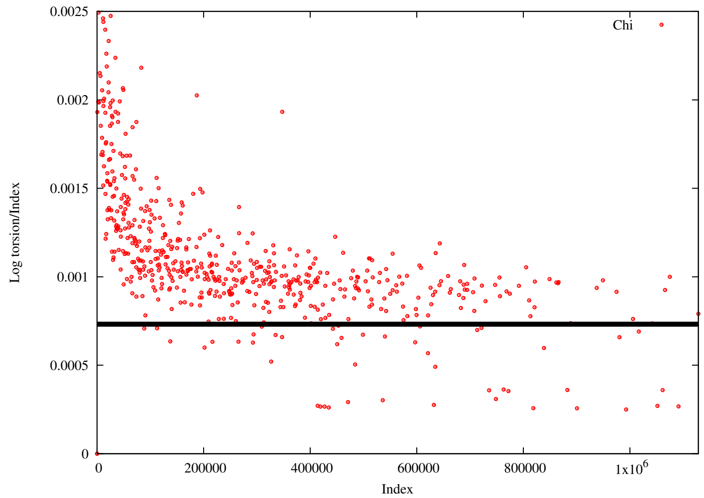


FIGURE 8. The "Euler characteristic" for subgroups of $\mathrm{GL}_3(\mathbb{Z})$

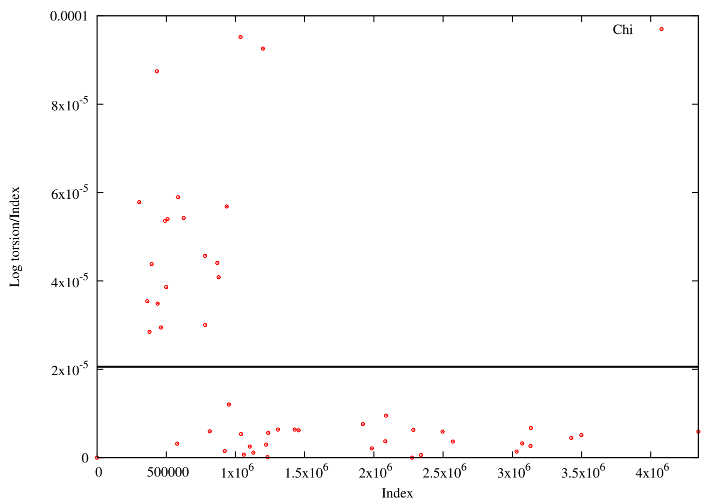


FIGURE 9. The "Euler characteristic" for subgroups of $\mathrm{GL}_4(\mathbb{Z})$

Since the exotic torsion appears to grow and dominates the order of the H_{tors}, the contributions from H_2 and H_3 eventually *cancel* on average. This forces the alternating sum to tend to zero instead of the B-V limit, as seen in Figure 10.

Indeed, the locally symmetric space for GL_2 will in general be a nontrivial S^1-bundle over the locally symmetric space for SL_2. On the other hand, one can choose subgroups of Γ for which this bundle is trivial, and the Künneth theorem then does imply the result. In general, a spectral sequence for the S^1-bundle should give the same result.

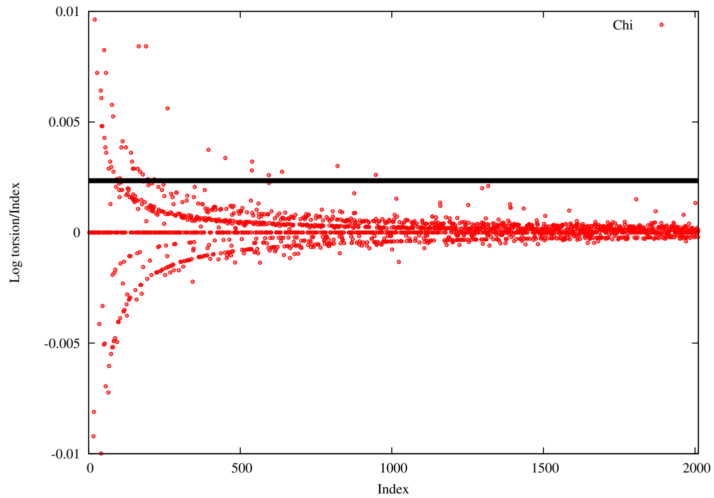


FIGURE 10. The "Euler characteristic" for subgroups of $\mathrm{GL}_2(\mathscr{O}_F)$ for the cubic field of discriminant -23

7. Towers of subgroups and prime levels

7.1. Conjecture 1.1 refers to taking a limit as Γ_k goes up a tower of congruence subgroups, whereas our computations were performed for almost complete lists of congruence subgroups up to some bound on the level. Unfortunately, since the complexity of our computations grows quite rapidly with the level, and since levels grow quickly in towers, we were unable to systematically test the dependence of generalizations of Conjecture 1.1 using towers.

7.2. However, there is one case in which we can give some indications: the imaginary quadratic field $\mathbb{Q}(\sqrt{-1})$. As mentioned in §4.5, our computations overlap with earlier work of Şengün, and so we do not emphasize them here. But for $\mathrm{GL}_2(\mathscr{O}_{\mathbb{Q}(\sqrt{-1})})$ we were able to compute with very large levels, which gives us the chance to explore some towers. The result is shown in Figure 11, where we give the plot for six towers each of length six for H_1, superimposed on the plot for H_1 for other levels. For comparison, Figure 12 shows the same set of data points, with prime levels indicated. Figure 13 shows a similar prime level plot for H_2 for the -23 cubic, and Figures 14–15 show prime level plots for H_2 of $\mathrm{GL}_3(\mathbb{Z})$.

As one can see, in towers the ratios do appear to climb to the predicted constant, although they take their time in doing so. On the other hand, prime levels seem to do a much better job, in the sense that the ratios tend to remain closer to the predicted limit than composite levels do.

7.3. These plots together with Figures 1–7 suggest several stronger conjectures than the direct analogue of Conjecture 1.1:

Conjecture 7.1. *Let Γ be any arithmetic group. The limit*

$$\lim_{k\to\infty} \frac{\log |H^i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]} \tag{11}$$

should tend to the B-V limit when $\delta = 1$ and when i is at the top of the cuspidal range and Γ_k ranges over congruence subgroups of Γ of increasing prime level.

Conjecture 7.2. *Let Γ be any arithmetic group. The limit* (11) *should equal the B-V limit as long as Γ_k ranges over any set of congruence subgroups of* increasing level*. In particular, the lim inf*

$$\liminf_{\Gamma_k} \frac{\log |H^i(\Gamma_k; \mathscr{L})_{\mathrm{tors}}|}{[\Gamma : \Gamma_k]},$$

taken over all congruence subgroups, should equal the B-V limit.

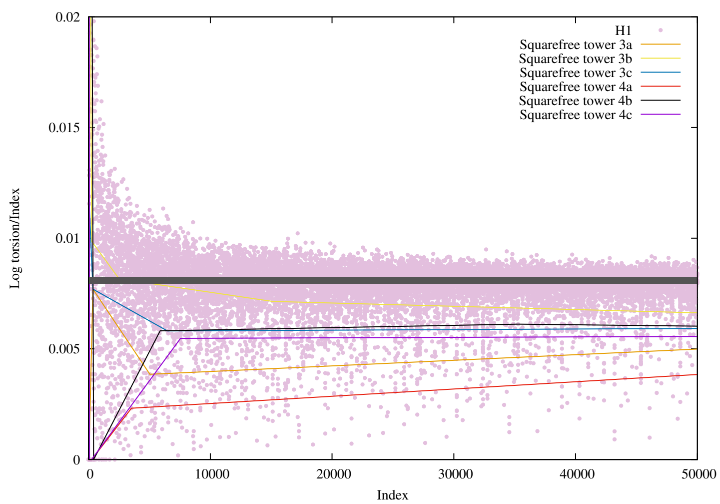


FIGURE 11. Six squarefree level towers of Voronoi homology group H_1 for congruence subgroups for $\mathrm{GL}_2(\mathbb{Z}[\sqrt{-1}])$. Each of tower is length six, and the groups in a tower are joined by straight lines, superimposed on the plot for H_1 for other levels.

8. Eisenstein phenomena

8.1. We now report on the Eisenstein cohomology phenomena we investigated. We begin with an overview of Eisenstein cohomology, the concept of which is due to Harder; for more information we refer to [22]. We restrict ourselves to trivial coefficients.

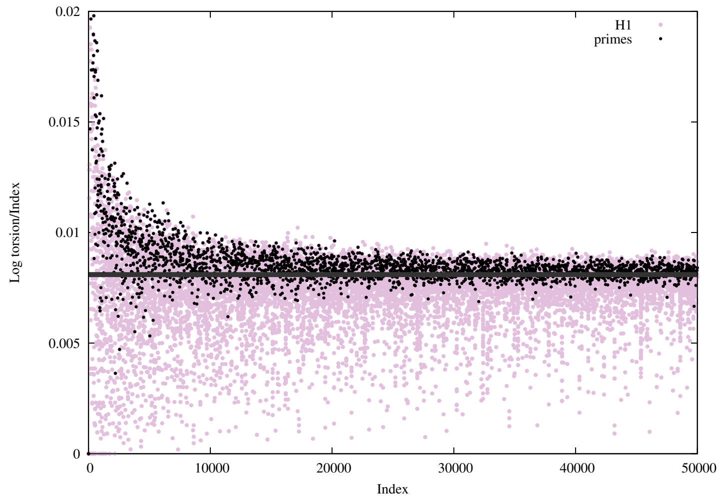


FIGURE 12. H_1 with prime levels indicated for the subgroups of $\mathrm{GL}_2(\mathbb{Z}[\sqrt{-1}])$.

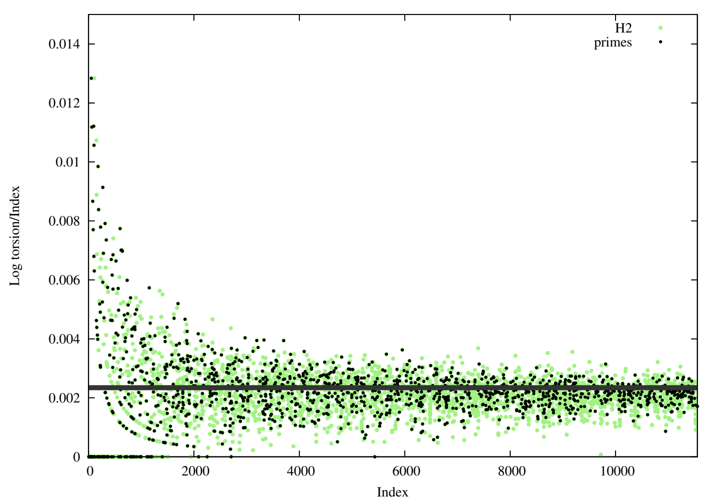


FIGURE 13. H_2 with prime levels indicated for the subgroups of $\mathrm{GL}_2(\mathscr{O}_F)$ for the cubic field of discriminant -23.

Recall that D is our global symmetric space. Let D^{BS} be the partial compactification constructed by Borel and Serre [11]. The quotient $Y := \Gamma\backslash D$ is an orbifold, and the quotient $Y^{\mathrm{BS}} := \Gamma\backslash D^{\mathrm{BS}}$ is a compact orbifold with corners. We have

$$H^*(\Gamma;\mathbb{C}) \simeq H^*(Y;\mathbb{C}) \simeq H^*(Y^{\mathrm{BS}};\mathbb{C}).$$

Let $\partial Y^{\mathrm{BS}} = Y^{\mathrm{BS}} \smallsetminus Y$. The inclusion of the boundary $\iota\colon \partial Y^{\mathrm{BS}} \to Y^{\mathrm{BS}}$ induces a map on cohomology $\iota^*\colon H^*(Y^{\mathrm{BS}};\mathbb{C}) \to H^*(\partial Y^{\mathrm{BS}};\mathbb{C})$. Moreover, this map is compatible

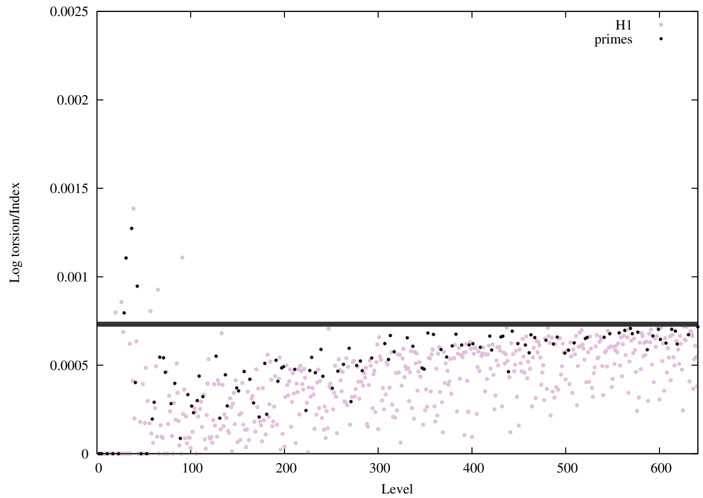


FIGURE 14. H_2 with prime levels indicated for the subgroups of $\mathrm{GL}_3(\mathbb{Z})$ (ordered by level).

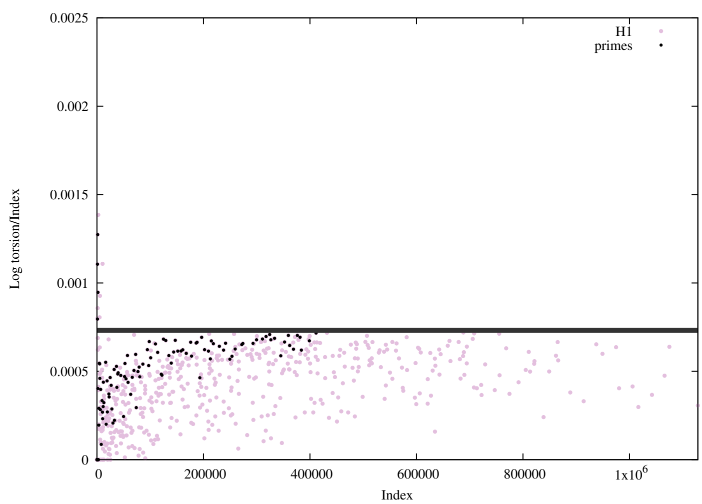


FIGURE 15. H_2 with prime levels indicated for the subgroups of $\mathrm{GL}_3(\mathbb{Z})$ (ordered by index).

with the action of the Hecke operators: the Hecke operators act naturally on the boundary ∂Y^{BS}. The kernel $H^*_!(Y^{\mathrm{BS}};\mathbb{C})$ of ι^* is called the *interior cohomology*; it equals the image of the cohomology with compact supports. The goal of Eisenstein cohomology is to use Eisenstein series and cohomology classes on the boundary to construct a Hecke-equivariant section $s\colon H^*(\partial Y^{\mathrm{BS}};\mathbb{C}) \to H^*(Y^{\mathrm{BS}};\mathbb{C})$ mapping onto a complement $H^*_{\mathrm{Eis}}(Y^{\mathrm{BS}};\mathbb{C})$ of the interior cohomology in the full cohomology. We call classes in the image of s *Eisenstein classes*.

8.2. The construction of Eisenstein cohomology provides a link between the cohomology of the Borel–Serre boundary ∂Y^{BS} and the cohomology of Y. In general it is very difficult to fully analyze the part of the cohomology coming from the boundary, and for groups of $\mathbb{Q}$-rank > 1 (such as $\mathrm{GL}_n/\mathbb{Q}$, $n \geq 3$), complete results are not known. However, one can typically predict the types of relationships one might see, and can observe them in practice by computing the Hecke action. For example, in the case of constant coefficients, one experimentally sees cohomology classes in H^5 of congruence subgroups of $\mathrm{GL}_4/\mathbb{Q}$ that correspond to classes in H^3 of congruence subgroups of $\mathrm{GL}_3/\mathbb{Q}$, as well as classes corresponding to weight 2 and weight 4 holomorphic modular forms [3]; one can refer there to see descriptions of the predicted mechanisms for the Eisenstein lifting.

In our experiments, we have one pair of groups where we can hope to see Eisenstein phenomena: $\mathrm{GL}_3/\mathbb{Q}$ and $\mathrm{GL}_4/\mathbb{Q}$. Since we are not computing Hecke operators on the torsion classes we computed, we instead try to see connections by looking for exotic primes that appear in different cohomological degrees. In particular, if we see the same exotic prime occurring as torsion in a pair of cohomology groups for $(\mathrm{GL}_3, \mathrm{GL}_4)$, and these cohomology groups can be related by a standard Eisenstein mechanism, we take that as evidence of Eisenstein phenomena. As before, the torsion size is given in factored form with exotic torsion in **bold**.

8.3. The first collection of Eisenstein lifts goes from H_2 of GL_3 to H_3 of GL_4. Recall that we index groups homologically by the Voronoi complex (4). Thus H_2 of GL_3 refers to H^3 and H_3 of GL_4 refers to H^6; both of these are in the vcd of their respective groups.

- At level 114, the size of the torsion in H_3 is $2^{12} \cdot 3^7 \cdot \mathbf{11^4}$. The corresponding torsion for GL_3 in H_2 is $2^5 \cdot 3^3 \cdot \mathbf{11^2}$.
- At level 118, the size of the torsion in H_3 is $2^{14} \cdot \mathbf{17^4}$. The corresponding torsion for GL_3 in H_2 is $\mathbf{17^2}$.
- At level 119, the size of the torsion in H_3 is $2^4 \cdot 3^3 \cdot \mathbf{31^4}$. The corresponding torsion for GL_3 in H_2 is $2^2 \cdot 3^1 \cdot \mathbf{31^2}$.

8.4. We also see lifts for H_3 of GL_3 to H_4 for GL_4; both of these correspond to cohomological degree one below the vcd of their respective groups.

- At level 49, the size of the torsion in H_4 is $3^1 \cdot \mathbf{7^2}$. The corresponding torsion for GL_3 in H_3 is $\mathbf{7}$.
- At level 98, the size of the torsion in H_4 is $\mathbf{7^5}$. The corresponding torsion for GL_3 in H_3 is $\mathbf{7}$.

Since Eisenstein series are differential forms defined over the complex numbers, they cannot directly be used to study torsion. Instead, a topological analysis of the

Borel-Serre boundary is required, which can mimic the Eisenstein phenomena, by showing how certain cohomology classes on the Borel-Serre boundary can be lifted to cohomology classes on the whole space. The topology could be carried out with integral coefficients and thus could deal with the torsion in theory. This has not been done for the boundary in the case of GL_4. Such a topological analysis poses a very interesting and probably very hard problem. Heuristically we can say that both of these examples can be explained by the topological Eisenstein mechanism of placing a class on $H^k(\partial_P Y)$, where $\partial_P Y$ is the Levi part of Borel–Serre boundary component corresponding to a maximal parabolic subgroup of type $(1, 3)$. Such a class has the potential of lifting to $H^{k+3}(Y^{\mathrm{BS}})$ through a spectral sequence with E_2 page given by $H^i(\partial_P Y, H^j N)$, where N is the nilmanifold part of the Borel–Serre boundary component. We do not know why we observe Eisenstein phenomena at these levels and not at other levels.

9. Conclusions and further questions

In this paper, we computed the torsion in the Voronoi homology of congruence subgroups of several arithmetic groups. The Voronoi homology is isomorphic to the group cohomology in dual dimensions. Our examples treated groups of deficiencies 1 and 2.

- We found excellent agreement in our results with the general heuristic espoused by Bergeron–Venkatesh [8], namely that groups with deficiency 1 should have exponential growth in the torsion in their cohomology. We also found excellent quantitative agreement with the predicted asymptotic limit from Conjecture 1.1, suitably interpreted for reductive groups.
- We found that, when the $\mathbb{Q}$-rank of a group is > 0 and the deficiency is 1, the explosive exotic torsion occurs in the top cohomological degree of the cuspidal range.
- When the deficiency is > 1, we still found exotic torsion in the top degree of the cuspidal range. However, the growth rate of the size of the torsion subgroup appears much lower than that in the deficiency 1 case. It would be interesting to formulate a quantitative estimate for the growth of the torsion in this case. Would this estimate be polynomial or subexponential?
- For groups of deficiency 1, the growth of the torsion in towers of congruence subgroups seems to agree with the predicted asymptotic limit, although oddly the convergence seems significantly slower than that experienced by families of congruence subgroups of increasing prime level or simply the family of all congruence subgroups ordered by increasing level.
- The exotic torsion in a group of deficiency 1 appears to tend to transfer to another via Eisenstein cohomology. What is the explanation of when this

transfer happens and when it doesn't? Could this be related to divisibility of special values of some L-function by the exotic primes in question?

- We made two new conjectures along the lines of Conjecture 1.1 but for not necessarily cocompact groups and for different families of congruence subgroups. These may be found in §7.3.

References

[1] A. Ash, *Galois representations attached to mod p cohomology of* $\mathrm{GL}(n, \mathbf{Z})$, Duke Math. J. **65** (1992), no. 2, 235–255.

[2] A. Ash, *Parity of mod p Betti numbers*, Bol. Soc. Mat. Mexicana (3) **5** (1999), no. 1, 79–85.

[3] A. Ash, P. E. Gunnells, and M. McConnell, *Cohomology of congruence subgroups of* $\mathrm{SL}_4(\mathbb{Z})$, J. Number Theory **94** (2002), no. 1, 181–212.

[4] A. Ash, P. E. Gunnells, and M. McConnell, *Torsion in the cohomology of congruence subgroups of* $\mathrm{SL}(4, \mathbb{Z})$ *and Galois representations*, J. Algebra **325** (2011), 404–415.

[5] A. Ash, P. E. Gunnells, and M. McConnell, *Resolutions of the Steinberg module for* $GL(n)$, J. Algebra **349** (2012), 380–390.

[6] A. Ash, P. E. Gunnells, and M. McConnell, Mod 2 *homology for* $\mathrm{GL}(4)$ *and Galois representations*, J. Number Theory **146** (2015), 4–22.

[7] A. Ash and M. McConnell, *Experimental indications of three-dimensional Galois representations from the cohomology of* $\mathrm{SL}(3, \mathbf{Z})$, Experiment. Math. **1** (1992), no. 3, 209–223.

[8] N. Bergeron and A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu **12** (2013), no. 2, 391–447.

[9] M. Bhargava, *Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants*, Int. Math. Res. Not. IMRN (2007), no. 17, Art. ID rnm052, 20.

[10] A. Borel, *Commensurability classes and volumes of hyperbolic* 3*-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), no. 1, 1–33.

[11] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.

[12] F. Calegari and A. Venkatesh, *A torsion Jacquet–Langlands correspondence*, submitted, 2012.

[13] J. E. Cremona, *Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields*, Compositio Math. **51** (1984), no. 3, 275–324.

[14] M. Dutour Sikirić, H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki, *On the cohomology of linear groups over imaginary quadratic fields*, J. Pure Appl. Algebra **220** (2016), no. 7, 2564–2589.

[15] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Quelques calculs de la cohomologie de* $\mathrm{GL}_N(\mathbb{Z})$ *et de la K-théorie de* $\mathbb{Z}$, C. R. Math. Acad. Sci. Paris **335** (2002), no. 4, 321–324.

[16] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math. **245** (2013), 587–624.

[17] J. Elstrodt, F. Grunewald, and J. Mennicke, *On the group* $\mathrm{PSL}_2(\mathbf{Z}[i])$, Number theory days, 1980 (Exeter, 1980), London Math. Soc. Lecture Note Ser., vol. 56, Cambridge Univ. Press, Cambridge, 1982, pp. 255–283.

[18] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, Harmonic analysis and number theory.

[19] J. Franke, *Harmonic analysis in weighted* L_2*-spaces*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 2, 181–279.

[20] P. E. Gunnells, F. Hajir, and D. Yasaki, *Modular forms and elliptic curves over the field of fifth roots of unity*, Exp. Math. **22** (2013), no. 2, 203–216.
[21] P. E. Gunnells and D. Yasaki, *Modular forms and elliptic curves over the cubic field of discriminant* -23, Int. J. Number Theory **9** (2013), no. 1, 53–76.
[22] G. Harder, *Eisenstein cohomology of arithmetic groups and its applications to number theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 779–790.
[23] A. Hurwitz, *Ueber die Reduction der binären quadratischen Formen*, Math. Ann. **45** (1894), no. 1, 85–117.
[24] M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivitätsbereichen. I*, Math. Ann. **141** (1960), 384–432.
[25] J.-S. Li and J. Schwermer, *Automorphic representations and cohomology of arithmetic groups*, Challenges for the 21st century (Singapore, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 102–137.
[26] J. Pfaff, *Exponential growth of homological torsion for towers of congruence subgroups of Bianchi groups*, Ann. Global Anal. Geom. **45** (2014), no. 4, 267–285.
[27] C. Priplata, *Cohomology and Homology of* PSL_2 *over Imaginary Quadratic Integers with General Coefficients* $M_{n,m}(O_K)$ *and Hecke Eigenvalues*, Ph.D. thesis, Heinrich Heine Universität, Düsseldorf, 2000.
[28] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015), no. 3, 945–1066.
[29] M. H. Şengün, *On the integral cohomology of Bianchi groups*, Exp. Math. **20** (2011), no. 4, 487–505.
[30] G. Voronoĭ, *Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math. **133** (1908), 97–178.

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