Name: ______________ Academic Integrity Signature: ______________

I have abided by the UNCG Academic Integrity Policy.

Read all of the following information before starting the exam:

- It is to your advantage to answer ALL of the 11 questions.
- It is your responsibility to make sure that you have all of the problems.
- There is no need to complete the test in order. The problems are independent.
- Correct numerical answers with insufficient justification may receive little or no credit.
- Clearly distinguish your final answer from your scratch work with a box or circle.
- *Budget your time!*
- If you have read all of these instructions, draw a happy face here.

Page:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

1. (10 points) Use the Intermediate Value Theorem to show that $f(x) = x^3 - x^2 - 1$ has a root in the interval $[1, 2]$.

Solution: First note that f is continuous on $[1, 2]$. (It is in fact continuous everywhere.) We have that $f(1) = -1$ and $f(2) = 8 - 4 - 1 = 3$. Since 0 is between -1 and 3, the Intermediate Value Theorem guarantees the existence of c between 1 and 2 such that $f(c) = 0$.

2. (5 points) (Precise definition of limit) Let $f(x)$ be defined on an open interval containing x_0, except possibly at x_0 itself. We say that the *limit of $f(x)$ as x approaches x_0 is L*, denoted $\lim_{x \to x_0} f(x) = L$, if

Solution: for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \neq x_0$,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

3. (5 points) Find a number $\delta > 0$ such that every number x in the interval $|x - 1| < \delta$ also satisfies $|(8x - 2) - 6| < \frac{1}{100}$.

Solution: We have that

$$|(8x - 2) - 6| = |8x - 8| = 8|x - 1| < \frac{1}{100}$$

as long as $|x - 1| < \frac{1}{800}$. Thus we can choose $\delta = \frac{1}{800}$.

Points earned: ______ out of 20.

4. Let $f(x) = \dfrac{x^2}{2x - 10}$.

   (a) (5 points) Evaluate $\lim_{x \to 5^-} f(x)$ and $\lim_{x \to 5^+} f(x)$.

   Solution: Note that if we tried to plug in $x = 5$, we would get "$\frac{25}{0}$". That means the limit is $+\infty$, $-\infty$, or does not exist. For values x near 5 but less than 5, x^2 is positive and $2x - 10$ is negative. Thus $\dfrac{x^2}{2x-10}$ is negative for such values. It follows that

   $$\lim_{x \to 5^-} \frac{x^2}{2x - 10} = -\infty.$$

   For values x near 5 but greater than 5, x^2 is positive and $2x - 10$ is positive. Thus $\dfrac{x^2}{2x-10}$ is positive for such values. It follows that

   $$\lim_{x \to 5^+} \frac{x^2}{2x - 10} = \infty.$$

   (b) (5 points) Does the graph $y = f(x)$ have a vertical asymptote? If it does, give the formula for the vertical asymptote. If not, explain why not.

   Solution: The only possible place this graph can have a vertical asymptote is where $2x - 10 = 0$. That means $x = 5$ is the only possibility. Since the numerator is not zero when $x = 5$, this shows that $x = 5$ is indeed a vertical asymptote.

5. Let $f(x) = \dfrac{3x^3 + 2x - 13}{7x^3 + 23x^2 + x - 1}$.

   (a) (5 points) Evaluate $\lim_{x \to \infty} f(x)$.

   Solution: Polynomials are dominated by their leading term. Hence

   $$\lim_{x \to \infty} \frac{3x^3 + 2x - 13}{7x^3 + 23x^2 + x - 1} = \lim_{x \to \infty} \frac{3\cancel{x^3}}{7\cancel{x^3}} = \lim_{x \to \infty} \frac{3}{7} = \frac{3}{7}$$

   (b) (5 points) Does the graph $y = f(x)$ have a horizontal asymptote? If it does, give the formula for the horizontal asymptote.

   Solution: Yes. The horizontal asymptote (from the computation above) is $y = \frac{3}{7}$.

6. Evaluate the following limits

(a) (5 points) $\lim_{t\to -2} \frac{t+2}{t^2+3t+2}$

Solution: Notice that if we plug in $t=-2$, we would get "$\frac{0}{0}$". That means we need to *do more work*. We compute

$$\lim_{t\to -2} \frac{t+2}{t^2+3t+2} = \lim_{t\to -2} \frac{\cancel{(t+2)}}{\cancel{(t+2)}(t+1)} = \lim_{t\to -2} \frac{1}{t+1} = -1.$$

(b) (5 points) $\lim_{x\to 0} \frac{\sin(5x)}{3x}$

Solution:

$$\lim_{x\to 0} \frac{\sin(5x)}{3x} = \frac{1}{3}\lim_{x\to 0} \frac{\sin(5x)}{x} = \frac{1}{3}\lim_{x\to 0} \frac{5\sin(5x)}{5x} = \frac{5}{3}\lim_{x\to 0} \frac{\sin(5x)}{5x}.$$

Let $\theta = 5x$. As $x \to 0$, we have $\theta \to 0$ Then

$$\lim_{x\to 0} \frac{\sin(5x)}{5x} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1,$$

so $\lim_{x\to 0} \frac{\sin(5x)}{3x} = \frac{5}{3}$.

7. (10 points) Suppose f is a function such that $\lim_{x\to 1} f(x) = 2$. Suppose g is a function such that $\lim_{x\to 1} g(x) = 4$. Use Limit Laws to compute $\lim_{x\to 1}(4f(x) - \sqrt{g(x)})$.

Solution:

$$\begin{aligned}
\lim_{x\to 1}(4f(x) - \sqrt{g(x)}) &= \lim_{x\to 1} 4f(x) - \lim_{x\to 1} \sqrt{g(x)} && \text{Difference Rule}\\
&= 4\lim_{x\to 1} f(x) - \lim_{x\to 1} \sqrt{g(x)} && \text{Constant Multiple Rule}\\
&= 4\lim_{x\to 1} f(x) - \sqrt{\lim_{x\to 1} g(x)} && \text{Root Rule}\\
&= 4(2) - \sqrt{4}\\
&= 6.
\end{aligned}$$

 Points earned: ______ out of 20.

8. (10 points) At what points is the function $f(x) = \dfrac{x+3}{x^2-3x-10}$ continuous?

Solution: Note that f is a rational function, and so it is continuous on its domain. The domain of a rational function is all real numbers, except the roots of the denominator. Since $x^2 - 3x - 10 = (x-5)(x+2)$, we have that f is continuous on

$$(-\infty, -2) \cup (-2, 5) \cup (5, \infty).$$

9. (10 points) For what value of a is

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < -2, \\ 5ax & \text{if } x \geq -2 \end{cases}$$

continuous at $x = -2$?

Solution: We have that f is continuous at $x = 3$ if $\lim_{x\to 3} f(x) = f(3)$. We compute $\lim_{x\to -2^-} f(x) = (-2)^2 + 1 = 5$ and $\lim_{x\to -2^+} f(x) = 5 \cdot a \cdot (-2) = -10a$. It follows that to arrange that f is continuous at $x = 3$, we must have $5 = -10a$. Thus choosing $a = -\frac{1}{2}$ will make f continuous at $x = 2$.

 Points earned: ______ out of 20.

10. (10 points) Suppose f and g are continuous functions such that

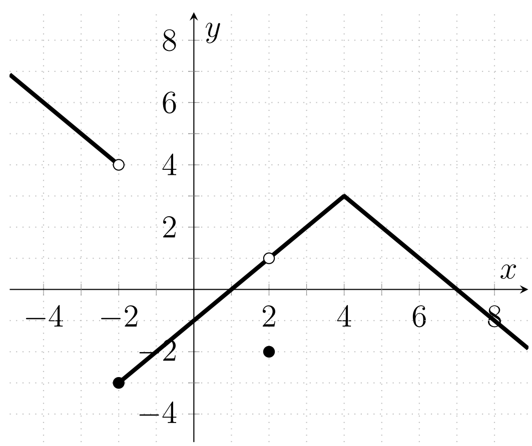
$$\lim_{x\to 0} f(x) = 2, \quad f(7) = -1, \quad \lim_{x\to 0} g(x) = 7, \quad \text{and} \quad g(2) = 3.$$

Compute $\lim_{x\to 0} g(f(x))$ or explain what additional information is needed to compute the limit.

Solution: Recall that the composition of continuous functions is continuous so that

$$\lim_{x\to 0} g(f(x)) = g(\lim_{x\to 0} f(x)) = g(2) = 3.$$

11. The graph of $y = f(x)$ is shown below. Compute the following or explain why it does not exist.



(a) (2 points) $\lim_{x\to -2^+} f(x)$

Solution: As we approach -2 from the right, the height of the graph approaches -3, so $\lim_{x\to -2^+} f(x) = -3$.

(b) (2 points) $\lim_{x\to -2^-} f(x)$

Solution: As we approach -2 from the left, the height of the graph approaches 4, so $\lim_{x\to -2^-} f(x) = 4$

(c) (2 points) $f(-2)$

Solution: The height at -2 is -3, so $f(-2) = -3$.

(d) (2 points) $\lim_{x \to 4} f(x)$

Solution: As we approach 4 from the left and from the right, the height of the graph approaches 3, so $\lim_{x \to 4} f(x) = 3$.

(e) (2 points) $\lim_{x \to 2} f(x)$

Solution: As we approach 2 from the left and from the right, the height of the graph approaches 1, so $\lim_{x \to 2} f(x) = 1$.