Name:	Academic Integrity Signature:	
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1. General Proof Exercise

The Product of Two Odd Numbers is Odd

The Product of an Even Number and an Odd Number is Even

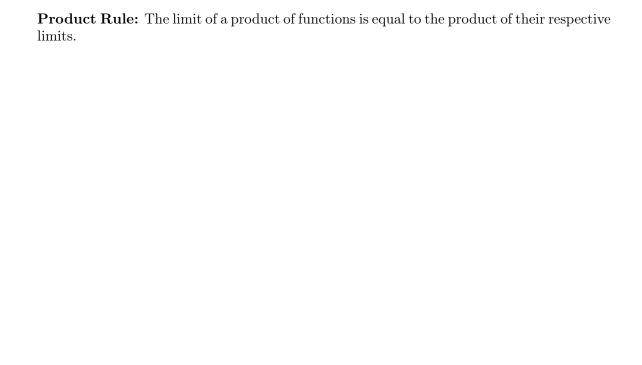
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1.	LIMIT	LAWS

(1)	Theorem 1	Limit Laws	Constant	Function	Rule:	The limi	t of a	constant-va	alue :	func-
	tion is equal	that constant.								

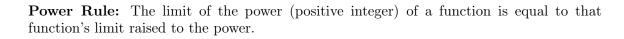
Sum and Difference Rules: The limit of the sum or difference of two functions is the sum or difference of their respective limits.

Constant Multiple Rule: The limit of a constant-multiple of a function is equal to the product of that constant multiplier and the limit of the original function. [Direct Proof]

Constant Multiple Rule: [Proof Using Product Rule]



**Quotient Rule:** The limit of a ratio of two functions is equal to the quotient of their respective limits (provided the limit of the denominator term is not 0).



Root Rule: The limit of the root (positive integer) of a function is equal to the root of that function's limit.

(2) Limits of Polynomials: If  $f(x) = a_n x^n + \dots + a_1 x + a_0$ , then  $\lim_{x \to c} = a_n c^n + \dots + a_1 c + a_0 c$ .

(3) Limits of Rational Functions: If P(x) and Q(x) are polynomials and  $Q(c) \neq 0$ , then  $\lim_{x\to c} P(x)/Q(x) = P(c)/Q(c)$ .

(4) Comparison of Left and Right Limits: The limit of a function f(x) exists at x = c if and only if the left and right-side Limits exist at x = c and are equal to one another.

(5) **Inequality Comparison of Limits:** If  $f(x) \leq g(x)$  for all x in an open interval containing c, except possibly c itself, and the limit of both functions exist as  $x \to c$ , then  $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$ .

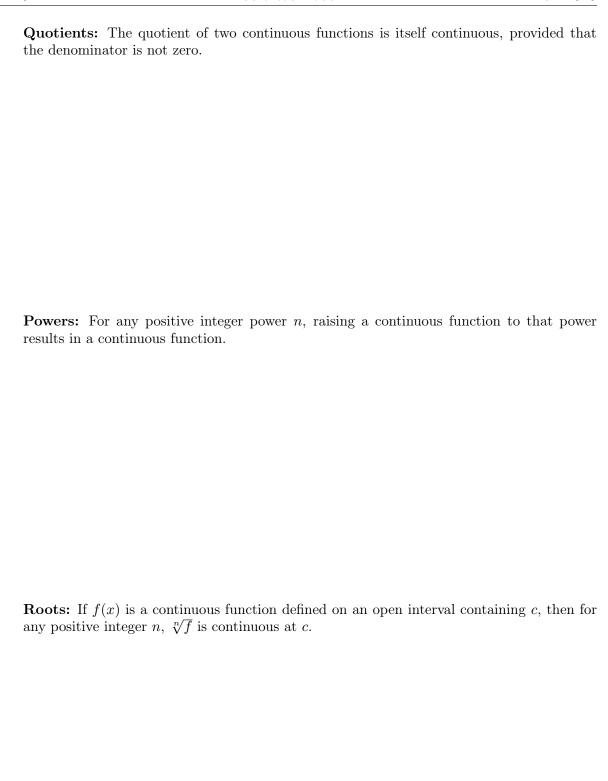
(6) The Sandwich Theorem: Suppose that  $g(x) \leq f(x) \leq h(x)$  for all x in an open interval containing c, except possibly c itself and that  $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$ . Then  $\lim_{x\to c} = L$ .

(7) Trigonometric Limits:  $\lim_{x\to 0} \sin x/x = 1$ 

(8) Trigonometric Limits:  $\lim_{x\to 0} (1-\cos x)/x = 0$ 

(9)	Properties of Continuous Functions: Sums and Differences: The sum or difference of two continuous functions is itself continuous.
	Constant Multipliers: A constant multiple of a continuous function is itself continuous

**Products:** The product of two continuous functions is itself continuous.



(10) Theorem 10 Limits of Continuous Functions: If g is continuous at b and  $\lim_{x\to c} f(x) = b$ , then

$$\lim_{x\to c}g(f(x))=g(b)=g(\lim_{x\to c}f(x)).$$

(11) Composition of Continuous Functions: If f is continuous at c and g is continuous at g(c), then  $g \circ f$  is continuous at c.

(12) **Intermediate Value Theorem:** If f is continuous on closed interval [a, b], and  $y_0$  is any value between f(a) and f(b), then  $y_0 = f(c)$  for some c in [a, b].

(13) **Fixed Point Theorem (2.5 - 67):** Suppose that f is continuous on [0,1] and  $0 \le f(x) \le 1$ . Show there must be point c on [0,1], called the fixed point, such that f(c) = c.

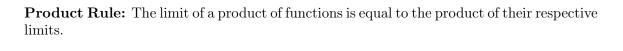
(14) Sign Preservation of Continuous Functions (2.5 -68): Let f be defined on (a, b) and continuous at some c with  $f(c) \neq 0$ . Show there is an interval  $(c - \delta, c + \delta)$  where f has the same sign as f(c).

(15) Continuity and Infintessimal Change in x (2.5 - 69): f is continuous at c if and only if  $\lim_{h\to 0} f(c+h) = f(c)$ .

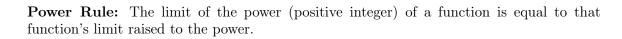
(16) Continuity of  $\sin(x)$  (2.5 - 70): Using the result above and addition formulas, show that  $\sin(x)$  is continuous at every x = c.

(17) Continuity of cos(x) (2.5-70): Using the result above and addition formulas, show that cos(x) is continuous at every x = c.

(18)	All limit laws apply as $x \to \pm \infty$ Constant Function Rule: The limit of a constant-value function is equal that constant.
	Sum and Difference Rules: The limit of the sum or difference of two functions is the sum or difference of their respective limits.
	Constant Multiple Rule: The limit of a constant-multiple of a function is equal to the product of that constant multiplier and the limit of the original function. [Direct Proof]
	Constant Multiple Rule: [Proof Using Product Rule]



**Quotient Rule:** The limit of a ratio of two functions is equal to the quotient of their respective limits (provided the limit of the denominator term is not 0).



Root Rule: The limit of the root (positive integer) of a function is equal to the root of that function's limit.

(19) Intriguing Theory (Chapter 2 - 17): Show that the function

$$f(x) = \begin{cases} x, & \text{if } x \text{ irrational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is continous at x = 0.

(20) Centered Difference Quotient: Assuming the limits exist, show that

$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

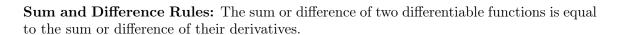
## 3. Derivatives

(1) **Differentiability implies Continuity:** If f has a derivative at x = c, then f is continuous at x = c.

(2) Derivative Rules:

Constant Rule: The derivative of a constant is 0.

Constant Multiplier Rule (Direct): The derivative of a constant multiple of a differentiable function is equal that constant times the derivative of that differentiable function.



Power Rule (positive integer) (Direct): For any positive integer n, the derivative of  $x^n$  is  $df/dx = nx^{n-1}$ .

**Product Rule:** For differentiable functions f and g, the derivative of their product is equal to f'(x)g(x) + f(x)g'(x).

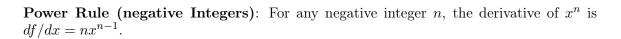
Quotient Rule (Direct): For differentiable functions f and g, with  $g(c) \neq 0$ , the derivative of their ratio f(x)/g(x) at x = c is

$$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

**Reciprocal Rule:** Prove that the derivative of  $f^{-1}(x)$  is  $-f'(x)/f^2(x)$  using the product and constant rules.

Quotient Rule: Prove the quotient rule using the Product Rule and the Reciprocal Rule.

Reciropcal Rule: Prove the reciprocal rule using the quotient rule.



**Power Rule (positive integer):** Prove the power rule using the product rule and inductive reasoning.

Constant Multiplier Rule: Using the product and constant rules, prove the constant multiplier rule.

**General Polynomial Derivative:** The derivative of the polynomial  $P(x) = a_n x^n + \cdots + a_1 x + a_0$  is  $P'(x) = n a_n x^{n-1} + \cdots + a_1$ .

Generalized Product Rule: The derivative of  $F(x) = f(x)g(x) \dots z(x)$  is given by  $\frac{dF}{dx} = f'(x)g(x) \dots z(x) + f(x)g'(x) \dots z(x) + \dots + f(x)g(x) \dots z'(x)$ 

(3) Derivatives of Trigonometric Functions

Sine: Prove that the derivative of  $\sin x$  is  $\cos x$ .

**Cosine:** Prove that the derivative of  $\cos x$  is  $-\sin x$ .

**Tangent:** Prove that the derivative of  $\tan x$  is  $\sec^2 x$ .

**Cotangent:** Prove that the derivative of  $\cot x$  is  $-\csc^2 x$ .

**Secant:** Prove that the derivative of  $\sec x$  is  $\sec x \tan x$ .

**Cosecant:** Prove that the derivative of  $\csc x$  is  $-\csc x \cot x$ 

(4) Chain Rule: If g(x) is differentiable at x, and f(u) is differentiable at u = g(x), then the composite function  $(f \circ g)(x) = f(g(x))$  is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

(5) **Power Chain Rule:** The derivative of  $u^n$  with respect to x is  $df/dx = nu^{n-1}du/dx$ .

(6) Power Rule (rational exponents): Beginning from  $y^q = x^p$ , both p and q integers and  $q \neq 0$ , use implicit differentiation to show that

$$\frac{d}{dx}\left(x^{p/q}\right) = \frac{p}{q}x^{(p/q)-1}$$

(7) **Derivative of Cosine:** Beginning with the trigonometric identity  $\cos^2 x + \sin^2 x = 1$ , use implicit differentiation to show  $(\cos x)' = -\sin x$ .

(8) **Derivative of Inverses (Part A):** If f'(x) exists and is never zero on interval I, then  $f^{-1}$  is differentiable at every point in the domain of  $f^{-1}$  (the range of f).

(9) **Derivative of Inverses (Part b):** Under the conditions above, the value of the derivative of  $f^{-1}$  at b is the reciprocal of the value of f' evaluated at  $a = f^{-1}(b)$ :

$$\frac{df^{-1}}{dx}|_{x=b} = \frac{1}{\frac{df}{dx}|_{x=f^{-1}(b)}}$$

(10) **Derivative of the Natural Log:** Using the theorem for the derivative of inverse functions and the fact that  $(e^x)' = e^x$ , prove that  $(\ln x)' = 1/x$ .

(11) **Derivative of the Exponential Function:** Using the theorem for the derivative of inverse functions and the fact that  $(\ln x)' = 1/x$ , prove that  $(e^x)' = e^x$ .

(12) The Number e as a Limit: Prove that the number e satisfies the limit  $e = \lim_{x \to (1+x)^{1/x}}$ .

(13) Derivative of Inverse Trigonometric Functions

**Derivative of Arcsine:** Prove that  $(\arcsin x)' = 1/\sqrt{1-x^2}$ .

**Derivative of Arccosine:** Prove that  $(\arccos x)' = -1/\sqrt{1-x^2}$ .

**Derivative of Arctangent:** Prove that  $(\arctan x)' = 1/(1+x^2)$ .

**Derivative of Arccotangent:** Prove that  $(\cot^{-1} x)' = -1/(1+x^2)$ .

**Derivative of Arcsecant:** Prove that  $(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$ .

**Derivative of Arccosecant:** Prove that  $(\csc^{-1} x)' = \frac{-1}{|x|\sqrt{x^2-1}}$ .

(14) **Derivative of an Odd Function:** If f(x) is a differentiable odd function, then f'(x) is an even function.

(15) **Derivative of an Even Function:** If f(x) is a differentiable even function, then f'(x)' is an odd function.

## 4. OPTIMIZATION AND APPLICATION

(1) **Global Extrema are Local Extrema:** Prove every absolute extremum is also a local extremum.

(2) The First Derivative Theorem for Local Extremes: If f has a local maximum or minimum at an interior point x = c, and if f'(c) is defined, then f'(c) = 0.

(3) **Rolle's Theorem:** If f(x) is continuous on closed interval [a, b] and differentiable on open interval (a, b), and if f(a) = f(b), then there exists some c in (a, b) such that f'(c) = 0.

(4) **The Mean Value Theorem:** If f(x) is continuous on closed interval [a,b] and differentiable on open interval (a,b), then there exists some c in (a,b) such that f'(c) = [f(b) - f(a)]/(b-a).

(5) Corollary 1 of the Mean Value Theorem: If f'(x) = 0 for all x on open interval (a, b), then f(x) = C constant for all x in the open interval.

(6) Antiderivatives Differ by Only a Constant: If f'(x) = g'(x) for all x in (a, b), then there exists a constant C such that f(x) = g(x) + C for all xin(a, b).

(7) **Parallel Tangents (4.2 - 62):** Assume that f and g are differentiable on [a,b] and that f(a) = g(a) and f(b) = g(b). Prove there is at least one point c between a and b where the tangesnts of f and g are parallel.

(8) Indeterminate Powers: Prove that if  $\lim_{x\to a} \ln f(x) = L$ , then  $\lim_{x\to a} f(x) = e^L$ .