Name: ______________________ Academic Integrity Signature: ______________________

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1. General Proof Exercise

The Product of Two Odd Numbers is Odd

The Product of an Even Number and an Odd Number is Even

2. Limit Laws

(1) **Theorem 1 Limit Laws Constant Function Rule:** The limit of a constant-value function is equal that constant.

Sum and Difference Rules: The limit of the sum or difference of two functions is the sum or difference of their respective limits.

Constant Multiple Rule: The limit of a constant-multiple of a function is equal to the product of that constant multiplier and the limit of the original function. [Direct Proof]

Constant Multiple Rule: [Proof Using Product Rule]

Product Rule: The limit of a product of functions is equal to the product of their respective limits.

Quotient Rule: The limit of a ratio of two functions is equal to the quotient of their respective limits (provided the limit of the denominator term is not 0).

Power Rule: The limit of the power (positive integer) of a function is equal to that function's limit raised to the power.

Root Rule: The limit of the root (positive integer) of a function is equal to the root of that function's limit.

(2) **Limits of Polynomials:** If $f(x) = a_n x^n + \cdots + a_1 x + a_0$, then $\lim_{x \to c} = a_n c^n + \cdots + a_1 c + a_0 c$.

(3) **Limits of Rational Functions:** If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x \to c} P(x)/Q(x) = P(c)/Q(c)$.

(4) **Comparison of Left and Right Limits:** The limit of a function $f(x)$ exists at $x = c$ if and only if the left and right-side Limits exist at $x = c$ and are equal to one another.

(5) **Inequality Comparison of Limits:** If $f(x) \leq g(x)$ for all x in an open interval containing c, except possibly c itself, and the limit of both functions exist as $x \to c$, then $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$.

(6) **The Sandwich Theorem:** Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in an open interval containing c, except possibly c itself and that $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$. Then $\lim_{x \to c} = L$.

(7) **Trigonometric Limits:** $\lim_{x\to 0} \sin x/x = 1$

(8) **Trigonometric Limits:** $\lim_{x\to 0}(1 - \cos x)/x = 0$

(9) **Properties of Continuous Functions:**
Sums and Differences: The sum or difference of two continuous functions is itself continuous.

Constant Multipliers: A constant multiple of a continuous function is itself continuous.

Products: The product of two continuous functions is itself continuous.

Quotients: The quotient of two continuous functions is itself continuous, provided that the denominator is not zero.

Powers: For any positive integer power n, raising a continuous function to that power results in a continuous function.

Roots: If $f(x)$ is a continuous function defined on an open interval containing c, then for any positive integer n, $\sqrt[n]{f}$ is continuous at c.

(10) **Theorem 10 Limits of Continuous Functions:** If g is continous at b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

(11) **Composition of Continuous Functions:** If f is continuous at c and g is continuous at $g(c)$, then $g \circ f$ is continuous at c.

(12) **Intermediate Value Theorem:** If f is continuous on closed interval $[a, b]$, and y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

(13) **Fixed Point Theorem (2.5 - 67):** Suppose that f is continuous on $[0, 1]$ and $0 \leq f(x) \leq 1$. Show there must be point c on $[0, 1]$, called the fixed point, such that $f(c) = c$.

(14) **Sign Preservation of Continuous Functions (2.5 -68):** Let f be defined on (a, b) and continuous at some c with $f(c) \neq 0$. Show there is an interval $(c - \delta, c + \delta)$ where f has the same sign as $f(c)$.

(15) **Continuity and Infintessimal Change in** x **(2.5 - 69):** f is continuous at c if and only if $\lim_{h \to 0} f(c+h) = f(c)$.

(16) **Continuity of** $\sin(x)$ **(2.5 - 70):** Using the result above and addition formulas, show that $\sin(x)$ is continuous at every $x = c$.

(17) **Continuity of** $\cos(x)$ **(2.5-70):** Using the result above and addition formulas, show that $\cos(x)$ is continuous at every $x = c$.

(18) **All limit laws apply as $x \to \pm\infty$ Constant Function Rule:** The limit of a constant-value function is equal that constant.

Sum and Difference Rules: The limit of the sum or difference of two functions is the sum or difference of their respective limits.

Constant Multiple Rule: The limit of a constant-multiple of a function is equal to the product of that constant multiplier and the limit of the original function. [Direct Proof]

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(19) **Intriguing Theory (Chapter 2 - 17):** Show that the function

$$f(x) = \begin{cases} x, & \text{if } x \text{ irrational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is continous at $x = 0$.

(20) Centered Difference Quotient: Assuming the limits exist, show that

$$\lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

3. Derivatives

(1) **Differentiability implies Continuity:** If f has a derivative at $x = c$, then f is continuous at $x = c$.

(2) **Derivative Rules:**
Constant Rule: The derivative of a constant is 0.

Constant Multiplier Rule (Direct): The derivative of a constant multiple of a differentiable function is equal that constant times the derviative of that differentiable function.

Sum and Difference Rules: The sum or difference of two differentiable functions is equal to the sum or difference of their derivatives.

Power Rule (positive integer) (Direct): For any positive integer n, the derivative of x^n is $df/dx = nx^{n-1}$.

Product Rule: For differentiable functions f and g, the derivative of their product is equal to $f'(x)g(x) + f(x)g'(x)$.

Quotient Rule (Direct): For differentiable functions f and g, with $g(c) \neq 0$, the derivative of their ratio $f(x)/g(x)$ at $x = c$ is

$$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Reciprocal Rule: Prove that the derivative of $f^{-1}(x)$ is $-f'(x)/f^2(x)$ using the product and constant rules.

Quotient Rule: Prove the quotient rule using the Product Rule and the Reciprocal Rule.

Reciropcal Rule: Prove the reciprocal rule using the quotient rule.

Power Rule (negative Integers): For any negative integer n, the derivative of x^n is $df/dx = nx^{n-1}$.

Power Rule (positive integer): Prove the power rule using the product rule and inductive reasoning.

Constant Multiplier Rule: Using the product and constant rules, prove the constant multiplier rule.

General Polynomial Derivative: The derivative of the polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ is $P'(x) = n a_n x^{n-1} + \cdots + a_1$.

Generalized Product Rule: The derivative of $F(x) = f(x)g(x)\dots z(x)$ is given by

$$\frac{dF}{dx} = f'(x)g(x)\dots z(x) + f(x)g'(x)\dots z(x) + \cdots + f(x)g(x)\dots z'(x)$$

(3) **Derivatives of Trigonometric Functions**
Sine: Prove that the derivative of $\sin x$ is $\cos x$.

Cosine: Prove that the derivative of $\cos x$ is $-\sin x$.

Tangent: Prove that the derivative of $\tan x$ is $\sec^2 x$.

Cotangent: Prove that the derivative of $\cot x$ is $-\csc^2 x$.

Secant: Prove that the derivative of $\sec x$ is $\sec x \tan x$.

Cosecant: Prove that the derivative of $\csc x$ is $-\csc x \cot x$

(4) **Chain Rule:** If $g(x)$ is differentiable at x, and $f(u)$ is differentiable at $u = g(x)$, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

(5) **Power Chain Rule:** The derivative of u^n with respect to x is $df/dx = nu^{n-1}du/dx$.

(6) **Power Rule (rational exponents):** Beginning from $y^q = x^p$, both p and q integers and $q \neq 0$, use implicit differentiation to show that

$$\frac{d}{dx}\left(x^{p/q}\right) = \frac{p}{q}x^{(p/q)-1}$$

(7) **Derivative of Cosine:** Beginning with the trigonometric identity $\cos^2 x + \sin^2 x = 1$, use implicit differentiation to show $(\cos x)' = -\sin x$.

(8) **Derivative of Inverses (Part A):** If $f'(x)$ exists and is never zero on interval I, then f^{-1} is differentiable at every point in the domain of f^{-1} (the range of f).

(9) **Derivative of Inverses (Part b):** Under the conditions above, the value of the derivative of f^{-1} at b is the reciprocal of the value of f' evaluated at $a = f^{-1}(b)$:

$$\frac{df^{-1}}{dx}|_{x=b} = \frac{1}{\frac{df}{dx}|_{x=f^{-1}(b)}}$$

(10) **Derivative of the Natural Log:** Using the theorem for the derivative of inverse functions and the fact that $(e^x)' = e^x$, prove that $(\ln x)' = 1/x$.

(11) **Derivative of the Exponential Function:** Using the theorem for the derivative of inverse functions and the fact that $(\ln x)' = 1/x$, prove that $(e^x)' = e^x$.

(12) **The Number e as a Limit:** Prove that the number e satisfies the limit $e = \lim_{x\to}(1+x)^{1/x}$.

(13) **Derivative of Inverse Trigonometric Functions**
Derivative of Arcsine: Prove that $(\arcsin x)' = 1/\sqrt{1-x^2}$.

Derivative of Arccosine: Prove that $(\arccos x)' = -1/\sqrt{1-x^2}$.

Derivative of Arctangent: Prove that $(\arctan x)' = 1/(1+x^2)$.

Derivative of Arccotangent: Prove that $(\cot^{-1} x)' = -1/(1+x^2)$.

Derivative of Arcsecant: Prove that $(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$.

Derivative of Arccosecant: Prove that $(\csc^{-1} x)' = \frac{-1}{|x|\sqrt{x^2-1}}$.

(14) **Derivative of an Odd Function:** If $f(x)$ is a differentiable odd function, then $f'(x)$ is an even function.

(15) **Derivative of an Even Function:** If $f(x)$ is a differentiable even function, then $f'(x)'$ is an odd function.

4. Optimization and Application

(1) **Global Extrema are Local Extrema:** Prove every absolute extremum is also a local extremum.

(2) **The First Derivative Theorem for Local Extremes:** If f has a local maximum or minimum at an interior point $x = c$, and if $f'(c)$ is defined, then $f'(c) = 0$.

(3) **Rolle's Theorem:** If $f(x)$ is continuous on closed interval $[a, b]$ and differentiable on open interval (a, b), and if $f(a) = f(b)$, then there exists some c in (a, b) such that $f'(c) = 0$.

(4) **The Mean Value Theorem:** If $f(x)$ is continuous on closed interval $[a, b]$ and differentiable on open interval (a, b), then there exists some c in (a, b) such that $f'(c) = [f(b) - f(a)]/(b - a)$.

(5) **Corollary 1 of the Mean Value Theorem:** If $f'(x) = 0$ for all x on open interval (a, b), then $f(x) = C$ constant for all x in the open interval.

(6) **Antiderivatives Differ by Only a Constant:** If $f'(x) = g'(x)$ for all x in (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x in (a, b)$.

(7) **Parallel Tangents (4.2 - 62):** Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Prove there is at least one point c between a and b where the tangesnts of f and g are parallel.

(8) **Indeterminate Powers:** Prove that if $\lim_{x\to a} \ln f(x) = L$, then $\lim_{x\to a} f(x) = e^L$.